MECHANICS APPLIED TO ENGINEERING

MECHANICS APPLIED TO ENGINEERING

BY

JOHN GOODMAN

WH. SCH., A.M.I.C.E., M.I.M.E.

PROFESSOR OF ENGINEERING IN THE YORKSHIRE COLLEGE, LEEDS
(VICTORIA UNIVERSITY)

With 620 Illustrations and Numerous Examples

LONGMANS, GREEN, AND CO.

39 PATERNOSTER ROW, LONDON

NEW YORK AND BOMBAY

1899

Phys. lib.
Prof. Alex. Ziwet
gt.
4-27-1923

PRINTED BY
WILLIAM CLOWES AND SONS, LIMITED,
LONDON AND BECCLES.

PREFACE

THIS book has been written expressly for engineers and students who have a fair knowledge of Theoretical Mechanics and Elementary Mathematics, to assist them in applying their knowledge to engineering problems. Those who are intending to enter for the examinations mentioned below, will find that much of the work required for them is covered by the following pages.

The Associate Membership Examination of the Institution of Civil Engineers;

The B.Sc. and B.A. degrees in Engineering that are conferred by some of the British Universities;

The Advanced and Honours Stages of the Science and Art Department Examinations in Applied Mechanics and Machine Construction;

The City and Guilds of London Technological Examination in Mechanical Engineering.

In preparing this book I have endeavoured to be concise, and have taken great pains to make every point clear without indulging in the wearisome diffuseness often found in text-books. At the same time, in certain cases I have deliberately avoided some of the short and direct methods in common use, because I believe that they do not appeal so forcibly to the student or bring home to him the principles involved so well as the methods adopted.

A friend who has read my proofs suggests that Chapters I., II., III. might have been omitted or very much curtailed. I feel, however, that so much of the work in the chapters that follow depends upon the early Mensuration and Moment work, that it is most important for the reader to get well grounded in the methods of arriving at such results, and not to blindly make use of mere tables.

My chapter on Mechanisms is much shorter than I originally intended it to be, mainly because my space was very limited; but when considering where to curtail, I thought it better to do so here rather than in other portions because

there is so much excellent literature on this subject available to students. Probably they cannot do better than turn to "The Mechanics of Machinery" by Professor Kennedy.

I have introduced the calculus very sparingly, and then only in its most elementary form. For the sake of those who avoid the calculus because they imagine that its use necessarily demands high attainments in Mathematics, I have given a popular and very elementary treatment in the Appendix, mainly for the sake of persuading such readers to take up the subject seriously by reading such books on the calculus as Perry's, Barker's, Smith's, and others. It is hoped that the instances in which it has been used in the Mensuration and Moment chapters may serve as useful examples to the beginner.

In order to assist readers who only occasionally refer to my pages, I have indexed all the symbols used throughout the book, except in the cases in which the meaning is perfectly obvious.

Readers should refer to the Appendix, as many notes have been added since the body of the book has been in the press. A large number of examples will also be found there. I much wish that I could have introduced a tolerably full account of the beautiful hydraulic experiments that have recently been made by Professor Hele-Shaw, LL.D. They, however, were not published until after the Hydraulic chapters were in print.

I am under great obligations to my friend Mr. A. H. Barker, B.A., B.Sc., who has taken a great deal of trouble in correcting my proofs, and verifying the work throughout; to him, also to Professor Hele-Shaw, I owe much for many valuable suggestions.

For assistance in making the drawings and for other valuable help, I wish to thank my colleague Mr. Andrew Forbes, also Mr. E. R. Verity and Mr. J. W. Jukes, two of my old students.

JOHN GOODMAN.

LEEDS, *March*, 1899.

CONTENTS

ERRATA

One or two errors have unfortunately escaped notice while passing through the press. On page 111 the values for p and p_1 are erroneous; they form a couple, and are therefore equal to one another, their value being $p = \frac{Wl}{x}$. On page 145, the = sign has been omitted halfway down the page. At the top of page 170, $\tan\theta = \frac{1 \cdot 5}{3 \cdot 5}$ instead of $\frac{1 \cdot 5}{5 \cdot 3}$.

MECHANICS APPLIED TO ENGINEERING

CHAPTER I.

INTRODUCTORY.

THE province of science is to ascertain truth from sources far and wide, to classify the observations made, and finally to embody the whole in some brief statement or formula. If some branches of truth have been left untouched or unclassified, the formula will only represent a part of the truth; such is the cause of discrepancies between theory and practice.

A scientific treatment of a subject is only possible when our statements with regard to the facts and observations are made in definite terms; hence, in an attempt to treat such a subject as Applied Mechanics from a scientific standpoint, we must at the outset have some means of making definite statements as to *quantity*. This we shall do by simply stating how many arbitrarily chosen units are required to make up the quantity in question.

We shall find that some of our units will be of a very complex character, but in every instance we shall be able to express them in terms of three fundamental units, viz. those of *time*, *mass*, and *space*. The complex units are usually termed "derived" units.

Units.

Time (t).—Unless otherwise stated, we shall take one second as the unit of time, but sometimes we shall find it convenient to take minutes and hours.

Mass (m).—Unit, one pound; occasionally hundredweights and tons.

1 pound (lb.)	= 0·454 kilogramme.
1 kilogramme	= 2·2 lbs.
1 hundredweight (cwt.)	= 50·8 kilos.
1 ton	= 1016 „ (tonneau or Millier).
1 tonneau or Millier	= 0·984 ton.

Space (s).—Unit, one foot; occasionally inches, yards, and miles. Such terms as distance, length, breadth, width, thickness, are frequently used to denote space in various directions.

1 foot	= 0·305 metre.
1 metre	= 3·28 feet.
1 inch	= 25·4 millimetres (nearly).
1 millimetre	= 0·0394 inch.
1 yard	= 0·914 metre.
1 metre	= 3·28 feet.
1 mile	= 1609·3 metres.
1 kilometre	= 1093·63 yards
	= 0·621 mile.

Dimensions.—The relation which exists between any given complex unit and the fundamental units is termed the dimensions of the unit. As an example, see p. 20, Chap. II.

Speed.—When a body changes its position relatively to surrounding objects, it is said to be in motion. The rate at which a body changes its position is termed the speed of the body.

Uniform Speed.—A body is said to have uniform speed when it traverses equal spaces in equal intervals of time. The

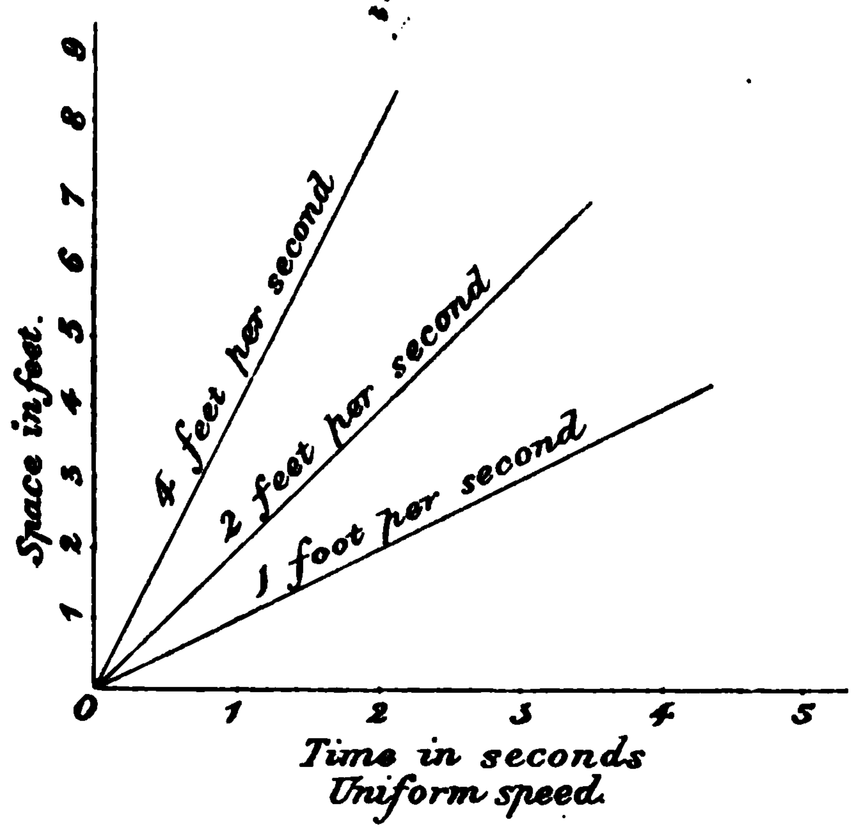


FIG. 1.

body is said to have unit speed when it traverses unit space in unit time.

$$\text{Speed (in feet per second)} = \frac{\text{space traversed (feet)}}{\text{time (seconds)}} = \frac{s}{t}$$

Varying Speed.—When a body does not traverse equal spaces in equal intervals of time, it is said to have a varying speed. The speed at any instant is the space traversed in an exceedingly short interval of time divided by that interval; the shorter the interval taken, the more nearly will the true speed be arrived at.

In Fig. 1 we have a diagram representing the distance travelled by a body moving with uniform speed, and in Fig. 2, varying speed. The speed at any instant, *a*, can be found by drawing a tangent to the curve as shown. From the slope of this tangent we see that, if the speed had been

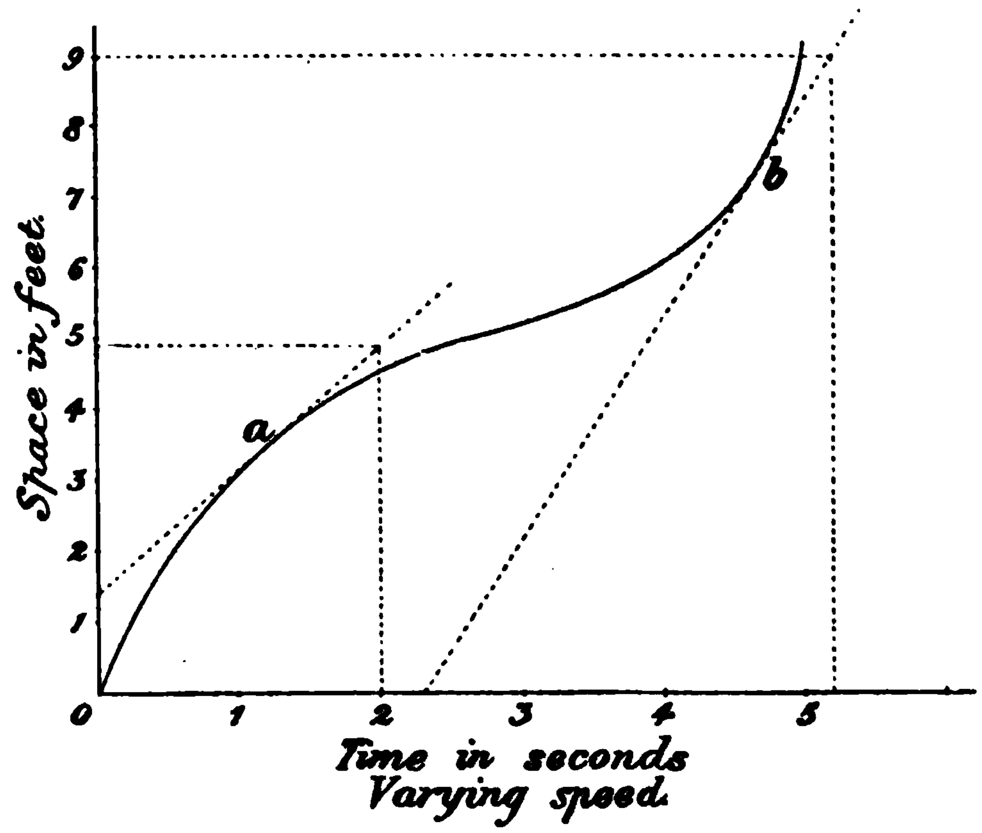


FIG. 2.

uniform, a space of $4 \cdot 9 - 1 \cdot 4 = 3 \cdot 5$ ft. would have been traversed in 2 secs., hence the speed at *a* is $\frac{3 \cdot 5}{2} = 1 \cdot 75$ ft. per second. Similarly, at *b* we find that 9 ft. would have been traversed in $5 \cdot 2 - 2 \cdot 3 = 2 \cdot 9$ secs., or the speed at *b* is $\frac{9}{2 \cdot 9} = 3 \cdot 1$ ft. per second. The same result will be obtained by taking any point on the tangent. For a fuller discussion of variable quantities, the reader is referred to either Perry's or Barker's Calculus.

Velocity (v).—The velocity of a body is the magnitude of its speed in any given direction; thus the velocity of a body may be changed by altering the *speed* with which it is moving, or by altering the *direction* in which it is moving. It does not

follow that if the speed of a body be uniform the velocity will be also. The idea of velocity embodies *direction* of motion, that of speed does not.

The speed of a point on a uniformly revolving wheel is constant, but the velocity is changing at every instant. Velocity and speed, however, have the same dimensions. The unit of velocity is usually taken as one foot per second.

$$\left.\begin{array}{r}\text{Velocity in feet}\\ \text{per second}\end{array}\right\} = \frac{\text{space (feet) traversed in a given direction}}{\text{time (seconds)}}$$

$$v = \frac{s}{t}, \text{ or } s = vt$$

1 ft. per second	= 0·305 metre per second
„ „	= 0·682 mile per hour
„ „	= 1·1 kilo. „
1 metre per second	= 3·28 ft. per second
1 mile per hour	= 1·467 „ „
	= 0·447 metre per second
1 kilo. „	= 0·912 ft. „
	= 0·278 metre „

Angular Velocity (ω), *or Velocity of Spin.*—Suppose a body to be spinning about an axis. The rate at which an angle is described by any line perpendicular to the axis is termed the angular velocity of the line or body, or the velocity of spin; the direction of spin must also be specified, as in the case of linear velocity. When a body spins round in the direction of the hands of a watch, it is termed a + or positive spin; and in the reverse direction, a − or negative spin.

As in the case of linear velocity, angular velocity may be uniform or varying.

The unit of angular measure is a "radian;" that is, an angle subtending an arc equal in length to the radius. The length of a circular arc subtending an angle $\theta°$ is $2\pi r \times \frac{\theta°}{360°}$, where π is the ratio of the circumference to the diameter ($2r$) of a circle, and θ is the angle subtended (see p. 22).

Then, when the arc is equal to the radius, we have—

$$\frac{2\pi r\theta}{360} = r$$

$$\theta = \frac{360}{2\pi} = 57·296°$$

practically $57·3°$

Thus, if a body be spinning in such a manner that a radius describes 100 degrees per second, its angular velocity is—

$$\omega = \frac{100}{57 \cdot 3} = 1 \cdot 75 \text{ radians per second}$$

It is frequently convenient to convert angular into linear velocities, and the converse. When one radian is described per second, the extremity of the radius vector describes every second a space equal to the radius, hence the space described in one second is $\omega r = v$, or $\omega = \frac{v}{r}$.

$$\text{Angular velocity in radians per sec.} = \frac{\text{linear velocity (ft. per sec.)}}{\text{radius (ft.)}}$$

The radius is a space quantity, hence—

$$\omega = \frac{s}{ts} = \frac{1}{t}$$

Thus an angular velocity is not affected by the unit of space adopted, and only depends on the time unit, but the time unit is one second in all systems of measurement, hence all angular measurements are the same for all systems of units—an important point in favour of using angular measure.

Acceleration (f) is the rate at which the velocity of a body increases in unit time—that is, if we take feet and seconds units, the acceleration is the number of feet per second that the velocity increases in one second; thus, unit acceleration is an increase of velocity of one foot per second per second. It should be noted that acceleration is the rate of change of *velocity*, and not merely change of speed. The speed of a body in certain cases does not change, yet there is an acceleration due to the change of direction (see p. 18).

As in the case of speed and velocity, acceleration may be either uniform or varying.

$$\left.\begin{array}{l}\text{Uniform ac-}\\\text{celeration}\\\text{in feet per}\\\text{sec. per sec.}\end{array}\right\} = \frac{\text{increase of velocity in ft. per sec. in a given time}}{\text{time in seconds}}$$

$$f = \frac{v_2 - v_1}{t} = \frac{v}{t}$$

$$\text{hence } v = ft, \text{ or } v_2 - v_1 = ft \quad . \quad . \quad . \quad . \quad \text{(i.)}$$

where v_2 is the velocity at the end of the interval of time, and v_1 at the beginning, and v is the increase of velocity. In Fig. 3, the vertical distance of any point on the line *ab* from the base line shows the velocity of a body at the corresponding instant; it is straight because the acceleration is assumed constant, and therefore the velocity increases directly as the time. If the body start from rest, when v_1 is zero, the mean velocity over any interval of time will be $\frac{v_2}{2}$, and the space traversed in the interval will be the mean velocity × time, or—

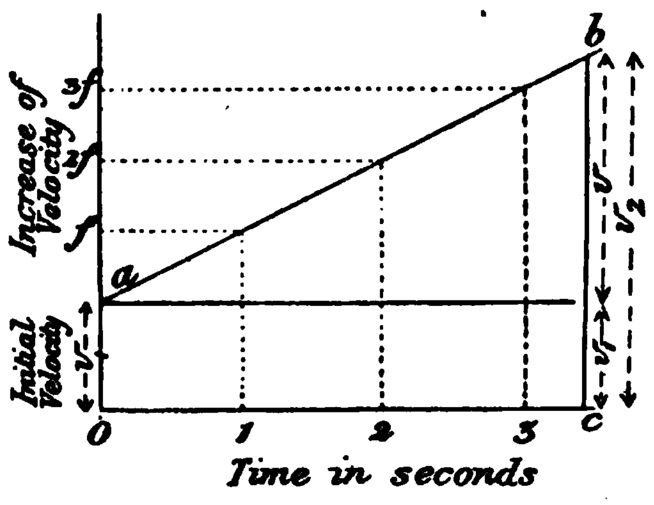


FIG. 3.

$$s = \frac{v_2}{2}t = \frac{ft^2}{2} \quad \text{(see equation i.)}$$

$$\text{and } f = \frac{2s}{t^2}$$

$$\text{Acceleration in feet per sec. per sec.} = \frac{\text{constant} \times \text{space (in } feet)}{(\text{time})^2 \text{ (in } seconds)}$$

When the body has an initial velocity v_1, the mean velocity during the time t is the mean height of the figure *oabc*.

$$\text{Mean velocity} = \frac{v_2 + v_1}{2} = \frac{v + 2v_1}{2} = v_1 + \frac{ft}{2} \quad . \quad \text{(ii.)}$$

(see equation i.)

The space traversed in the time t—

$$s = \left(v_1 + \frac{ft}{2}\right)t \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(iii.)}$$

which is represented in the diagram by the area of the diagram *oabc*.

By multiplying equations i. and ii., we get—

$$\frac{v_2^2 - v_1^2}{2} = \left(v_1 + \frac{ft}{2}\right)ft$$

Substituting from iii., we get—

$$\frac{v_2^2 - v_1^2}{2} = \left(\frac{s}{t}\right)ft = fs$$

$$v_2^2 - v_1^2 = 2fs$$

$$\text{or } v_2^2 = v_1^2 + 2fs$$

When a body falls freely due to gravity, $f = g = 32{\cdot}2$ ft. per second per second, it is then usual to use the letter h, the height through which the body has fallen, instead of s.

When the body starts from rest, we have $v_1 = 0$, and $v_2 = v$; then by substitution from above, we have—

$$v^2 = 2gh$$
$$v = \sqrt{2gh} = 8{\cdot}02\sqrt{h} \quad . \quad . \quad . \quad . \quad . \quad \text{(iv.)}$$

Momentum (M). If a body of mass m[1] move with a velocity v, the moving mass is said to possess *momentum*, or *quantity of motion*, $= mv$.

Unit momentum is that of unit mass moving with unit velocity—

$$\text{M} = mv = \frac{ms}{t}$$

Impulse.—Consider a ball of mass m travelling through the air with a velocity v_1 and let it receive a fair blow (without causing it to spin) as it travels along, in such a manner that its velocity is suddenly increased from v_1 to v_2.

The momentum before the blow $= mv_1$
" after " $= mv_2$
The change of momentum due to the blow $= m(v_2 - v_1)$

The effect of the blow is termed an *impulse*, and is measured by the change of momentum.

Impulse = change of momentum $= m(v_2 - v_1)$

Force (F).—If the ball in the paragraph above had received a very large number of very small impulses instead of a single blow, its velocity would have been gradually changed, and we should have—

The whole impulse per second = the change of momentum per second.

When the impulses become infinitely rapid the whole impulse per second is termed the *force* acting on the body. Hence the momentum may be changed gradually from v_1 to v_2 by a force acting for t seconds. Then—

[1] For a rational definition of *mass*, the reader is referred to Prof. Karl Pearson's "Grammar of Science," p. 357.

$$Ft = m(v_2 - v_1)$$

$$\text{where } F = \frac{\text{total change of momentum}}{\text{time}}.$$

$$\text{But } \frac{v_2 - v_1}{t} = f \text{ (acceleration) (see p. 5)}$$

$$\text{hence } F = mf = \frac{mv}{t}$$

Hence the dimensions of this unit are—

$$F = \frac{ms}{t^2}$$

$$\text{Force} = \text{mass} \times \text{acceleration}$$
$$\text{unit force} = \text{unit mass} \times \text{unit acceleration}$$

Thus unit force is that force which, when acting on a mass of one $\begin{Bmatrix}\text{pound}\\ \text{gram}\end{Bmatrix}$ for one second, will change its velocity by one $\begin{Bmatrix}\text{foot}\\ \text{centimetre}\end{Bmatrix}$ per second, and is termed one $\begin{Bmatrix}\text{poundal.}\\ \text{dyne.}\end{Bmatrix}$

We are now in a position to appreciate the words of Newton—

Change of momentum is proportional to the impressed force, and takes place in the direction of the force; . . . also, *a body will remain at rest, or, if in motion, will move with a uniform velocity in a straight line unless acted upon by some external force.*

Our only knowledge of force is from the accelerative effect it produces; force simply describes *how* motion takes place, *not why* it takes place.

It does not follow, because the velocity of a body is not changing, or because it is at rest, that no forces are acting upon it; for suppose the ball mentioned above had been acted upon by two equal and opposite forces at the same instant, the one would have tended to accelerate the body backwards (termed a negative acceleration or retardation) just as much as the other tended to accelerate it forwards, with the result that the one would have just neutralized the other and the velocity, and consequently the momentum would have remained unchanged. We say then, in this case, that the positive acceleration is equal and opposite to the negative acceleration.

If a railway train be running at a constant velocity, it must not be imagined that no force is required to draw it; the force exerted by the engine produces a positive acceleration, while

the friction on the axles, tyres, etc., produces an equal and opposite negative acceleration. If the velocity of the train be constant, the whole effort exerted by the engine is expended in overcoming the frictional resistance, or the negative acceleration. If the positive acceleration at any time exceeds the negative acceleration due to the friction, the positive or forward force exerted by the engine will still be equal to the negative or backward force or the total resistance overcome; but the resistance now consists partly of the frictional resistance, and partly the resistance of the train to having its velocity increased. The work done by the engine over and above that expended in overcoming friction is stored up in the moving mass of the train as energy of motion, or kinetic energy (see p. 14).

UNITS OF FORCE.

Force.	Mass.	Acceleration.
Poundal.	One pound.	One foot per second per second.
Dyne.	One gram.	One centimetre per second per second.

1 poundal = 13,825 dynes.
1 pound = 445,000 dynes.

Weight (W).—The weight of a body is the force that gravity exerts on that body. It depends (1) on the mass of the body; (2) on the acceleration of gravity (g), which varies inversely as the square of the distance from the centre of the earth, hence the weight of a body depends upon its position as regards the centre of the earth. The distance, however, of all inhabited places on the earth from the centre is so nearly constant, that for all practical purposes we assume that the acceleration of gravity is constant (the extreme variation is about one-third of one per cent.). Consequently for practical purposes we compare masses by their weights.

$$\text{Weight} = \text{mass} \times \text{acceleration of gravity}$$
$$W = mg$$

We have shown above that—

$$\text{Force} = \text{mass} \times \text{acceleration}^{1}$$

[1] Expressing this in absolute units, we have—

Weight or force (poundals) = mass (pounds) × acceleration (feet per second per second)

Then—

Force of gravity on a mass of one pound = 1 × 32·2 = 32·2 poundals

But, as poundals are exceedingly inconvenient units to use for practical

hence we speak of forces as being equal to the weight of so many pounds; but for convenience of expression we shall speak of forces of so many pounds, or of so many tons, as the case may be.

VALUES OF g.[1]

		In foot-pounds, secs.		In centimetre-grammes, secs.
The equator	... ...	32·091	...	978·10
London	... ...	32·191	...	981·17
The pole	... ...	32·255	...	983·11

Work.—When a body is moved so as to overcome a resistance, we know that it must have been acted upon by a force acting in the direction of the displacement. The force is then said to perform work, and the measure of the work done is the product of the force and the displacement. The absolute unit of work is unit force (one poundal) acting through unit displacement (foot), *or one foot-poundal.* Such a unit of work is, however, never used by engineers; the unit nearly always used in England is the "foot-pound," *i.e.* one pound weight lifted one foot high.

$$\begin{aligned}\text{Work} &= \text{force} \times \text{displacement}\\ &= \text{FS}\end{aligned}$$

The dimensions of the unit of work are therefore $\dfrac{ms^2}{t^2}$

purposes, we shall adopt the engineer's unit of one pound, *i.e.* a unit 32·2 times as great; then, in order that the fundamental equation may hold for this unit, viz.—

$$\text{Weight or force (pounds)} = \text{mass} \times \text{acceleration}$$

we must divide our weight or force expressed in poundals by 32·2, and we get—

$$\text{Weight or force (pounds)} = \frac{\text{weight or force (poundals)}}{32{\cdot}2} = \frac{\text{mass} \times \text{acceleration}}{32{\cdot}2}$$

or—

$$\text{weight or force (pounds)} = \frac{\text{mass in pounds}}{32{\cdot}2} \times \text{acceleration in ft.-sec. per sec.}$$

thus we must take our new unit of mass as 32·2 times as great as the absolute unit of mass.

Readers who do not see the point in the above had better leave it alone—at any rate, for the present, as it will not affect any question we shall have to deal with. As a matter of fact, engineers always do (probably unconsciously) make the assumption, but do not explicitly state it.

[1] Hicks's "Elementary Dynamics," p. 45.

Frequently we shall have to deal with a variable force acting through a given displacement; the work done is then the average[1] force multiplied by the displacement. Methods of finding such averages will be discussed later on. In certain cases it will be convenient to remember that the work done in lifting a body is the weight of the body multiplied by the height the centre of gravity of the body is lifted.

UNITS OF WORK.

Force.	Displacement.	Unit of work.
Poundal.	Foot.	Foot-poundal.
Pound.	Foot.	Foot-pound.
Kilogram.	Metre.	Kilogrammetre.
Dyne.	Centimetre.	Erg.

1 foot-pound = 32·2 foot-poundals.
" = 13,560,000 ergs.

Power.—Power is the rate of doing work. Unit power is unit work done in unit time, or one foot-pound per second.

$$\text{Power} = \frac{\text{total work done}}{\text{time taken to do it}} = \frac{FS}{t}$$

The dimensions of the unit of power are therefore $\dfrac{ms^2}{t^3}$.

The unit of power commonly used by engineers is an arbitrary unit established by James Watt, viz. a horse-power, which is 33,000 foot-pounds of work done per minute.

$$\text{Horse-power} = \frac{\text{foot-pounds of work done in a given time}}{\text{time (in minutes) occupied in doing the work} \times 33{,}000}$$

1 horse-power = 33,000 foot-pounds per minute
= $7·46 \times 10^9$ ergs per second.
1 French horse-power = 32,500 foot-pounds per minute
= $7·36 \times 10^9$ ergs per second.
1 watt = 746 foot-pounds per minute
= 10^7 ergs per second.

Couples.—When forces act upon a body in such a manner as to tend to give it a spin or a rotation about an axis without any tendency to shift its c. of g., the body is said to be acted

[1] Space-average.

upon by a couple. Thus, in the figure the force F tends to turn the body round about the point O, its c. of g. If, however, this were the only force acting on the body, it would have a motion of translation in the direction of the force as well as a spin round the axis; in order to prevent this motion of translation, another force, F_1, equal and parallel but opposite in direction to F, must be applied to the body in the same plane. Thus, a couple is said to consist of two parallel forces of equal magnitude acting in opposite directions, but not in the same straight line.

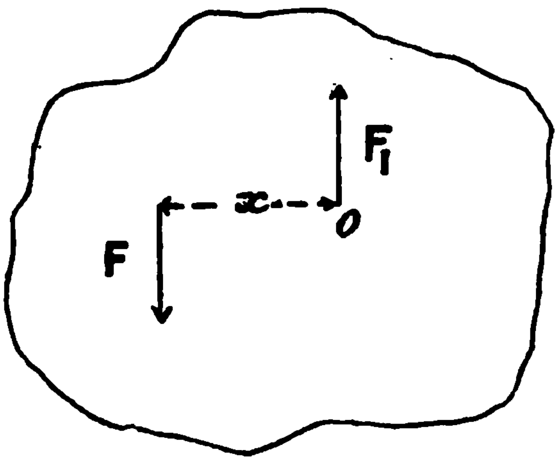


FIG. 4.

The perpendicular distance x between the forces is termed the *arm* of the couple. The tendency of a couple is to turn the body to which it is applied in the plane of the couple. When it tends to turn it in the direction of the hands of a watch, it is termed a clockwise or positive (+) couple, and in the contrary direction, a contra-clockwise or negative (−) couple.

It is readily proved[1] that not only may a couple be shifted anywhere in its own plane, but its arm may be altered (as long as its moment is kept the same) without affecting the equilibrium of the body.

Moments.—The moment of a couple is the product of one of the forces and the length of the arm. It is usual to speak of the moment of a force about a given point—that is, the product of the force and the perpendicular distance from its line of action to the point in question.

As in the case of couples, moments are spoken of as clockwise, contra-clockwise, etc.

If a rigid body be in equilibrium under any given system of moments, the algebraic sum of all the moments in any given plane must be zero, or the clockwise moments must be equal to the contra-clockwise moments in any given plane.

$$\begin{aligned} \text{Moment} &= \text{force} \times \text{arm} \\ &= \text{FS} \end{aligned}$$

The dimensions of a moment are therefore $\frac{ms^2}{t^2}$.

[1] See Hicks's "Elementary Mechanics."

Centre of Gravity (c. of g.).—The gravitation forces acting on the several particles of a body may be considered to act parallel to one another.

If a point be so chosen in a body that the sum of the moments of all the gravitation forces acting on the several particles about the one side of any straight line passing through that point be equal to the sum of the moments on the other side of the line, that point is termed the centre of gravity of the body.

Thus, the resultant of all the gravitation forces acting on a body passes through its centre of gravity, however the body may be tilted about.

Centroid.—In certain cases in which parallel systems of forces are concerned, the point referred to in the last paragraph is frequently termed the centroid; such cases are fully dealt with in Chapter III.

Energy.—Capacity for doing work is termed energy.

Conservation of Energy.—Experience shows us that energy cannot be created or destroyed; it may be dissipated, or it may be transformed from any one form to any other, hence the whole of the work supplied to any machine must be equal to the work got out of the machine, together with the work converted into heat,[1] either by the friction or the impact of the parts one on the other.

Mechanical Equivalent of Heat.—It was experimentally shown by Joule that in the conversion of mechanical into heat energy,[2] 772 foot-lbs. of work have to be expended in order to produce one thermal unit.

Efficiency of a Machine.—The efficiency of a machine is the ratio of the useful work got out of the machine to the gross work supplied to the machine.

$$\text{Efficiency} = \frac{\text{work got out of the machine}}{\text{work supplied to the machine}}$$

This ratio is necessarily less than unity.

The counter-efficiency is the reciprocal of the efficiency, and is always greater than unity.

$$\text{Counter-efficiency} = \frac{\text{work supplied to the machine}}{\text{work got out of the machine}}$$

[1] To be strictly accurate, we should also say light, sound, electricity, etc.

[2] By far the most accurate determination that has ever been made is that recently accomplished by Professor Osborne Reynolds and Mr. W. H. Moorby, who obtained the value 776·94 (see *Phil. Trans.*, vol. 190, pp. 301–422).

Kinetic Energy.—From the principle of the conservation of energy, we know that when a body falls freely by gravity, the work done on the falling body must be equal to the energy of motion stored in the body (neglecting friction).

The work done by gravity on a weight of W pounds in falling through a height h ft. = Wh foot-lbs. But we have shown above that $h = \frac{v^2}{2g}$, where v is the velocity after falling through a height h; whence—

$$Wh = \frac{Wv^2}{2g}, \text{ or } \frac{mv^2}{2}$$

This quantity, $\frac{Wv^2}{2g}$, is known as the kinetic energy of the body, or the energy due to its motion.

Inertia.—Since energy has to be expended when the velocity of a body is increased, a body may be said to offer a resistance to having its velocity increased, this resistance is known as the inertia of the body. Inertia is sometimes defined as the capacity of a body to possess momentum.[1]

Moment of Inertia (I).—We have defined inertia as the capacity of a body to possess momentum, and momentum as the product of mass and velocity (mv). If we have a very small body of mass m rotating about an axis at a radius r, with an angular velocity ω, the linear velocity of the body will be $v = \omega r$, and the momentum will be mv. But if the body be shifted further from the axis of rotation, and r be thereby increased, the momentum will also be increased in the same ratio. Hence, when we are dealing with a rotating body, we have not only to deal with its mass, but with the *arrangement* of the body about the axis of rotation, *i.e.* with its moment about the axis.

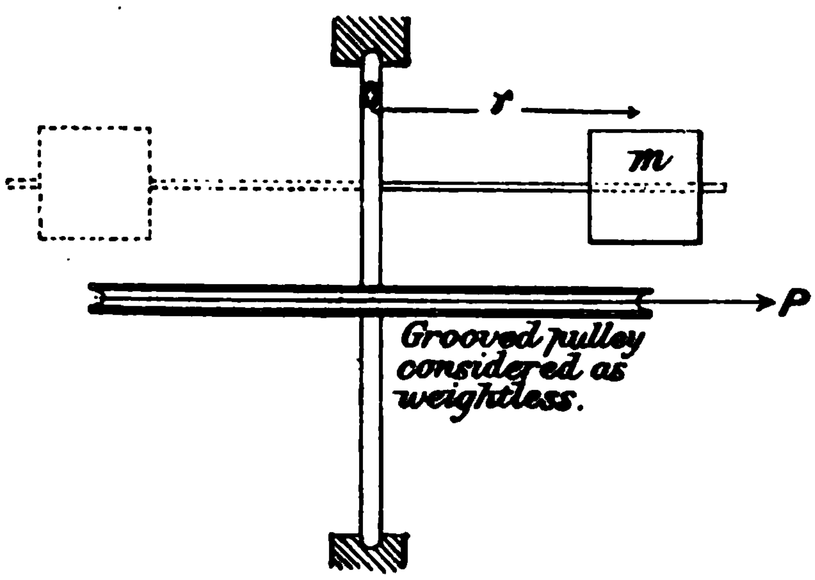


FIG. 5.

Let the body be acted upon by a twisting moment, Pr = T,

[1] Hicks's "Elementary Dynamics," p. 28.

then, as the force P acts at the same radius as that of the body, it may be regarded as acting on the body itself. The force P acting at a radius r will produce the same effect as a force nP acting at a radius $\frac{r}{n}$. The force P acting on the mass m gives it a linear acceleration f, where $P = mf$, or $f = \frac{P}{m}$. The angular velocity ω is $\frac{1}{r}$ times the linear velocity, hence the angular acceleration is $\frac{1}{r}$ times the linear acceleration. Let A = the angular acceleration; then—

$$A = \frac{f}{r} = \frac{P}{mr} = \frac{Pr}{mr^2} = \frac{T}{mr^2}$$

$$\text{or angular acceleration} = \frac{\text{twisting moment}^1}{\text{mass} \times (\text{radius})^2}$$

In the case we have just dealt with, the mass m is supposed to be exceedingly small, and every part of it at a distance r from the axis. When the body is great, it may be considered to be made up of a large number of small masses, m_1, m_2, etc., at radii r_1, r_2, etc., respectively; then the above expression becomes—

$$A = \frac{T}{(m_1r_1^2 + m_2r_2^2 + m_3r_3^2 +, \text{etc.})}$$

The quantity in the denominator is termed the "moment of inertia" of the body.

We stated above that the capacity of a body to possess momentum is termed the "inertia of the body." Now, in a case in which the capacity of the body to possess angular momentum depends upon the moment of the several portions of the body about a given axis, we see why the capacity of a rotating body to possess momentum should be termed the "moment of inertia."

Let M = mass of the whole body, then $M = m_1 + m_2 + m_3$, etc.; then the moment of inertia of the body, I, $= M\kappa^2 = (m_1r_1^2 + m_2r_2^2$, etc.).

Radius of Gyration (κ).—The κ in the paragraph above is known as the radius of gyration of the body. Thus, if we could condense the whole body into a single particle at a distance κ from the axis of rotation, the body would still have

[1] The reader is advised to turn back to the paragraph on "couples," so that he may not lose sight of the fact that a couple involves *two* forces.

the same capacity for possessing energy, due to rotation about that axis.

Representation of Displacements, Velocities,[1] **Accelerations, Forces by Straight lines.**—Any $\left\{\begin{array}{l}\text{displacement}\\ \text{velocity}\\ \text{acceleration}\\ \text{force}\end{array}\right\}$ is fully represented when we state its magnitude and its direction.

Hence a straight line may be used to represent any $\left\{\begin{array}{l}\text{displacement}\\ \text{velocity}\\ \text{acceleration}\\ \text{force}\end{array}\right\}$, the length of which represents its magnitude, and the direction of the line the direction in which the force, etc., acts.

Two or more $\left\{\begin{array}{l}\text{displacements}\\ \text{velocities}\\ \text{accelerations}\\ \text{forces}\end{array}\right\}$, meeting at a point, may be replaced by one force, etc., passing through the same point, which is termed the resultant force, etc.

If two $\left\{\begin{array}{l}\text{displacements}\\ \text{velocities}\\ \text{accelerations}\\ \text{forces}\end{array}\right\}$, not in the same straight line, meeting at a point *a*, be represented by two straight lines, *ab*, *ac*, and if two other straight lines, *dc*, *bd*, be drawn parallel to them from their extremities to form a parallelogram, *abdc*, the diagonal of the parallelogram *ad* which passes through that point will represent the resultant $\left\{\begin{array}{l}\text{displacement}\\ \text{velocity}\\ \text{acceleration}\\ \text{force}\end{array}\right\}$ in magnitude and direction.

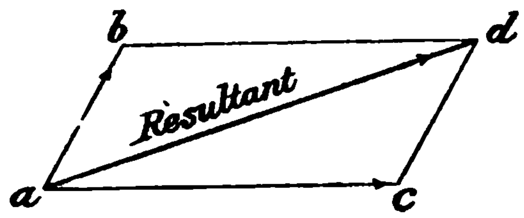


Fig. 6.

Hence, if a force equal and opposite to *ad* act on the point, the point will be in equilibrium.

It is evident from the figure that *bd* is equal in every respect to *ac*; then the three forces are represented by the three sides of the triangle *ab*, *bd*, *ad*. Hence we may say that if three

[1] Including angular velocities or spins.

forces act upon a point in such a manner that they are equal and parallel to the sides of a triangle, the point is in equilibrium under the action of those forces. This is known as the theorem of the "triangle of forces."

Many special applications of this method will be dealt with in future chapters.

The proof of the above statements will be found in all elementary books on Mechanics.

Hodograph.—The motion of a body moving in a curved path may be very conveniently analyzed by means of a curve called a "hodograph." In Fig. 7, suppose a point moving along the path P, P_1, P_2, with varying velocity. If a line, *op*, known as a "radius vector," be drawn so that its length represents on any given scale the speed of the point at P, and the direction of the radius vector the direction in which P is moving, the line *op* completely represents the velocity of the point P. If other radii are drawn in the same manner, the curve traced out by their extremities is known as the "hodograph" of the point P. The change of velocity of the point P in passing from P to P_1 is represented on the hodograph by the distance pp_1, consisting of a change in the length of the line, viz. q_1p_1 representing the change in speed of the point P, and pq_1 the change of velocity due to change of direction, if a radius vector be drawn each second; then pp_1 will represent the change of velocity per second, or the rate of change of velocity of the point P, or, in other words, the acceleration (see p. 5) of the point P; thus the velocity of p represents the acceleration of the point P.

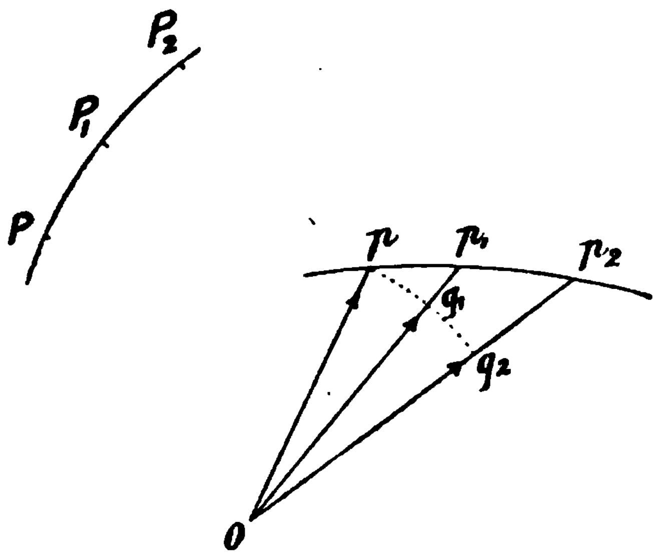


FIG. 7.

If any interval of time other than a second be taken, the space pp_1 must be divided by the actual time, in order to get the velocity with which the hodograph is described.

If the speed of the point P remained constant, then the length of the line *op* would also be constant, and the hodograph would become the arc of a circle, and the only change in the velocity would be the change in direction pq_1.

Centrifugal Force.—If a heavy body be attached to the end of a piece of string, and the body be caused to move round

in the arc of a circle, the string will be found to be in tension, the amount of which will depend upon (1) the mass of the body, (2) the length of the string, and (3) the velocity with which the body moves. The tension in the string is termed the centrifugal force. We will now show how the exact value of this force may be calculated in any given instance.[1]

Let the speed with which the body describes the circle be constant; then the radius vector of the hodograph will be of constant length, and the hodograph itself will be a circle. Let the body describe the outer of the two circles shown in the figure, with a velocity v, and let its velocity at A be represented by the radius OP, the inner circle being the hodograph of A. Now let A move through an extremely small space to A_1, and the corresponding radius vector to OP_1; then the line PP_1 represents the change in velocity of A while it was moving to A_1. (The reader should never lose sight of the fact that change of velocity involves change of direction as well as change of speed, and as the speed is constant in this case, the change of velocity is wholly a change of direction.)

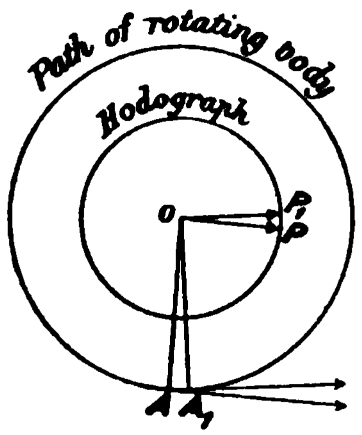


FIG. 8.

As the distance AA_1 becomes smaller, PP_1 becomes more nearly perpendicular to OP, and in the limit it does become perpendicular, and parallel to OA; thus the change of velocity is radial and towards the centre.

We have shown on p. 17 that the velocity of P represents the acceleration of the point A; then, as both circles are described in the same time—

$$\frac{\text{velocity of P}}{\text{velocity of A}} = \frac{\text{OP}}{\text{OA}}$$

But OP was made equal to the velocity of A, viz. v, and OA is the radius of the circle described by the body. Let OA = R; then—

$$\frac{\text{velocity of P}}{v} = \frac{v}{\text{R}}$$

$$\text{or velocity of P} = \frac{v^2}{\text{R}}$$

[1] For another method of treatment, see Barker's "Graphic Methods of Engine Design."

$$\text{and acceleration of A} = \frac{v^2}{R}$$

$$\text{and since force} = \text{mass} \times \text{acceleration}$$

$$\text{we have centrifugal force C} = \frac{mv^2}{R}$$

$$\text{or in gravitational units C} = \frac{Wv^2}{gR}$$

We have shown above that this force acts radially inwards towards the centre.

Sometimes it is convenient to have the centrifugal force expressed in terms of the angular velocity of the body. We have—

$$v = \omega R$$

$$\text{hence C} = m\omega^2 R$$

$$\text{or C} = \frac{W\omega^2 R}{g}$$

Change of Units.—It frequently happens that we wish to change the units in a given expression to some other units more convenient for our immediate purpose; such an alteration in units is very simple, provided we set about it in systematic fashion. The expression must first be reduced to its fundamental units; then each unit must be multiplied by the required constant to convert it into the new unit. For example, suppose we wish to convert foot-pounds of work to ergs, then—

The dimensions of work are $\frac{ms^2}{t^2}$

$$\text{work (in ft.-poundals)} = \frac{\text{pounds} \times (\text{feet})^2}{(\text{seconds})^2}$$

$$\text{work in ergs} = \frac{\text{grams} \times (\text{centimetres})^2}{(\text{seconds})^2}$$

$$1 \text{ pound} = 453{\cdot}6 \text{ grams}$$

$$1 \text{ foot} = 30{\cdot}48 \text{ centimetres}$$

Hence—

$$1 \text{ foot-poundal} = 453{\cdot}6 \times 30{\cdot}48^2 = 421{,}390 \text{ ergs}$$

$$\text{and } 1 \text{ foot-pound} = 32{\cdot}2 \text{ foot-poundals}$$

$$= 32{\cdot}2 \times 421{,}390 = 13{,}560{,}000 \text{ ergs}$$

CHAPTER II.

MENSURATION.

MENSURATION consists of the measurement of lengths, areas, and volumes, and the expression of such measurements in terms of a simple unit of length.

Length.—If a point be shifted through any given distance, it traces out a line in space, and the length of the line is the distance the point has been shifted. A simple statement in units of length of this *one* shift completely expresses its only dimension, *length;* hence a line is said to have but *one* dimension, and when we speak of a line of length l, we mean a line containing l length units.

Area.—If a straight line be given a side shift in any given plane, the line sweeps out a surface in space. The area of the surface swept out is dependent upon *two* distinct shifts of the generating point: (1) on the length of the original shift of the point, *i.e.* on the length of the generating line (l); (2) on the length of the side shift of the generating line (d).

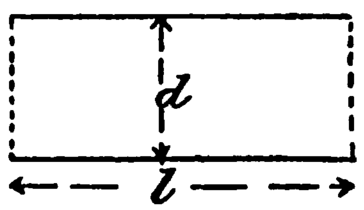


FIG. 9.

Thus a statement of the area of a given surface must involve *two* length quantities, l and d, both expressed in the same units of length. Hence a surface is said to have *two* dimensions, and the area of a surface ld must always be expressed as the product of two lengths, each containing so many length units, viz.—

$$\text{Area} = \text{length units} \times \text{length units}$$
$$= (\text{length units})^2$$

Volume.—If a plane surface be given a side shift to bring it into another plane, the surface sweeps out a volume in space.

The volume of the space swept out is dependent upon *three* distinct shifts of the generating point: (1) on the length of the original shift of the generating point, *i.e.* on the length of the generating line l; (2) on the length of the side shift of the generating line d; (3) on the side shift of the generating surface t. Thus the statement of the volume of a given body or space must involve *three* length quantities, l, d, t, all expressed in the same units of length.

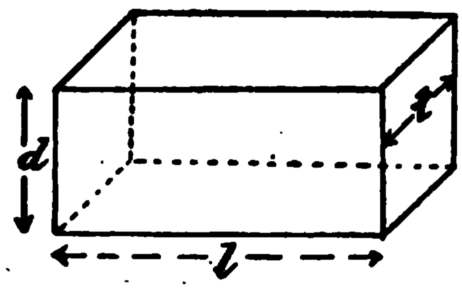


FIG. 10.

Hence a volume is said to have *three* dimensions, and the volume of a body must always be expressed as the product of *three* lengths, each containing so many length units, viz.—

$$\begin{aligned}\text{Volume} &= \text{length units} \times \text{length units} \times \text{length units}\\ &= (\text{length units})^3\end{aligned}$$

Lengths.

Straight line.

Circumference of circle.

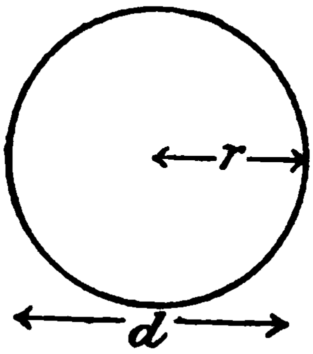


FIG. 11.

Length of circumference $= \pi d$
$= 3{\cdot}1416d$
or $= 2\pi r$
$= 6{\cdot}2832r$

The last two decimals above may usually be neglected; the error will be less than $\frac{1}{8}$ in. on a 10-ft. circle.

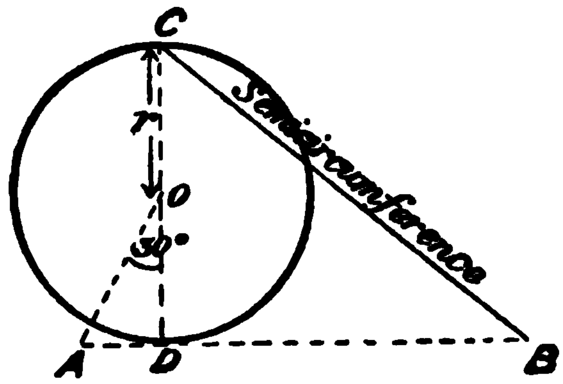


FIG. 12.

Ceradini's Approximate Graphical Method.—Draw diameter CD and tangent AB; with a 30° set square, set off OA. Make AB $= 3r$. Join BC. Then BC $= \pi r$, *i.e.* the semicircumference (nearly).

Arc of circle.

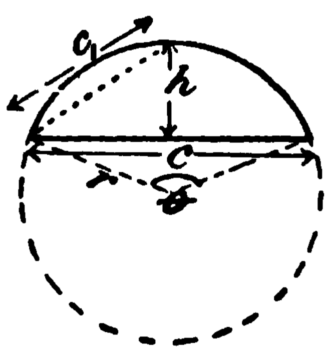


FIG. 13.

$$\text{Length of arc} = \frac{\pi d\theta}{360}$$

$$\text{or} = \frac{2\pi r\theta}{360} = \frac{r\theta}{57{\cdot}3}$$

For an arc less than a semicircle—

$$\text{Length} = \frac{8C_1 - C}{3} \text{ approximately}$$

The length of lines can be measured to within $\frac{1}{100}$ in. with a scale divided into either tenths or twentieths of an inch. With special appliances lengths can be measured to within $\frac{1}{1000000}$ in. if necessary.

The mathematical process by which the value of π is determined is too long for insertion here. One method consists of calculating the perimeter of a many-sided polygon described about a circle, also of one inscribed in a circle. The perimeter of the outer polygon is greater, and that of the inner less, than the perimeter of the circle. The greater the number of sides the smaller is the difference. The value of π has been found to 750 places of decimals, but it is rarely required for practical purposes beyond three or four places. For a simple method of finding the value of π, see "Longmans' School Mensuration," p. 48.

In the figure we have DC $= 2r$, AB $= 3r$ (by construction), DA $= \dfrac{r}{\sqrt{3}}$ (Euc. I. 47).

$$DB = AB - DA = 3r - \frac{r}{\sqrt{3}} = 2{\cdot}423r$$

$$BC = \sqrt{DB^2 + DC^2} = \sqrt{(2{\cdot}423r)^2 + (2r)^2} = \sqrt{9{\cdot}869r^2}$$

$BC = 3{\cdot}1416r$, *i.e.* the semicircumference (nearly).

The length of the arc is less than the length of the circumference in the ratio $\dfrac{\theta}{360}$.

$$\text{Length of arc} = \pi d \times \frac{\theta}{360} = \frac{\pi d\theta}{360}$$

The approximate formula given is extremely near when h is not great compared with c; even for a semicircle the error is only about 1 in 80. The proof is given in Lodge's "Mensuration for Senior Students" (Longmans).

Arc of circle.

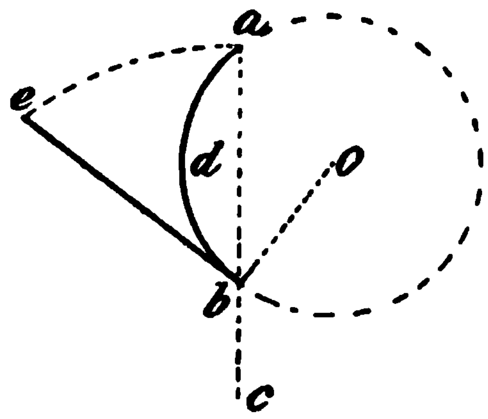


FIG. 14.

Rankine's Approximate Method for Short Arcs.—Produce chord ab. Make $bc = \frac{ab}{2}$; describe arc ae from centre c and radius ac; draw be perpendicular to ob. Length of arc $adb = be$.

Irregular curved line abc.

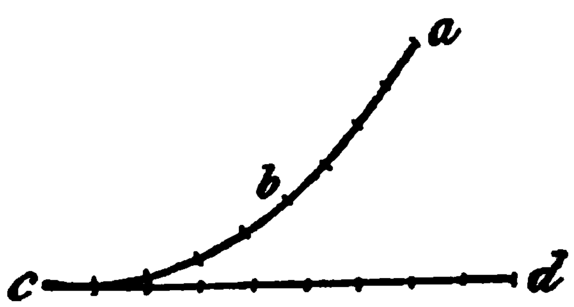


FIG. 15.

Set off tangent cd. With pair of dividers start from a, making small steps till the point c is reached, or nearly so. Count number of steps, and step off same number along tangent.

Areas.

Parallelograms.

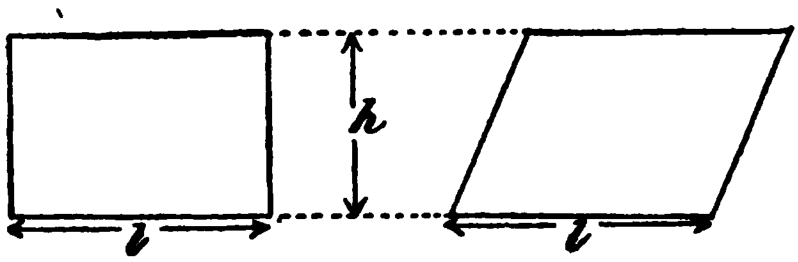


FIG. 16.

Area of figure $= lh$

Triangles.

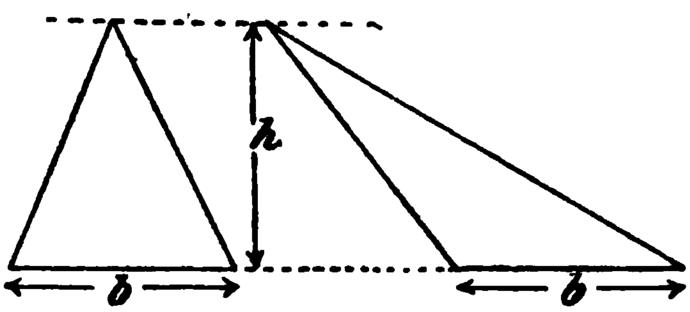


FIG. 17.

Area of figure $= \frac{bh}{2}$

Equilateral triangle.

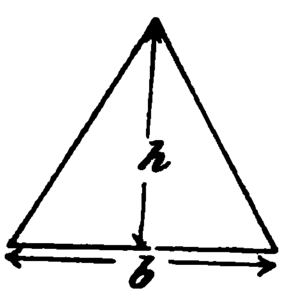


FIG. 18.

Area of figure $= \frac{\sqrt{3}}{4}b^2 = 0{\cdot}433b^2$

For short arcs this method is very accurate; the error is only about 1 in 1000 when $adb = bo$, but it increases somewhat rapidly as the arc gets greater than the radius (see Rankine's "Machinery and Millwork," p. 28).

The stepping should be commenced at the end remote from the tangent; then if the last step does not exactly coincide with c, the backward stepping can be commenced from the last point without causing any appreciable error. The greater the accuracy required, the greater must be the number of steps.

Areas.

See Euc. I. 35.

See Euc. I. 41.

$$h = \sqrt{b^2 - \left(\frac{b}{2}\right)^2} = \sqrt{\frac{3b^2}{4}} = \frac{\sqrt{3}b}{2}$$

$$\text{area} = \frac{bh}{2} = \frac{\sqrt{3}b^2}{4} = 0{\cdot}433b^2$$

Quadrilateral.

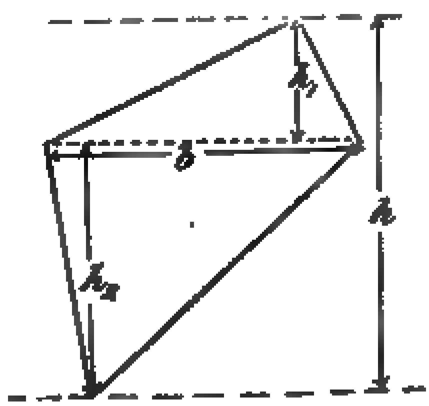


FIG. 20.

Area of figure $= \dfrac{bh}{2}$

Trapezium.

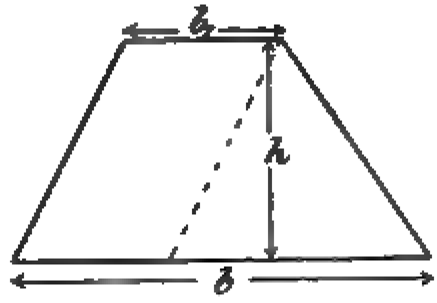


FIG. 21.

Area of figure $= \left(\dfrac{b+b_1}{2}\right)h$

Irregular straight-lined figure.

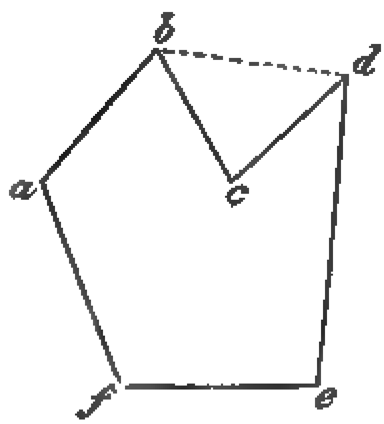


FIG. 22.

Area of figure = area *abdef* − area *bcd* or area of triangles (*acb* + *acf* + *cfe* + *ced*)

The proof is somewhat lengthy, but perfectly simple (see "Longmans' Mensuration," p. 18).

$$\text{Area of upper triangle} = \frac{bh_1}{2}$$

$$\text{,, lower triangle} = \frac{bh_2}{2}$$

$$\text{,, both triangles} = b\left(\frac{h_1 + h_2}{2}\right) = \frac{bh}{2}$$

$$\text{Area of parallelogram to left} = b_1h$$

$$\text{Area of triangle to right} = \frac{(b - b_1)h}{2}$$

$$\text{Area of whole figure} = \frac{(b - b_1 + 2b_1)h}{2} = \frac{(b + b_1)h}{2}$$

Simple case of addition and subtraction of areas.

Circle.

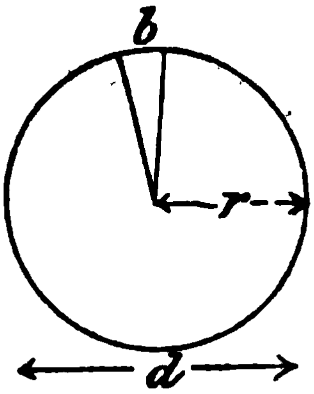


FIG. 23.

$$\text{Area of figure} = \pi r^2 = 3{\cdot}1416r^2$$
$$\text{or } \frac{\pi d^2}{4} = 0{\cdot}785d^2$$

Sector of circle.

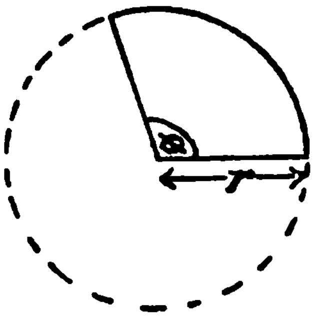


FIG. 24.

$$\text{Area of figure} = \frac{\pi r^2\theta}{360}$$

Segment of circle.

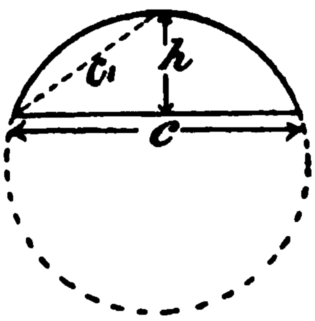


FIG. 25.

$$\text{Area of figure} = \tfrac{2}{3}\text{C}h \text{ when } h \text{ is small}$$
$$= \frac{h}{15}(6\text{C} + 8\text{C}_1) \text{ nearly}$$

Hollow circle.

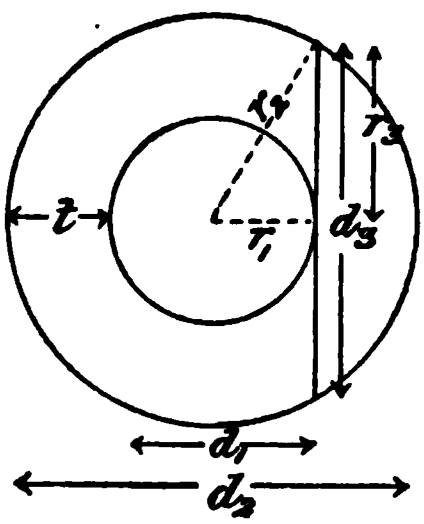


FIG. 26.

Area of figure = area of outer circle − area of inner circle

$$= \pi r_2^2 - \pi r_1^2$$
$$= \pi(r_2^2 - r_1^2)$$
$$= \pi r_3^2$$
$$\text{or} = 0{\cdot}785d_3^2$$
$$\text{or} = \frac{\pi(d_1 + d_2)}{2}t, \textit{ i.e.} \text{ mean circumf.} \times \text{thickness}$$

The circle may be conceived to be made up of a great number of tiny triangles, such as the one shown, the base of each little triangle being b units, then the area of each triangle is $\frac{br}{2}$; but the sum of all the bases equal the circumference, or $\Sigma b = 2\pi r$, hence the area of all the triangles put together, *i.e.* the area of the circle, $= \frac{2\pi r \cdot r}{2} = \pi r^2$.

The area of the sector is less than the area of the circle in the ratio $\frac{\theta}{360}$, hence the area of the sector $= \frac{\pi r^2 \theta}{360}$; if β be the angle expressed in circular measure, then the above ratio becomes $\frac{\beta}{2\pi}$.

$$\text{The area} = \frac{r^2\beta}{2}$$

When h is less than $\frac{c}{4}$, the arc of the circle very nearly coincides with a parabolic arc (see p. 31). For proof of second formula, see Lodge's "Mensuration for Senior Students" (Longmans).

Simple Case of Subtraction of Areas.—The substitution of r_3^2 for $r_2^2 - r_1^2$ follows from the properties of the right-angled triangle (Euc. I. 47).

The mean circumference × thickness is a very convenient form of expression; it is arrived at thus:

$$\text{Mean circumference} = \frac{\pi(d_2 + d_1)}{2}$$

$$\text{thickness} = \frac{d_2 - d_1}{2}$$

$$\text{product} = \frac{\pi(d_2 + d_1)}{2} \times \frac{d_2 - d_1}{2} = \frac{\pi}{4}(d_2^2 - d_1^2)$$

Ellipse.

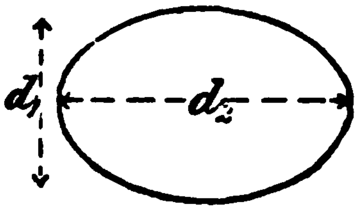


FIG. 27.

Area of figure $= \pi r_1 r_2$

or $= \frac{\pi}{4} d_1 d_2$

Parabolic segments.

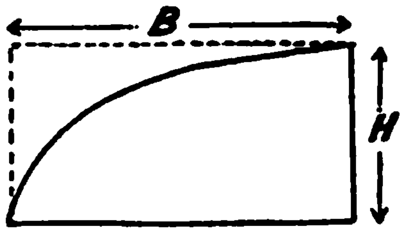


FIG. 28.

Area of figure = $\frac{2}{3}$BH

i.e. $\frac{2}{3}$(area of circumscribing rectangle)

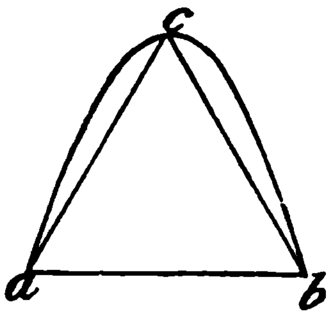


FIG. 29.

Area of figure = $\frac{4}{3}$ area of $\triangle$ *abc*

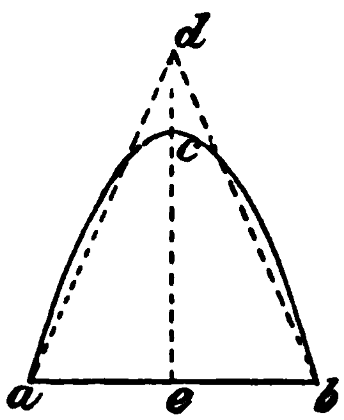


FIG. 30.

Make $dc = \frac{1}{3}ce$

area of figure = area of $\triangle$ *abd*

An ellipse may be regarded as a flattened or an elongated circle; hence the area of an ellipse is $\left\{\begin{matrix}\text{less}\\ \text{greater}\end{matrix}\right\}$ than the area of a circle whose diameter is the $\left\{\begin{matrix}\text{major}\\ \text{minor}\end{matrix}\right\}$ axis of an ellipse $\left\{\begin{matrix}d_2\\ d_1\end{matrix}\right\}$, in the ratio $\left\{\begin{matrix}d_1 \text{ to } d_2\\ d_2 \text{ to } d_1\end{matrix}\right\}$.

$$\text{Area} = \frac{\pi d_2^{\,2}}{4} \times \frac{d_1}{d_2} = \frac{\pi}{4} d_1 d_2, \text{ or } \frac{\pi d_1^{\,2}}{4} \times \frac{d_2}{d_1} = \frac{\pi d_1 d_2}{4}$$

From the properties of the parabola, we have—

$$\frac{h^2}{\mathrm{H}^2} = \frac{b}{\mathrm{B}}$$

$$h = \sqrt{\frac{b\mathrm{H}^2}{\mathrm{B}}} = \frac{b^{\frac{1}{2}}\mathrm{H}}{\mathrm{B}^{\frac{1}{2}}}$$

$$\text{area of strip} = h \,.\, db = \left(\frac{\mathrm{H}b^{\frac{1}{2}}}{\mathrm{B}^{\frac{1}{2}}}\right) db$$

$$\text{area of whole figure} = \frac{\mathrm{H}}{\mathrm{B}^{\frac{1}{2}}} \int_{b=0}^{b=\mathrm{B}} b^{\frac{1}{2}} \,.\, db = \frac{\mathrm{H}}{\mathrm{B}^{\frac{1}{2}}} \times \frac{\mathrm{B}^{\frac{3}{2}}}{\frac{3}{2}}$$

$$= \tfrac{2}{3}\mathrm{HB}$$

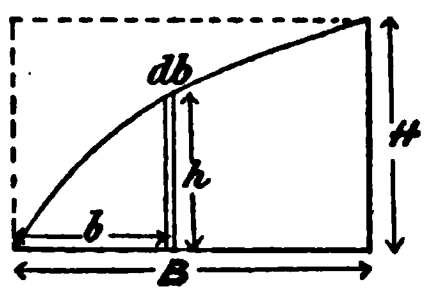


FIG. 28*a*.

The area *acbg* has been shown to be $\frac{2}{3}$HB. Take from each the area of the $\triangle\, abg$, then the remainder $abc = \frac{1}{3}$ the $\triangle\, abe$; but, from the properties of the parabola, we have $ed = \frac{1}{2}eb$, hence the area $abc = \frac{2}{3}$ area of the circumscribing $\triangle\, abd$.

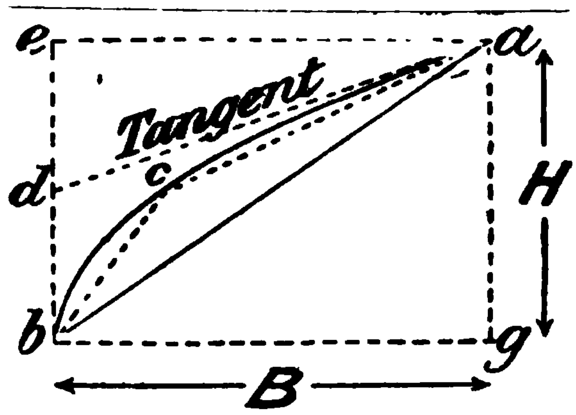


FIG. 29*a*.

From the properties of the parabola, we also have the height of the $\triangle\, abd$ = 2(height of the $\triangle\, abc$); hence the area of the $\triangle\, abd$ = 2(area of $\triangle\, abc$), and the area of the parabolic segment $= 2 \times \frac{2}{3}$ area $\triangle\, abc = \frac{4}{3}$ area $\triangle\, abc$.

By increasing the height of the $\triangle\, abc$ to $\frac{4}{3}$ its original height, we increase its area in the same ratio, and consequently make it equal to the area of the parabolic segment.

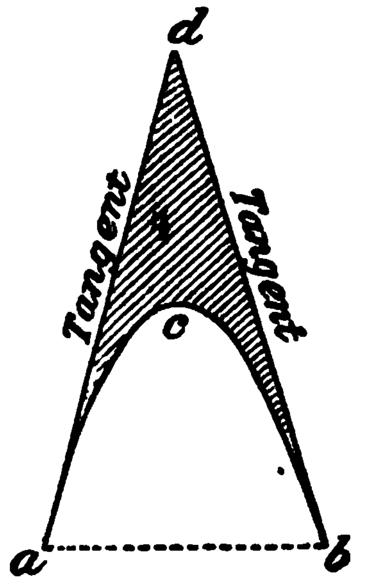


FIG. 31.

Area of shaded figure = $\frac{1}{3}$ area of $\triangle$ *abd*

Surfaces bounded by an irregular curve.

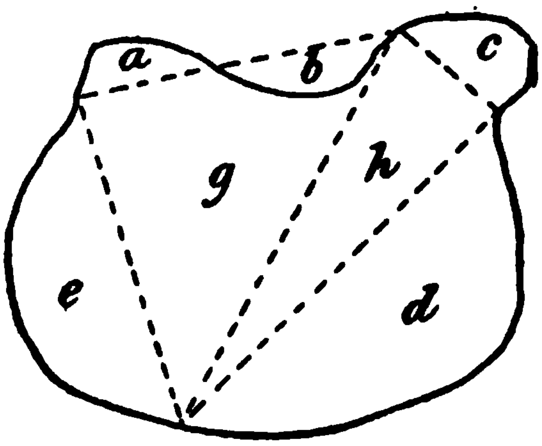


FIG. 32.

Area of figure = areas of parabolic segments $(a - b + c + d + e)$ + areas of triangles $(g + h)$.

Mean Ordinate method.—

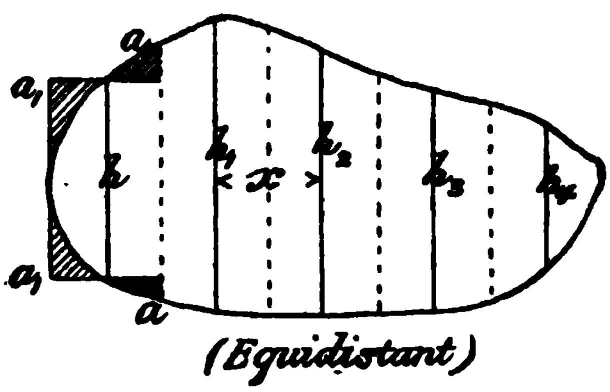


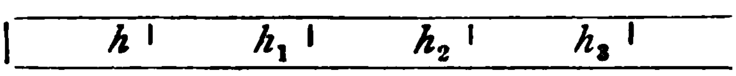
FIG. 33.

Area of figure $= (h + h_1 + h_2 + h_3 +, \text{etc.})x$

The area $abc = \frac{2}{3}$ area of triangle abd, hence the remainder $= \frac{1}{3}$ of triangle abd.

Simply a case of addition and subtraction of areas. It is a somewhat clumsy and tedious method, and is not recommended for general work. One of the following methods are considered to be better.

This is a fairly accurate method if a large number of ordinates are taken. The measurement of h_1, h_2, h_3, etc., is most



and so on.

FIG. 33 *a*.

easily done by marking them off continuously on a strip of paper, then divide by the total length to get the mean height.

The value of x must be accurately found; thus, If n be the number of ordinates, then $x = \frac{l}{n}$.

The method assumes that the areas a, a cut off are equal to the areas a_1, a_1 put on.

Simpson's Method.—

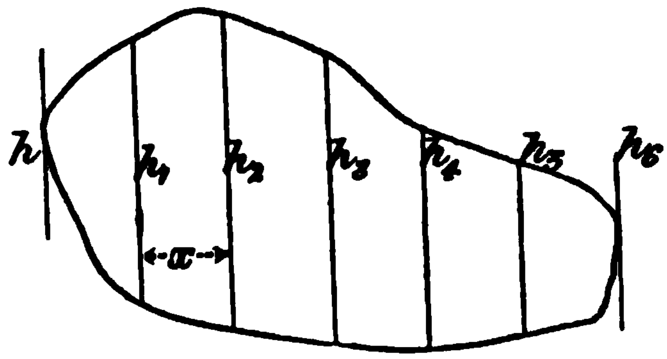


FIG. 34.

Area of figure $= \frac{x}{3}(h + 4h_1 + 2h_2 + 4h_3 + 2h_4 + 4h_5 + h_6)$

N.B.—In this case, the ordinate h being a tangent to the curve, its height is of course $= 0$; it should be written down, however, to avoid slips, thus:

$$\frac{x}{3}(0 + 4h_1 +, \text{ and so on})$$

Any *odd* number of ordinates may be taken—the greater the number the greater will be the accuracy.

This is by far the most accurate and useful of all methods of measuring such areas. The proof is as follows :—

The curve *gfedc* is assumed to be a parabolic arc.

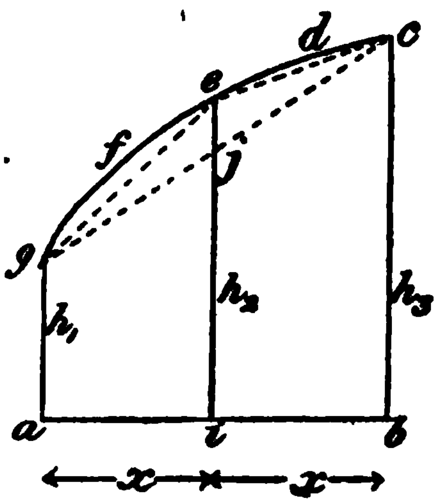


FIG. 34*a*.

$$\text{Area } aieg = x\left(\frac{h_1 + h_2}{2}\right) \quad . \quad . \quad . \quad . \quad \text{(i.)}$$

$$\text{,, } ibce = x\left(\frac{h_2 + h_3}{2}\right) \quad . \quad . \quad . \quad . \quad \text{(ii.)}$$

$$\text{,, } abceg = \frac{x}{2}(h_1 + 2h_2 + h_3) \quad . \quad \text{(i.+ii.)}$$

$$\text{,, } abcjg = 2x\left(\frac{h_1 + h_3}{2}\right) = x(h_1 + h_3) \text{(iii.)}$$

$$\text{Area of } \triangle\, gce = \text{(i.)} + \text{(ii.)} - \text{(iii.)}$$

$$\text{,, } \quad = \frac{x}{2}(h_1 + 2h_2 + h_3) - x(h_1 + h_3)$$

$$\text{,, } \quad = \frac{x}{2}(2h_2 - h_1 - h_3) \quad . \quad . \quad . \quad . \quad \text{(iv.)}$$

$$\left.\begin{array}{r}\text{Area of parabolic}\\ \text{segment } gcdef\end{array}\right\} = \frac{4}{3}\text{(iv.)} = \frac{2x}{3}(2h_2 - h_1 - h_3) \,. \quad \text{(v.)}$$

$$\text{Whole figure} = \text{(iii.)} + \text{(v.)} = x(h_1 + h_3) + \frac{2x}{3}(2h_2 - h_1 - h_3)$$

$$\text{,, } \quad = \frac{x}{3}(h_1 + 4h_2 + h_3)$$

If two more slices were added to the figure, the added area would be as above $= \frac{x}{3}(h_3 + 4h_4 + h_5)$, and when the two are added they become $= \frac{x}{3}(h_1 + 4h_2 + 2h_3 + 4h_4 + h_5)$.

Surfaces of revolution.

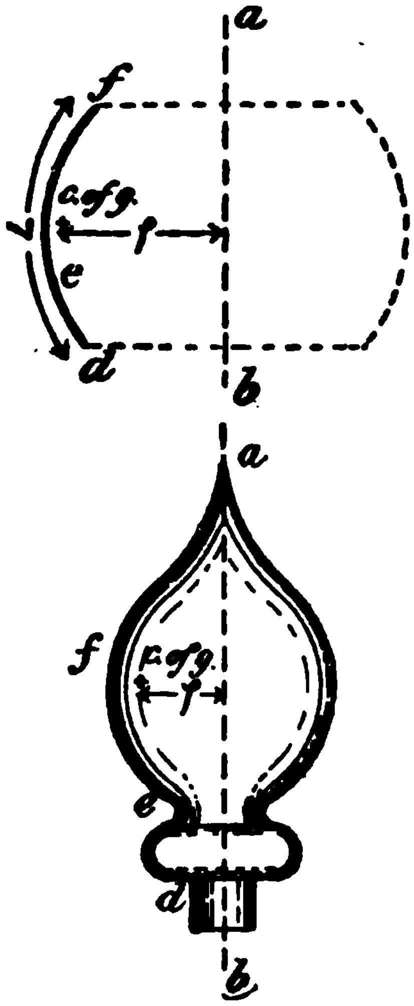


FIG. 35.

Pappus' or Guldinus' Method.—

$$\left.\begin{array}{l}\text{Area of surface swept out by} \\ \quad\text{the revolution of the line} \\ \quad def \text{ about the axis } ab\end{array}\right\} = L \times 2\pi\rho$$

$$\text{Length of line} = L$$

$$\left.\begin{array}{l}\text{Radius of c. of g. of line } def \\ \quad\text{considered as a fine wire}\end{array}\right\} = \rho$$

This method also holds for any part of a revolution as well as for a complete revolution. The area of such figures as circles, hollow circles, sectors, parallelograms ($\rho = \infty$), can also be found by this method.

Surface of sphere.

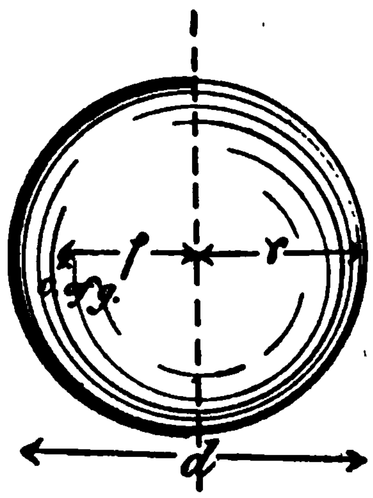


FIG. 36.

$$\text{Area of surface of sphere} = 4\pi r^2$$
$$= \pi d^2$$

The surface of a sphere is the same as the curved surface of a cylinder of same diameter and length $= d$.

Surface of cone.

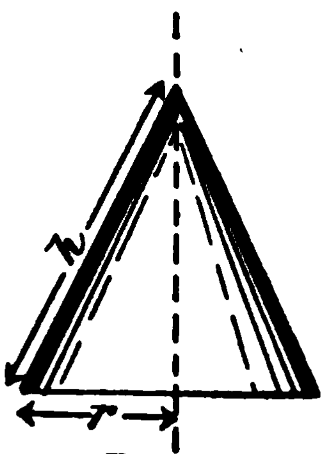


FIG. 37.

Area of surface of cone $= \pi r h$.

The area of the surface traced out by a narrow strip of length $\left\{ \begin{matrix} l_0 \text{ and radius } \rho_0 = 2\pi l_0\rho_0 \\ l_1 \quad ,, \quad \rho_1 = 2\pi l_1\rho_1 \end{matrix} \right\}$, and so on.

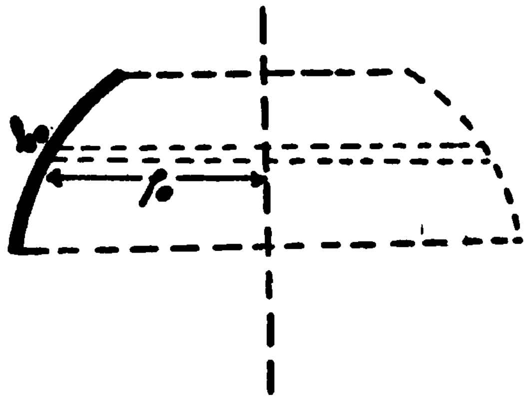


Fig. 35*a*.

Area of whole surface

$= 2\pi(l_0\rho_0 + l_1\rho_1 +$, etc.)

$= 2\pi$(each elemental length of wire $\times$ its distance from axis of revolution)

$= 2\pi$(total length of revolving wire $\times$ distance of c. of g. from axis of revolution) (see p. 58).

$=$ (total length of revolving wire $\times$ length of path described by its centre of gravity)

$= L2\pi\rho$

N.B.—The revolving wire must lie wholly on one side of the axis of revolution and in the same plane.

The distance of the c. of g. of any circular *arc*, or wire bent to a circular arc, from the centre of the circle is $y = \frac{rc}{a} = \rho$, where $r =$ radius of circle, c chord of arc, a length of arc (see p. 64).

In the spherical surface $a = L = \pi r$, $c = 2r$, $\rho = \frac{2r^2}{\pi r} = \frac{2r}{\pi}$

$$\text{Surface of sphere} = \pi r \,.\, 2\pi \,.\, \frac{2r}{\pi} = 4\pi r^2$$

$$\text{Length of revolving wire} = L = h$$

$$\text{radius of c. of g.} \quad ,, \quad ,, \quad = \rho = \frac{r}{2}$$

$$\text{surface of cone} = \frac{h2\pi r}{2} = \pi r h$$

In the hyperbola we have—

$$XY = X_1Y_1 = xy$$

$$\text{hence } y = \frac{XY}{x}$$

$$\text{area of strip} = y \,.\, dx = XY\frac{dx}{x}$$

$$\left.\begin{matrix}\text{Area of}\\ \text{whole}\\ \text{figure}\end{matrix}\right\} = XY\int_{x=X}^{x=X_1}\frac{dx}{x}$$

$$= XY\,(\log_e X_1 - \log_e X)$$

$$= XY \log_e \frac{X_1}{X}$$

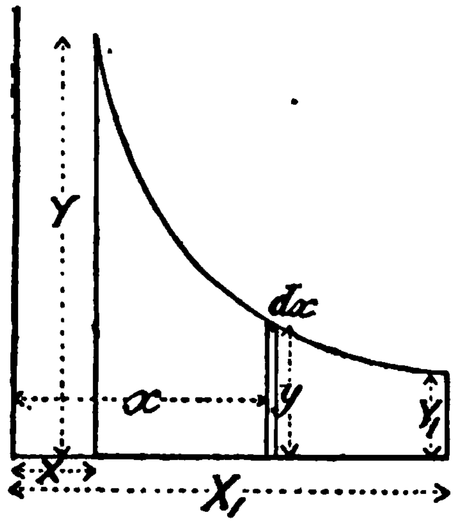


FIG. 38*a*.

Using the figure above, in this case we have—

$$YX^n = Y_1X_1^n = yx^n$$

$$\text{hence } y = \frac{YX^n}{x^n}$$

$$\text{area of strip} = y \,.\, dx = \frac{YX^n}{x^n}dx = YX^n x^{-n}dx$$

$$\text{area of whole figure} = YX^n\int_{x=X}^{x=X_1} x^{-n}dx = YX^n\left(\frac{X_1^{1-n} - X^{1-n}}{1-n}\right)$$

$$= \frac{YX^nX_1^{1-n} - YX}{1-n}$$

$$\text{But } Y_1X_1^n = YX^n$$

Multiply both sides by X_1^{1-n}

$$\text{then } Y_1X_1 = YX^nX_1^{1-n}$$

Substituting, we have—

$$\text{Area of whole figure} = \frac{Y_1X_1 - YX}{1-n}$$

$$\text{or} = \frac{YX - Y_1X_1}{n-1}$$

most accurately measured by a planimeter, such as Amsler's or Goodman's.

A very convenient method is to cut out a piece of thin cardboard or sheet metal to the exact dimensions of the area; weigh it, and compare with a known area (such as a circle or square) cut from the same cardboard or metal. A convenient method of weighing is shown on the opposite page, and gives very accurate results if reasonable care be taken.

Volumes.

Prisms.

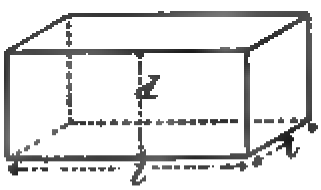


FIG. 41.

Let A = area of the end of prism;
l = length of prism.

Volume = lA

Parallelopiped.

Volume = ldt

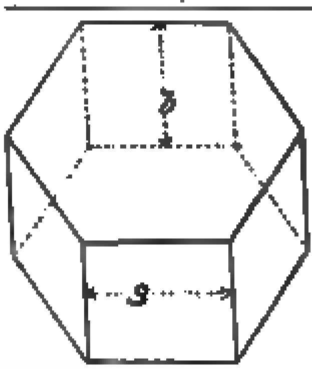


FIG. 42.

Hexagonal prism.

Volume = $2\cdot598s^2l$

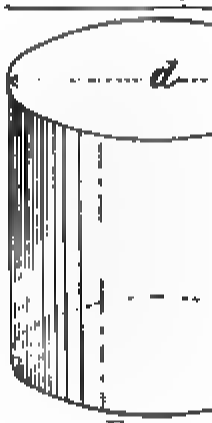


FIG. 43.

Suspend a knitting-needle or a straight piece of wire or wood by a piece of cotton, and accurately balance by shifting the cotton. Then suspend the two pieces of cardboard by pieces of cotton or silk; shift them till they balance; then measure the distances x and y.

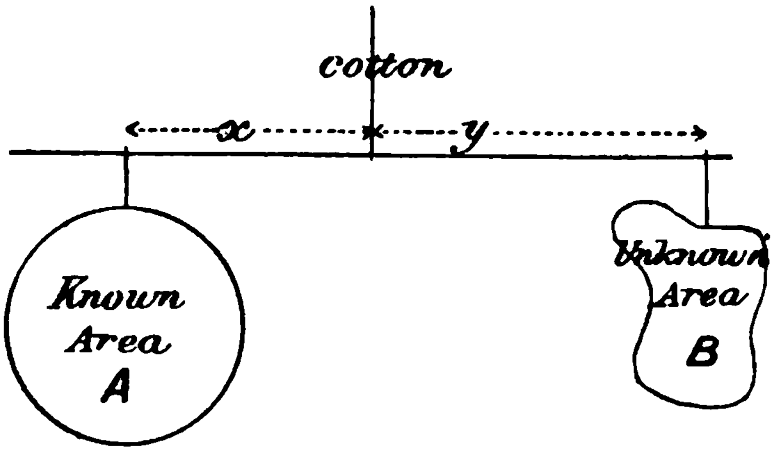


FIG. 40.

Then $Ax = By$

or $B = \dfrac{Ax}{y}$

The area of A should not differ very greatly from the area of B, or one arm becomes very short, and error is more likely to occur.

$$\text{Area of end} = td$$
$$\text{volume} = ltd$$

$$\text{Area of hexagon} = \text{area of six equilateral triangles}$$
$$= 6 \times 0{\cdot}433S^2 \text{ (see Fig. 18)}$$
$$\text{volume} = 2{\cdot}598S^2l$$
$$\text{or say } 2{\cdot}6S^2l$$

$$\text{Area of circular end} = \frac{\pi}{4}d^2$$
$$\text{volume} = \frac{\pi d^2 l}{4}$$

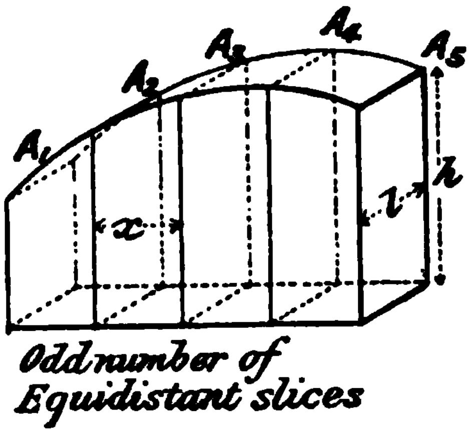


FIG. 44.

Prismoid.

Simpson's Method.—

$$\text{Volume} = \frac{x}{3}(A_1 + 4A_2 + 2A_3 + 4A_4 + A_5)$$

and so on for any number of slices.

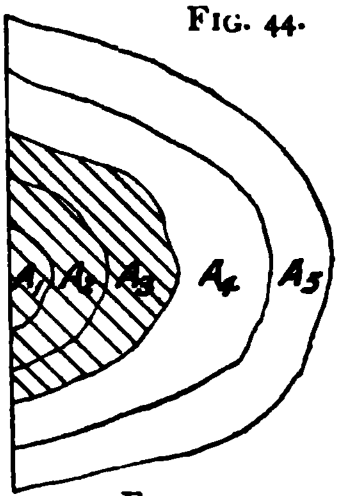


FIG. 45.

Contoured volume.

N.B.—Each area is to be taken to include those within it, *not* the area between the two contours. A_3 is shaded over to make this clear.

Solids of revolution.

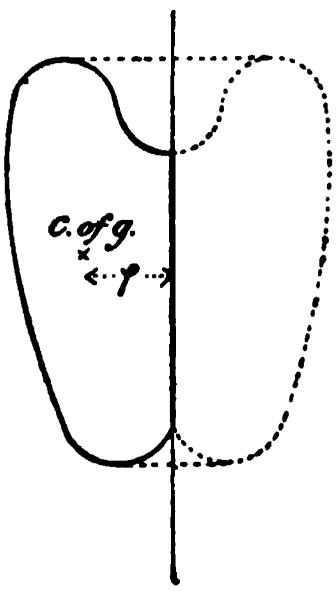


FIG. 46.

Method of Pappus or Guldinus.—

Let A = area of full lined surface;
ρ = radius of c. of g. of *surface.*

Volume of solid of revolution $= 2\pi\rho A$

N.B.—The surface must lie wholly on one side of the axis of revolution, and in the same plane.

This method is applicable to a great number of problems, spheres, cones, rings, etc.

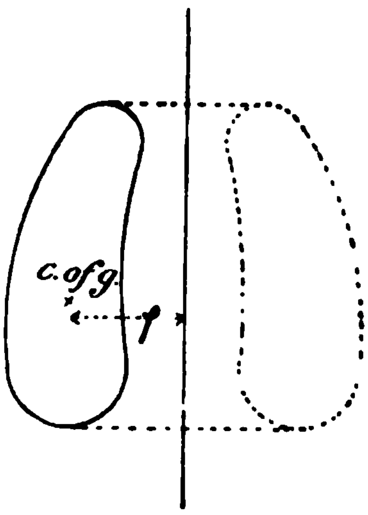


FIG. 47.

Area of end (or side) $= \frac{x}{3}(h_1 + 4h_2 + 2h_3 +$, etc. (see p. 34), where h_1, h_2, etc., are the heights of the sections.

$$\text{Volume} = \frac{x}{3}(h_1 l + 4h_2 l + 2h_3 l +, \text{ etc.})$$
$$= \frac{x}{3}(A_1 + 4A_2 + 2A_3 +, \text{ etc.})$$

The above proof assumes that the sections are parallelograms, *i.e.* the solid is flat-topped along its length. We shall later on show that the formula is accurate for many solids having surfaces curved in all directions, such as a sphere, ellipsoid, paraboloid, hyperboloid.

Let the area be revolved round the axis; then—

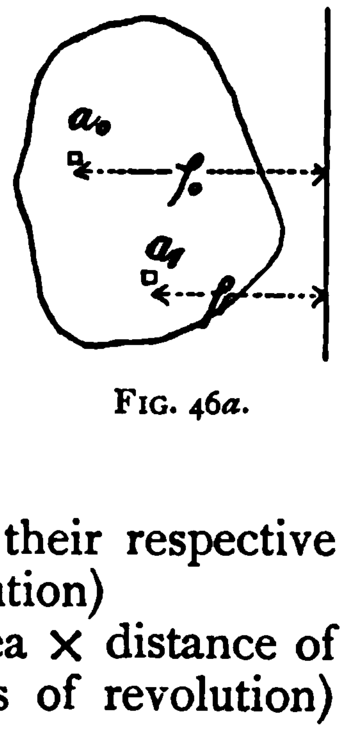


FIG. 46*a*.

The volume swept out by an elemental area a_0, when revolving round the axis at a distance ρ_0 $\Big\}$ $= a_0 \times 2\pi\rho_0$

Ditto ditto a_1 and $\rho_1 = a_1 \times 2\pi\rho_1$
and so on.

Whole volume swept out by all the elemental areas, a_0, a_1, etc., when revolving round the axis at their respective distances, ρ_0, ρ_1, etc. $\Big\}$ $= 2\pi(a_0\rho_0 + a_1\rho_1 +$, etc.)

$= 2\pi$(each elemental area, a_0, a_1, etc. $\times$ their respective distances, ρ_0, ρ_1, from the axis of revolution)

$= 2\pi$ (sum of elemental areas, or whole area $\times$ distance of c. of g. of whole area from the axis of revolution) (see p. 58)

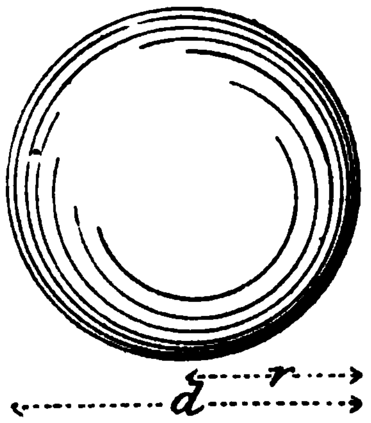
$= A \times 2\pi\rho = 2\pi\rho A$

But $2\pi\rho$ is the distance the c. of g. has moved through, or the length of the path of the c. of g.; hence—

Whole volume = area of generating surface $\times$ the length of the path of the c. of g. of the area

This proof holds for any part of a revolution, and for any value of ρ; when ρ becomes infinite, the path becomes a straight line, in such a case as a prism.

Sphere.



External diameter $= d_e$
Internal diameter $= d_i$

FIG. 48.

$$\text{Volume of sphere} = \frac{\pi d^3}{6}, \text{ or } \tfrac{4}{3}\pi r^3$$

$$\text{Volume of sphere} = \tfrac{2}{3} \text{ volume of circumscribing cylinder}$$

Hollow sphere.

$$\left.\begin{array}{l}\text{Volume of}\\ \text{hollow sphere}\end{array}\right\} = \begin{array}{l}\text{volume of outer sphere} -\\ \text{volume of inner sphere}\end{array}$$

$$= \frac{\pi d_e^3}{6} - \frac{\pi d_i^3}{6}$$

$$= \frac{\pi}{6}(d_e^3 - d_i^3)$$

Slice of sphere.

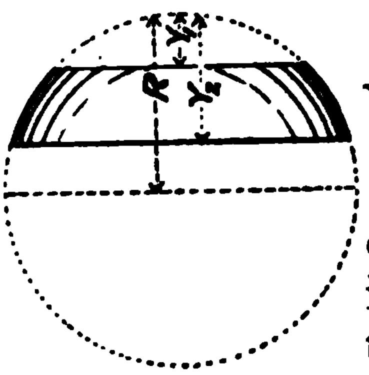


FIG. 49.

$$\text{Volume of slice} = \frac{\pi}{3}\{3R(Y_2^2 - Y_1^2) - Y_2^3 + Y_1^3\}$$

N.B.—The slice must be taken wholly on one side of the diameter; if the slice includes the diameter, it must be treated as two of the following slices.

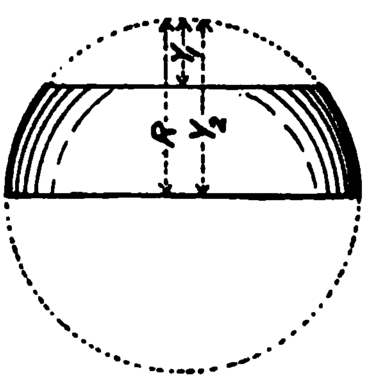


FIG. 50.

Special case in which $Y_2 = R$.

$$\text{Volume of slice} = \frac{\pi}{3}(2R^3 - 3RY_1^2 + Y_1^3)$$

Sphere.—The revolving area is a semicircle of area $\frac{\pi r^2}{2}$

$$\left.\begin{array}{l}\text{The distance of the c. of g.}\\ \quad\text{from the diameter}\end{array}\right\} = \rho = \frac{4r}{3\pi} \text{ (see p. 66)}$$

$$\text{volume swept out} = 2\pi \times \frac{4r}{3\pi} \times \frac{\pi r^2}{2} = \tfrac{4}{3}\pi r^3 = \frac{\pi d^3}{6}$$

Volume of elemental slice $= \pi c^2 dy$

$$= \pi\{R^2 - (R^2 + y^2 - 2Ry)\}dy$$

$$= \pi(2Ry - y^2)dy$$

$$\left.\begin{array}{l}\text{Volume of}\\ \text{whole}\\ \text{slice}\end{array}\right\} = \pi \int_{y=Y_1}^{y=Y_2} (2Ry - y^2)dy$$

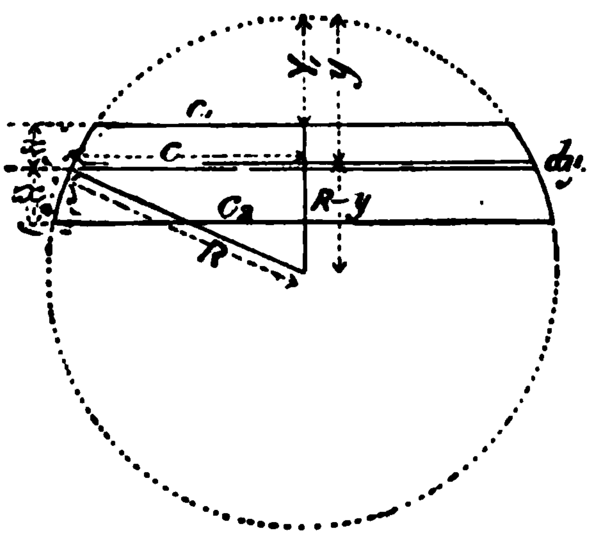


FIG. 49*a*.

$$= \pi\left[\frac{2Ry^2}{2} - \frac{y^3}{3}\right]_{y=Y_1}^{y=Y_2}$$

$$= \pi\left\{\frac{2R(Y_2^2 - Y_1^2)}{2} - \frac{Y_2^3 - Y_1^3}{3}\right\}$$

$$= \frac{\pi}{3}\{3R(Y_2^2 - Y_1^2) - Y_2^3 + Y_1^3\}$$

The same result can be obtained by Simpson's method—

$$\text{Volume} = \frac{x}{3}(\pi C_2^2 + 4\pi C^2 + \pi C_1^2)$$

For x substitute $\frac{Y_2 - Y_1}{2}$

c^2 „ $(2Ry - y^2)$ with the proper suffixes.

The algebraic work is long, but the results by the two methods will be found to be identical.

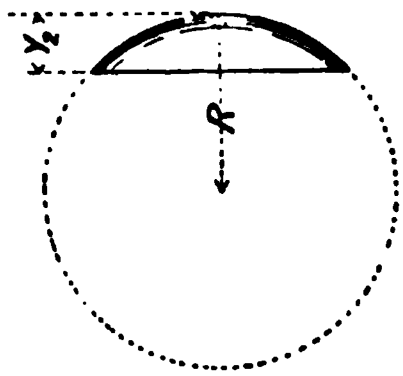


FIG. 51.

Special case in which $Y_1 = 0$.

$$\text{Volume of slice} = \frac{\pi}{3}(3RY_2^2 - Y_2^3)$$

When $Y_2 = R$, and $Y_1 = 0$, the slice becomes a hemisphere, and the—

$$\text{Volume of hemisphere} = \frac{\pi}{3}(2R^3)$$
$$= \tfrac{2}{3}\pi R^3$$

which is one-half the volume of the sphere found by the other method.

Paraboloid.

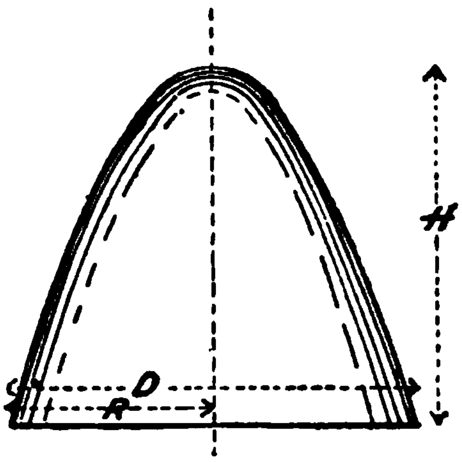


FIG. 52.

$$\left.\begin{array}{r}\text{Volume of}\\ \text{paraboloid}\end{array}\right\} = \frac{\pi}{2}R^2H, \text{ or } \frac{\pi}{8}D^2H$$
$$= 1{\cdot}57R^2H, \text{ or } 0{\cdot}39D^2H$$
$$= \tfrac{1}{2} \text{ volume circumscribing cylinder}$$

Cone.

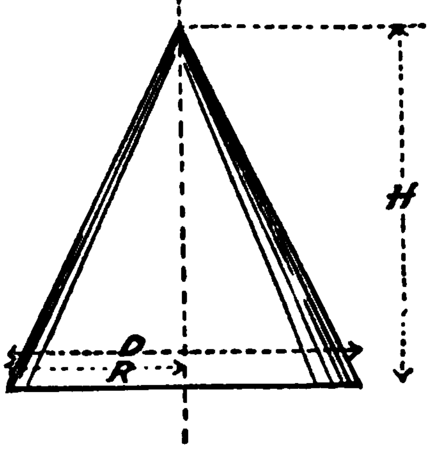


FIG. 53.

$$\text{Volume of cone} = \frac{\pi}{3}R^2H$$
$$= \frac{\pi}{12}D^2H$$
$$= \tfrac{1}{3} \text{ volume circumscribing cylinder}$$

(*Continued from page* 45.)

For the hemisphere it comes out very easily, thus—

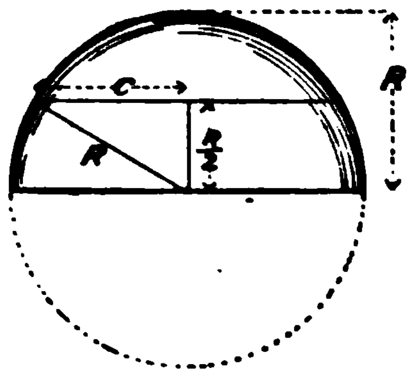


FIG. 51*a*.

$$x = \frac{R}{2} \qquad c^2 = R^2 - \frac{R^2}{4}$$

$$\text{Volume} = \frac{R}{6}\{0 + \pi(4R^2 - R^2) + \pi R^2\}$$
$$= \tfrac{2}{3}\pi R^3$$

From the properties of the parabola, we have—

$$\frac{r^2}{R^2} = \frac{h}{H}$$

$$r^2 = \frac{R^2h}{H}$$

$$\text{Volume of slice} = \pi r^2 dh = \frac{\pi R^2 h}{H}dh$$

$$\therefore \text{volume of solid} = \frac{\pi R^2}{H}\int_0^H h\,.\,dh$$

$$= \frac{\pi R^2 H^2}{2H} = \frac{\pi R^2 H}{2}$$

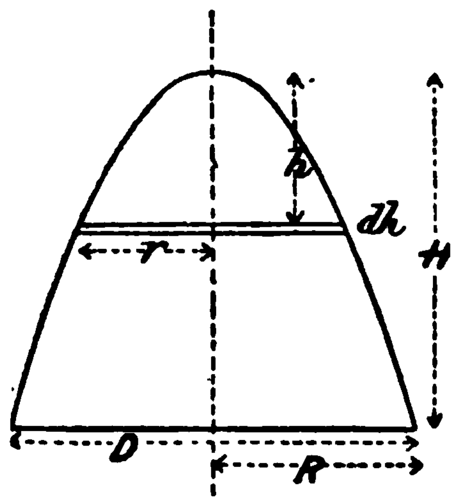


FIG. 52*a*.

$$\frac{r}{R} = \frac{h}{H}$$

$$r = \frac{Rh}{H}$$

$$\text{Volume of slice} = \pi r^2 dh = \frac{\pi R^2 h^2}{H^2}dh$$

$$\text{volume of cone} = \frac{\pi R^2}{H^2}\int_0^H h^2 dh$$

$$= \frac{\pi R^2}{H^2} \times \frac{H^3}{3} = \frac{\pi R^2 H}{3}$$

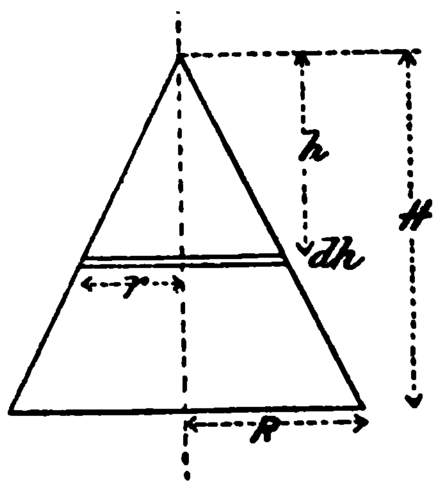


FIG. 53*a*.

Pyramid.

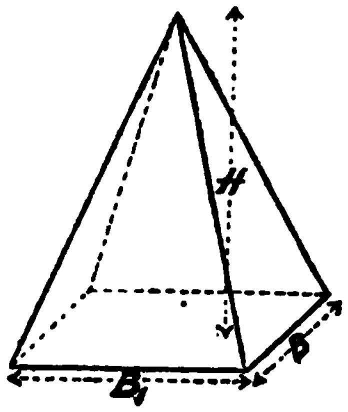


FIG. 54.

$$\text{Volume of pyramid} = \frac{B_1BH}{3}$$
$$= \frac{H^3}{3}, \text{ when } B_1 = B = H$$
$$= \tfrac{1}{3} \text{ volume circumscribing solid}$$

Slightly tapered body.

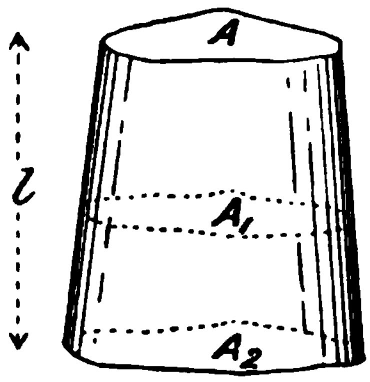


FIG. 55.

Mean Areas Method.—

$$\text{Volume of body} = \left(\frac{A + A_1 + A_2}{3}\right)l \text{ (approx.)}$$
$$= (\text{mean area})l$$

Ring.

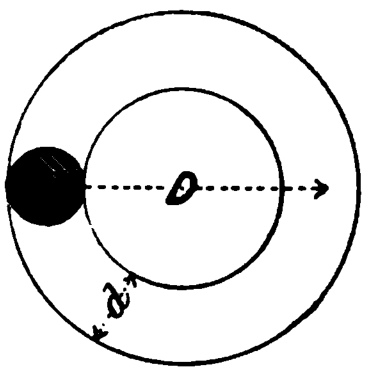


FIG. 56.

$$\text{Volume of ring} = \frac{\pi d^2}{4} \times \pi D = 2\cdot47 d^2 D$$

WEIGHT OF MATERIALS.

Brass and bronze	0·30	lbs.	per cubic	inch.
Copper	0·32		,,	,,
Iron—cast	0·26		,,	,,
,, wrought	0·278		,,	,,
Steel	0·283		,,	,,
Lead	0·412		,,	,,

This may be proved in precisely the same manner as the cone, or thus by Simpson's method—

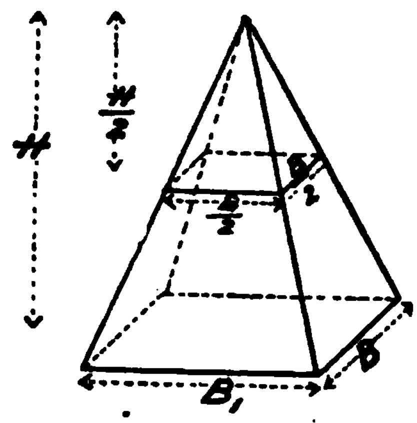


FIG. 54*a*.

$$\text{Volume} = \frac{H}{2 \times 3}\left\{o + 4\left(\frac{B}{2} \times \frac{B_1}{2}\right) + B \times B_1\right\}$$

$$= \frac{H}{6}(2BB_1) = \frac{BB_1H}{3}$$

This method is only approximately true when the taper is very slight. For such a body as a pyramid it would be seriously in error; the volume obtained by this method would be $\frac{5}{12}H^3$ instead of $\frac{4}{12}H^3$.

The diameter D is measured from centre to centre of the sections of the ring, *i.e.* their centres of gravity—

Volume = area of surface of revolution × length of path of c. of g. of section

$$= \frac{\pi d^2}{4} \times \pi D = \frac{\pi^2 d^2 D}{4}$$

CHAPTER III.

MOMENTS.

THAT branch of applied mechanics which deals with moments is of the utmost importance to the engineer, and yet perhaps it gives the beginner more trouble than any other part of the subject. The following simple illustrations may possibly help to make the matter clear. We have already (see p. 12) explained the meaning of the term "clockwise" and "contra-clockwise" moments.

In the figures that follow, the two pulleys of radii R and R_1 are attached to the same shaft, so that they rotate together. We shall assume that there is no friction on the axle.

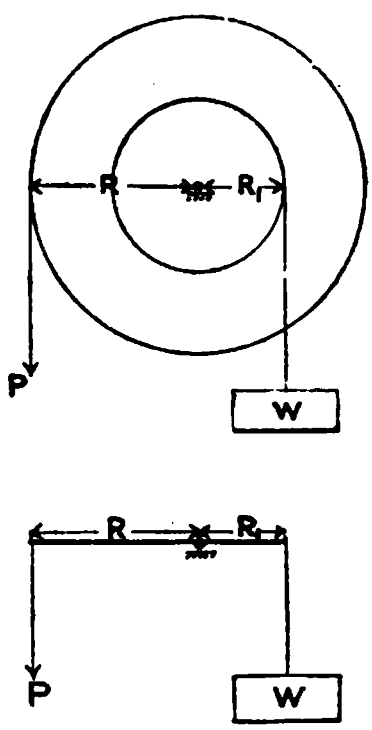


FIG. 57.

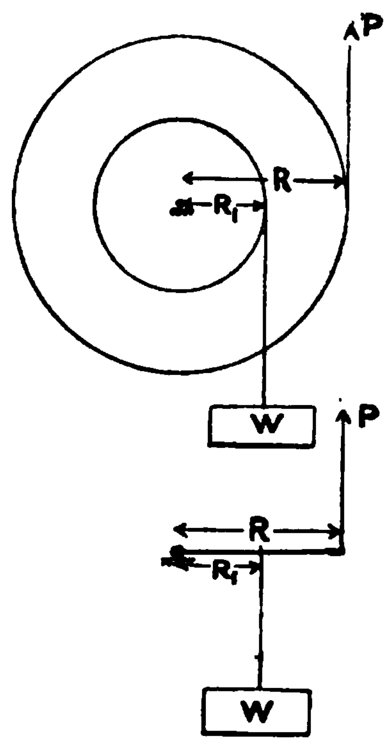


FIG. 58.

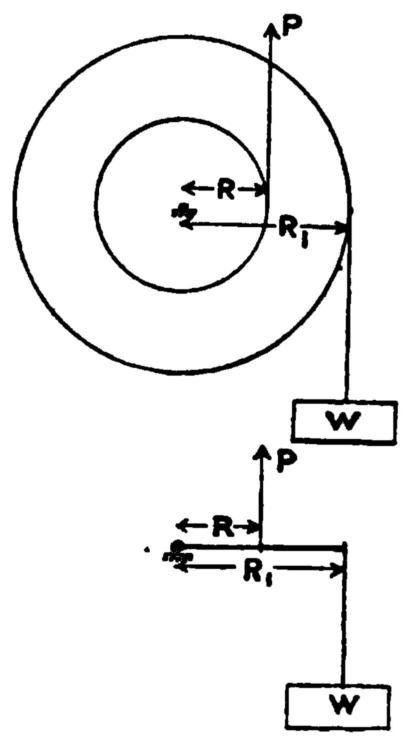


FIG. 59.

Let a cord be wound round each pulley in such a manner that when a force P is applied to one cord, the weight W will be lifted by the other.

Now let the cord be pulled through a sufficient distance to cause the pulleys to make one complete revolution; we shall then have—

$$\text{The work done by pulling the cord} = P \times 2\pi R$$
$$\text{,, ,, lifting the weight} = W \times 2\pi R_1$$

These must be equal, as no work is assumed to be wasted in friction; hence—

$$P 2\pi R = W 2\pi R_1$$
$$\text{or } PR = WR_1$$

or the contra-clockwise moment = the clockwise moment

It is clear that this relation will hold for any portion of a revolution, however small; also for any size of pulleys.

The levers shown in the same figures may be regarded as a small portion of the pulleys; hence the same relations hold in their case.

It may be stated as a general principle that if a rigid body be in equilibrium under any given system of moments, the algebraic sum of all the moments in any given plane must be zero, or the clockwise moments must be equal to the contra-clockwise moments.

First Moments.—The product of a $\left\{\begin{array}{l}\text{force } (f)\\ \text{mass } (m)\\ \text{area } (a)\\ \text{volume } (v)\end{array}\right\}$ by the length of its arm l, viz. $\left\{\begin{array}{c}fl\\ ml\\ al\\ vl\end{array}\right\}$, is termed the *first moment* of the $\left\{\begin{array}{l}\text{force}\\ \text{mass}\\ \text{area}\\ \text{volume}\end{array}\right\}$, or sometimes simply the *moment.*

A statement of the first moment of a $\left\{\begin{array}{l}\text{force}\\ \text{mass}\\ \text{area}\\ \text{volume}\end{array}\right\}$ must consist of the product of $\left\{\begin{array}{l}\text{force units} \times \text{length units.}\\ \text{mass units} \times \text{length units.}\\ \text{area units} \times \text{length units.}\\ \text{volume units} \times \text{length units.}\end{array}\right.$

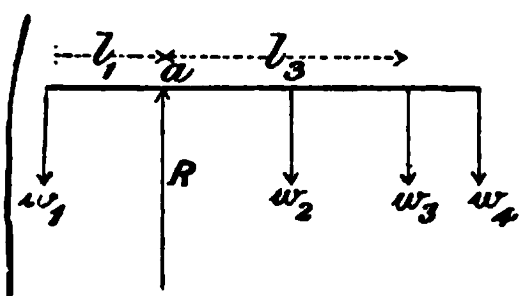
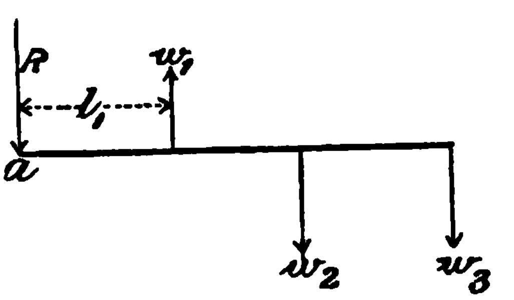
In speaking of moments, we shall always put the force, etc., units first, and the length units afterwards. For example, we shall speak of a moment as so many pounds-feet or tons-inches, to avoid confusion with work units.

Second Moments.—The product of a $\left\{\begin{matrix}\text{force } (f) \\ \text{mass } (m) \\ \text{area } (a) \\ \text{volume } (v)\end{matrix}\right\}$ by the square or second power of the length (l) of its arm, viz. $\left\{\begin{matrix} fl^2 \\ ml^2 \\ al^2 \\ vl^2 \end{matrix}\right\}$, is termed the *second* moment of the $\left\{\begin{matrix}\text{force} \\ \text{mass} \\ \text{area} \\ \text{volume}\end{matrix}\right\}$. The *second moment* of a volume or an area is sometimes termed the "moment of inertia" (see p. 78) of the volume or area. There is no objection to the use of the term when dealing with questions involving the inertia of bodies; but in other cases, where the second moment has nothing whatever to do with inertia, the term "second moment" is preferable.

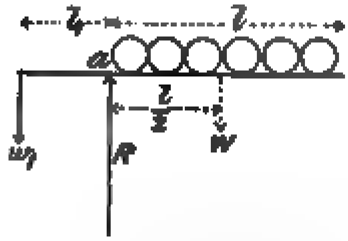
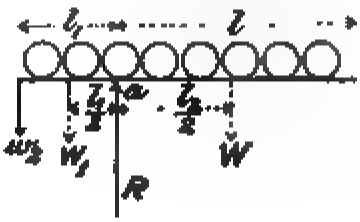
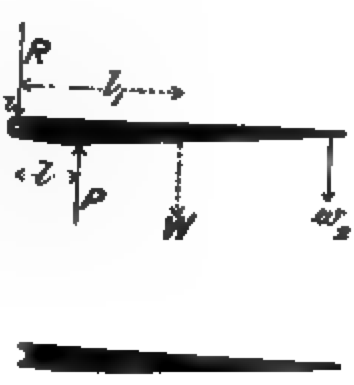
A statement of the second moment of a $\left\{\begin{matrix}\text{force} \\ \text{mass} \\ \text{area} \\ \text{volume}\end{matrix}\right\}$ must consist of the product of $\left\{\begin{matrix}\text{force units} \times (\text{length units})^2. \\ \text{mass units} \times (\text{length units})^2. \\ \text{area units} \times (\text{length units})^2. \\ \text{volume units} \times (\text{length units})^2.\end{matrix}\right.$

First Moments.

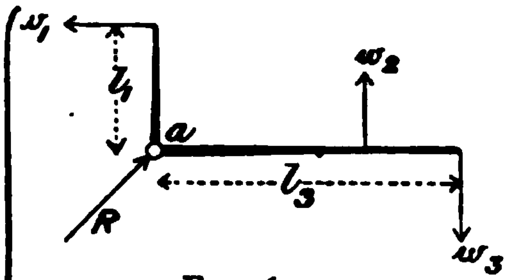
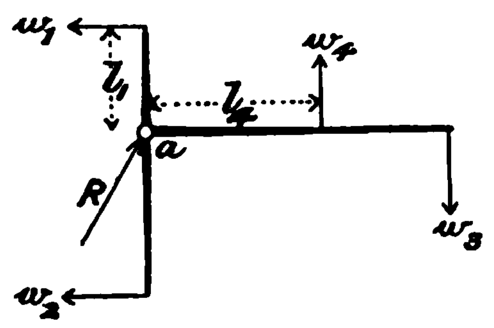
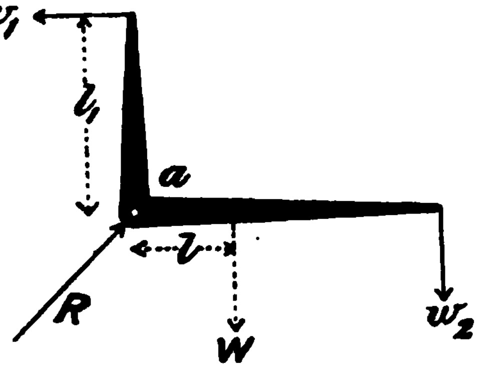
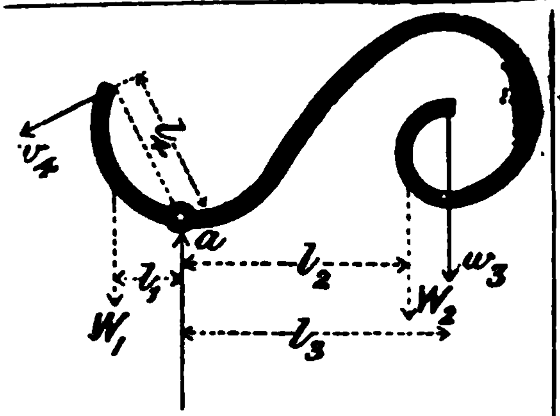
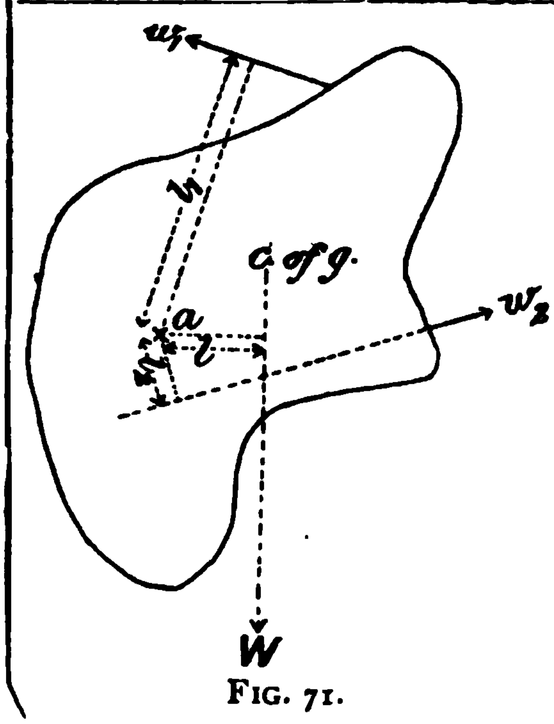
Neglecting weight of lever.

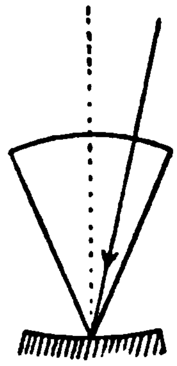
Lever S.	Clockwise moments ↻ about the point *a*.	Contra-clockwise moments ↺ about the point *a*.
 Fig. 60.	$w_2l_2 + w_3l_3 + w_4l_4$	$= w_1l_1$
 Fig. 61.	$w_2l_2 + w_3l_3$	$= w_1l_1$

Reaction R at fulcrum *a*, *i.e.* the resultant of all the forces acting on lever.	REMARKS.
$w_1+w_2+w_3+w_4$	To save confusion in the diagrams, the l has in some cases been omitted. In every case the suffix of l indicates the distance of the weight w bearing the same suffix from the fulcrum.
$w_1-w_2-w_3$	

	FIG. 62.		
Neglecting weight of lever	 FIG. 63.	$W\frac{l}{2} = \frac{wl^2}{2}$ for if w = distributed load per unit length $wl = W$	$= w_1 l_1$
	 FIG. 64.	$W\frac{l}{2} = \frac{wl^2}{2}$	$= W_1\frac{l_1}{2} + w_2 l_2$ or $\frac{wl_1^2}{2} + w_2 l_2$
Allowing for weight of lever.	 FIG. 65.	Wl W = weight of long arm of lever W_1 = weight of short arm of lever	$= W_1 l_1 + w_2 l_2 + w_2 l_2$ l = distance of c. of g. of long arm from a l_1 = distance of c. of g. of short arm from a
	 FIG. 66.	$w_2 l_2 + W l_1$ W = weight of whole lever Strictly speaking, W is the weight of the whole lever minus the weight of the eye, as shown in the lower figure.	$= Pl$ l_1 = distance of c. of g. of lever from a

Reaction R at fulcrum *a*, *i.e.* the resultant of all the forces acting on lever.	REMARKS.
$w_1+w_2-w_3+w_4-w_5$	
w_1+W	N.B.—The weight per unit length must be taken on the same length units as the length of the lever.
w_2+W_1+W	
$w_2+W_1+w_3+W$	This is the arrangement of the lever of the Buckton testing machine. Instead of using a huge balance weight on the short arm, the travelling weight w_2 has a contra-clockwise moment when the lever is balanced, and the load on the specimen, viz. w_3, is zero. As w_2 moves along its moment is decreased, and consequently the load w_3 is increased. When w_2 passes over the fulcrum its moment is clockwise; then we have $Wl + w_2l_2 = W_1l_1 + w_3l_3$.
$W+w_2-P$	This is the arrangement of an ordinary lever safety-valve, where P is the pressure on the valve.

		Clockwise moments ↻ about the point *a*.	Contra-clockwise moments ↺ about the point *a*.
Bell-crank levers. Neglecting weight of lever.	 FIG. 67.	w_3l_3	$= w_1l_1 + w_2l_2$
	 FIG. 68.	$w_2l_2 + w_3l_3$	$= w_1l_1 + w_4l_4$
Allowing for weight of lever.	 FIG. 69.	$Wl + w_2l_2$ W = weight of horizontal arm	$= w_1l_1$ l = distance of c. of g. from *a*
	 FIG. 70.	$W_2l_2 + w_3l_3$ W_2 = weight of long curved arm of lever W_1 = ditto short arm	$= W_1l_1 + w_4l_4$ l_2 = distance of c. of g. of long arm from *a* l_1 = ditto short arm l_4 = perpendicular distance of the line of w_4 from *a*
	 FIG. 71.	Wl W = weight of body	$= w_1l_1 + w_2l_2$ l = perpendicular distance of force W from *a*

Reaction R.	REMARKS.
In all these cases it must be found by the parallelogram of forces.	It should be noticed that the direction of the resultant R varies with the *position* of the weights; hence, if a bell crank lever be fitted with a knife-edge, and the weights travel along, as in some types of testing-machines, the resultant does not always pass fairly through the knife-edge, thus— which tends to make the knife-edge slide sideways on its seat, causing it to chimble away, or to damage its fine edge.  FIG. 68*a*.
	The shape of the lever makes no difference whatever to the leverage. Note that the moment of each force is the product of the force by the perpendicular distance of the line of force from the fulcrum.

Centres of Gravity, or Centroids.—We have already given the following definition of the centre of gravity (see p. 13). If a point be so chosen in a body that the sum of the moments of all the gravitational forces acting on the several particles about the one side of any straight line passing through that point, be equal to the sum of the moments on the other side of the line, that point is termed the centre of gravity; or if the moments on the one side of the line be termed positive (+), and the moments on the other side of the line be termed negative (−), the sum of the moments will be zero.

From this definition it will be seen that, as the particles of any body are acted upon by a system of parallel forces, viz. gravity acting upon each, the algebraic sum of the moments of these forces about a line must be zero when that line passes through the c. of g. of the body.

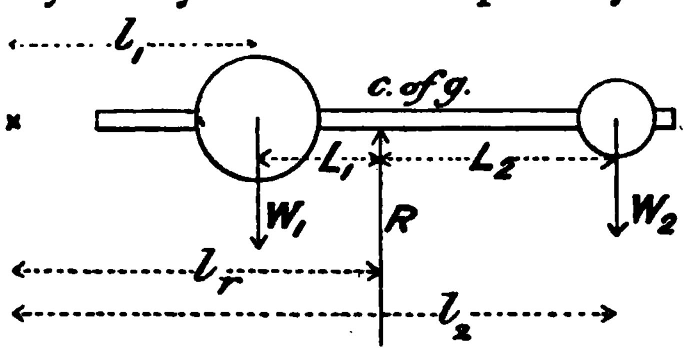


FIG. 72.

Let the weights W_1, W_2, be attached, as shown, to a balanced rod—we need not consider the rod itself, as it is balanced—then, by our definition of the c. of g., we have $W_1L_1 = W_2L_2$.

In finding the position of the c. of g., it will be more convenient to take moments about another point, say x, distant l_1 and l_2 from W_1 and W_2 respectively, and distant L_r (at present unknown) from the c. of g.

$$\text{Then } L_1 = l_r - l_1, \text{ and } L_2 = l_2 - l_r$$
$$\text{but } W_1L_1 = W_2L_2$$
$$\text{or } W_1l_r - W_1l_1 = W_2l_2 - W_2l_r$$
$$(W_1 + W_2)l_r = W_1l_1 + W_2l_2$$
$$l_r = \frac{W_1l_1 + W_2l_2}{W_1 + W_2}$$

If we are dealing with a thin sheet of uniform thickness and weighing K pounds per unit of area, the weight of any given portion will be Ka pounds. Then we may put $W_1 = Ka_1$, and $W_2 = Ka_2$;

$$\text{and } l_r = \frac{K(a_1l_1 + a_2l_2)}{K(a_1 + a_2)} = \frac{a_1l_1 + a_2l_2}{A}$$

or, expressed in words—

distance of c. of g. from the point x

$$= \frac{\text{the sum of the moments of all elemental surfaces about } x}{\text{area of surface}}$$

$$\text{or} = \frac{\text{the moment of surface about } x}{\text{area of surface}}$$

where $A = a_1 + a_2 =$ whole area.

In an actual case there will, of course, be a great number of elemental areas, a, a_1, a_2, a_3, etc., with their corresponding arms, l, l_1, l_2, l_3, etc. Only two have been taken above, they being sufficient to show the principle involved.

When dealing with a body at rest, we may consider its whole mass as being concentrated at its centre of gravity.

Position of Centre of Gravity, or Centroid.

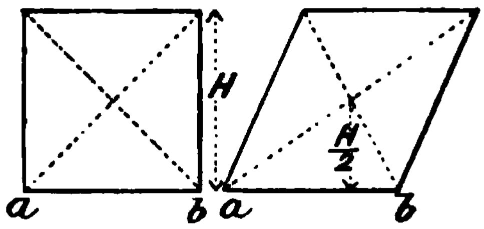


Fig. 73.

Parallelograms.

Intersection of diagonals.

Height above base $ab = \dfrac{H}{2}$

Triangle.

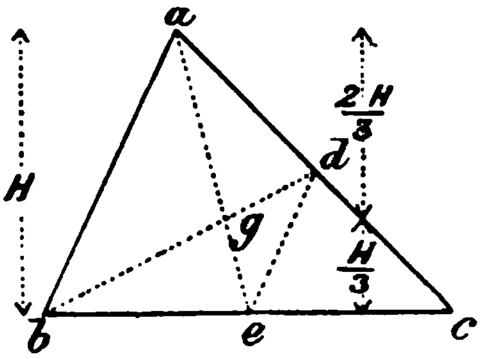


Fig. 74.

Intersection of *ae* and *bd*, where *d* and *e* are the middle points of *ac* and *bc* respectively.

Height above base $bc = \dfrac{H}{3}$

height below apex $a = \dfrac{2H}{3}$

Trapezium.

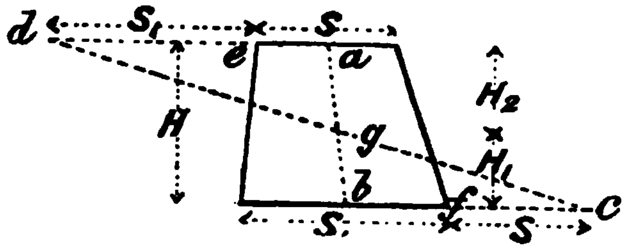


Fig. 75.

Intersection of *ab* and *cd*, where *a* and *b* are the middle points of S and S_1, and $ed = S_1$, $fc = S$.

$$\left.\begin{array}{l}\text{Height above}\\ \text{base } H_1\end{array}\right\} = \frac{H(2S + S_1)}{3(S + S_1)}$$

$$\left.\begin{array}{l}\text{height below}\\ \text{top } H_2\end{array}\right\} = \frac{H(S + 2S_1)}{3(S + S_1)}$$

In a symmetrical figure it is evident that the c. of g. lies on the axis of symmetry. A parallelogram has two axes of symmetry, viz. the diagonals; hence the c. of g. lies on each, and therefore at their intersection, and as they bisect one another, the intersection is at a height $\frac{H}{2}$ from the base.

Conceive the triangle divided up into a great number of very narrow strips parallel to one of the sides, viz. *bc*. It is evident that the c. of g. of each strip will be at the middle point of each, and therefore will lie on a line drawn from the opposite angle point *a* to the middle point of the side *e*, *i.e.* on *ae*; likewise it will lie on *bd*; therefore the c. of g. is at the intersection of *ae* and *bd*, viz. *g*.

Join *de*. Then by construction $ad = dc = \frac{ac}{2}$, and $be = ec = \frac{bc}{2}$; hence the triangles *acb* and *dce* are similar, and therefore $de = \frac{ab}{2}$. The triangles *agb* and *dge* are also similar, hence $eg = \frac{ag}{2} = \frac{ae}{3}$.

Draw the dotted line parallel to the sloping side of the trapezium in Fig. 75*a*.

Height of c. of g. of figure from base

$$= H_1 = \frac{\text{area of parallg.} \times \text{ht. of its c.of g.} + \text{area of } \triangle \times \text{ht. of its c. of g.}}{\text{area of whole figure}}$$

$$H_1 = \frac{\frac{SH \times H}{2} + (S_1 - S)\frac{H}{2} \times \frac{H}{3}}{\left(\frac{S + S_1}{2}\right)H} = \frac{H(2S + S_1)}{3(S + S_1)}$$

$$H_2 = \frac{\frac{SH \times H}{2} + (S_1 - S)\frac{H}{2} \times \frac{2H}{3}}{\left(\frac{S + S_1}{2}\right)H} = \frac{H(S + 2S_1)}{3(S + S_1)}$$

$$\frac{H_1}{H_2} = \frac{2S + S_1}{2S_1 + S} = \frac{S + \frac{S_1}{2}}{S_1 + \frac{S}{2}}$$

or $\frac{bg}{ga} = \frac{bc}{ad}$

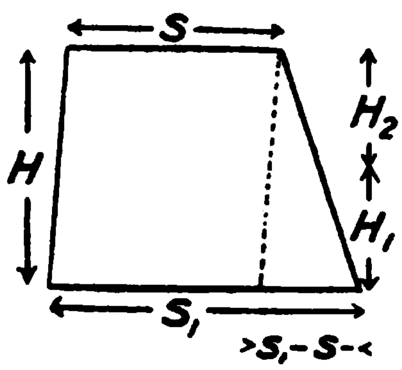


FIG. 75*a*.

POSITION OF CENTRE OF GRAVITY, OR CENTROID.

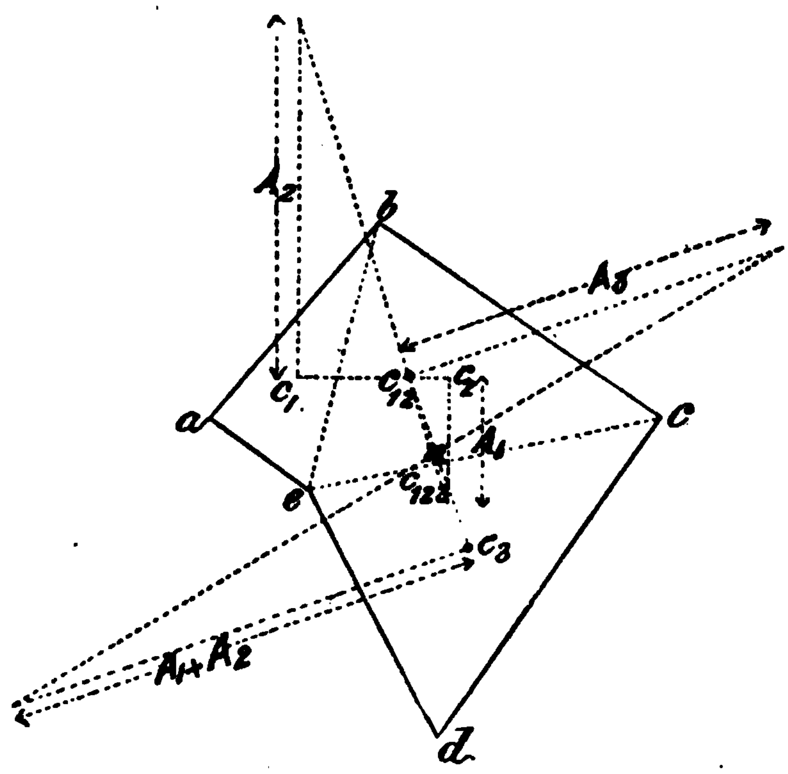


FIG. 76.

Area $abe = A_1$
$bce = A_2$
$cde = A_3$

c. of g. of area abe C_1
" " bce C_2
" " cde C_3
" " whole figure $C_{1.2.3.}$

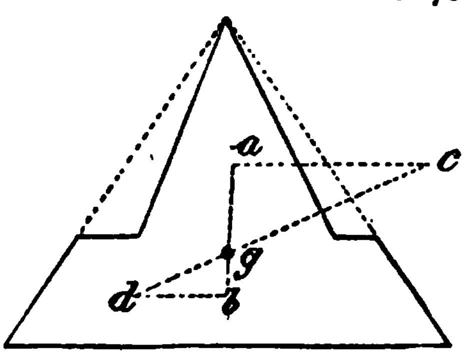


FIG. 77.

Trapezium and triangle.

Intersection of line joining c. of g. of triangle and c. of g. of trapezium, viz. ab and cd, where ac = area of trapezium, and db area of triangle. ac is parallel to bd.

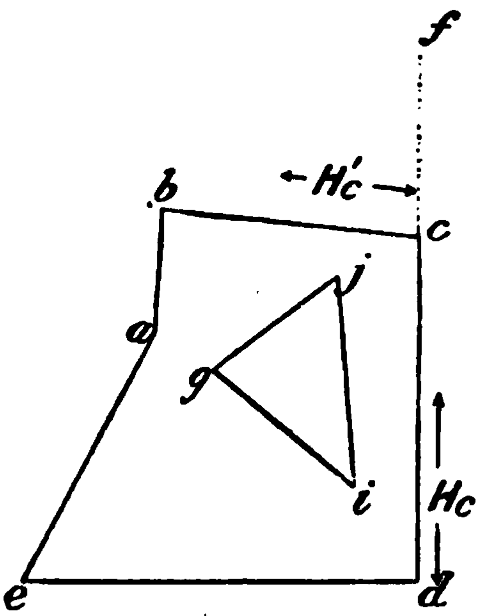


FIG. 78.

Lamina with hole.

Let A = area $abcde$;
H = height of its c. of g. from ed;
H_1 = height of its c. of g. from df drawn at right angles to ed;
a = area of hole gji;
h = height of its c. of g. from ed;
h_1 = height of its c. of g. from df.

Then—

Hc = height of c. of g. of whole figure from ed
$H'c$ = height of c. of g. of whole figure from df

$$Hc = \frac{AH - ah}{A - a}$$

$$H'c = \frac{AH_1 - ah_1}{A - a}$$

The principle of these graphic methods is as follows :—

Let the centres of gravity of two areas, A_1 and A_2, be situated at points C_1 and C_2 respectively, and let the common centre of gravity be situated at c, distant x_1 from C_1, and x_2 from C_2; then we shall have $A_1x_1 = A_2x_2$. From C_2, set off a line c_2b_2, whose length represents on some given scale the area A_1, and from C_1, a line c_1b_1 parallel to it, whose length represents on the same scale the area A_2. Join b_1, b_2. Then the intersection of b_1b_2 and C_1C_2 is the common centre of gravity c. The two triangles are similar, therefore—

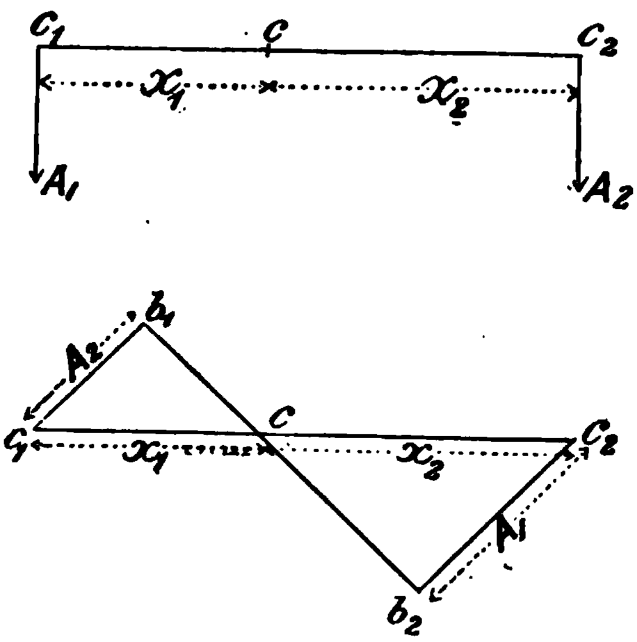


FIG. 76*a*.

$$\frac{A_1}{A_2} = \frac{x_2}{x_1}, \text{ or } A_1x_1 = A_2x_2$$

N.B.—The lines C_1b_1 and C_2b_2 are set off on *opposite* sides of C_1, C_2, and at opposite ends to their respective areas, at any convenient angle; but it is undesirable to have a very acute angle at c, otherwise the point will not be well defined. When one of the areas, say A_2, is negative, *i.e.* is the area of a hole or a part cut out of a lamina, then the lines c_1b_1 and c_2b_2 must be set off on the same side of the line, thus—

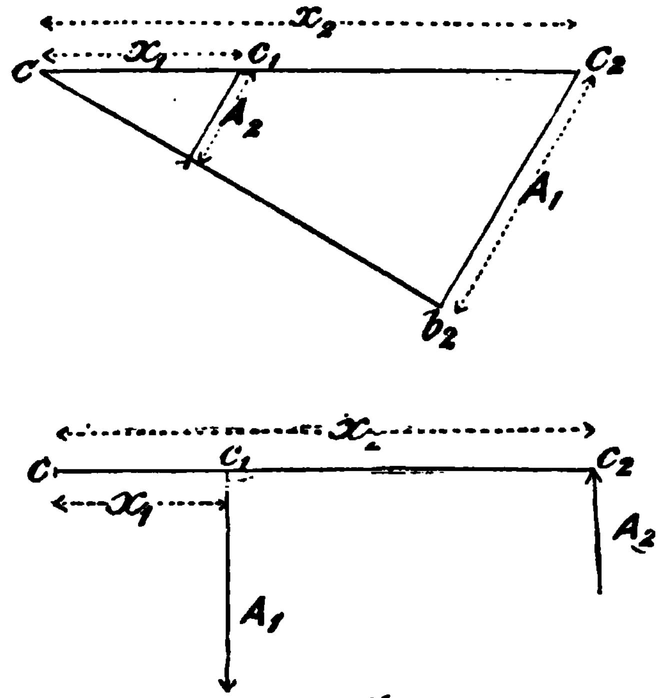


FIG. 76*b*.

Then—

$$\frac{A_1}{A_2} = \frac{x_2}{x_1}, \text{ or } A_1x_1 = A_2x_2$$

POSITION OF CENTRE OF GRAVITY, OR CENTROID.

Graphical method—

Lamina with hole.

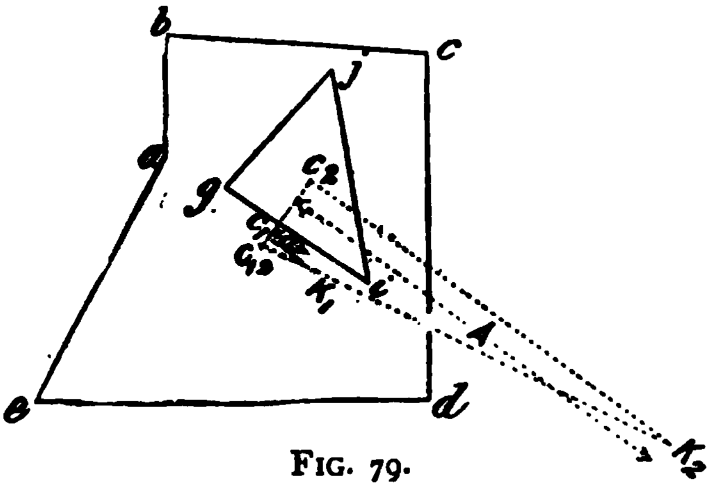


FIG. 79.

If c_1 be the c. of g. of *abcde*;
c_2 " " *gji*.

Join c_1, c_2, and produce; set off c_2K_2 and c_1K_1 parallel to one another, and equal to A and a respectively; through the end points K_2, K_1 draw a line to meet the line through c_1, c_2 in $c_{1.2}$, which is the c. of g. of the whole figure.

NOTE.—The lines c_2K_2, c_1K_1 need not be at right angles to the line c_1c_2, but the line K_2K_1 should not cut it at a very acute angle.

Portion of a regular polygon or an arc of a circle, considered as a thin wire.

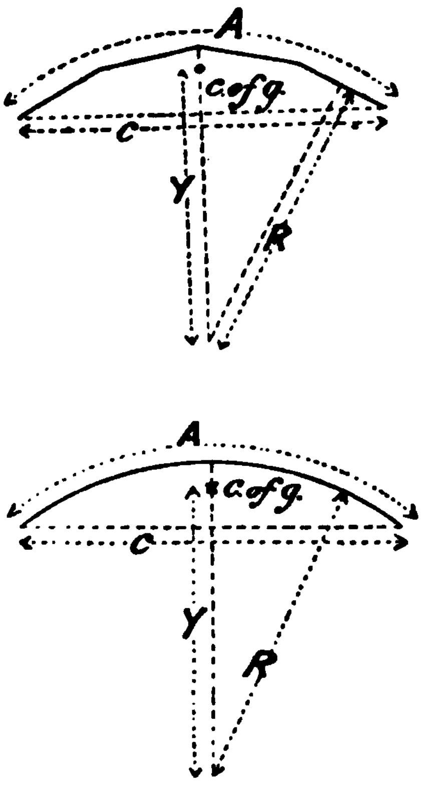


FIG. 80.

Let A = length of the sides of the polygon, or the length of the arc in the case of a circle;

R = radius of a circle inscribed in the polygon, or the radius of the circle itself;

C = chord of the arc of the polygon or circle;

Y = distance of the c. of g. from the centre of the circle.

Then A : R :: C : Y

$$\text{or } \frac{A}{R} = \frac{C}{Y}$$

$$\text{or } Y = \frac{RC}{A}$$

See p. 63.

Regard each side of the polygon as a piece of wire of length l; the c. of g. of each side will be at the middle point, and distant y_1, y_2, y_3, etc., from the diameter of the inscribed circle; and let the projected length of each side on the diameter be c_1, c_2, c_3, etc.

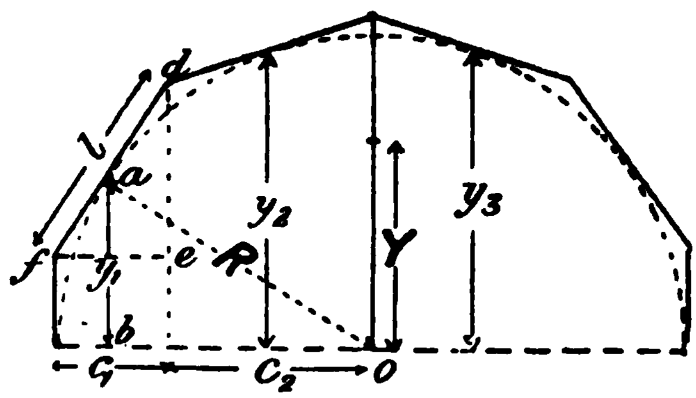


FIG. 80*a*.

The triangles *def* and O*ba* are similar;

therefore $\frac{fd}{fe} = \frac{Oa}{ab}$, or $\frac{l}{c_1} = \frac{R}{y_1}$

$$\text{and } y_1 = \frac{Rc_1}{l}$$

$$\text{likewise } y_2 = \frac{Rc_2}{l}, \text{ and so on}$$

Let Y = distance of c. of g. of portion of polygon from the centre O;

w = weight of each side of the polygon.

Then—

$$Y = \frac{wy_1 + wy_2 +, \text{etc.}}{wn}$$

where n = number of sides. The w cancels top and bottom.

Substituting the values of y_1, y_2, etc., found above, we have—

$$Y = \frac{R}{nl}(c_1 + c_2 +, \text{etc.})$$

(*Proof concluded on p.* 67.)

Position of Centre of Gravity or Centroid.

Semicircular arc or wire.

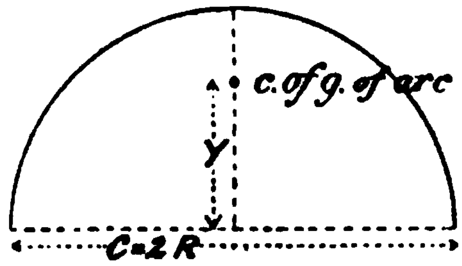


FIG. 81.

$$Y = \frac{2R}{\pi} = \frac{D}{\pi}$$

Circular sector considered as a thin sheet.

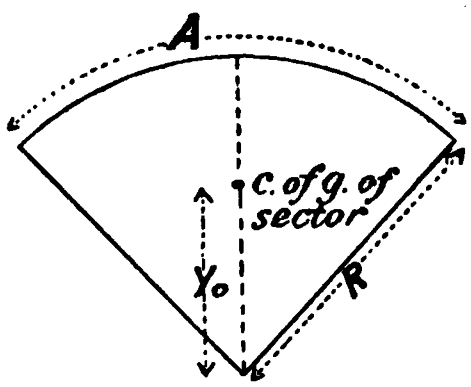


FIG. 82.

$$Y_0 = \frac{2RC}{3A}$$

Semicircular lamina or sheet.

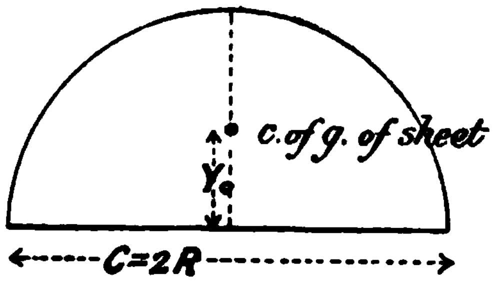


FIG. 83.

$$Y_0 = \frac{4R}{3\pi} = \frac{2D}{3\pi}$$

Parabolic segment.

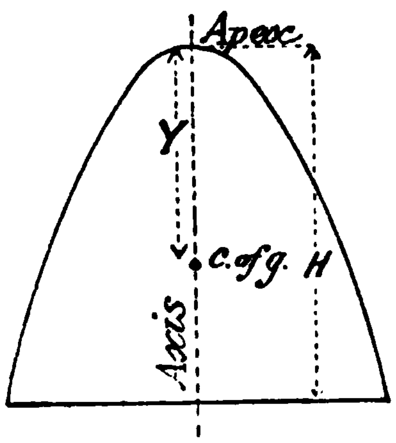


FIG. 84.

$$Y = \tfrac{3}{5}H$$

where Y = distance of c. of g. from apex.

The figure being symmetrical, the c. of g. lies on the axis.

but $c_1 + c_2 +$, etc. = the whole chord subtended by the sides of the polygon

$$= \mathrm{C}$$

$$\text{and } nl = \mathrm{A}$$

$$\text{then } \mathrm{Y} = \frac{\mathrm{RC}}{\mathrm{A}}$$

When n becomes infinitely great, the polygon becomes a circle.

The axis of symmetry on which the c. of g. lies is a line drawn from the centre of the circle at right angles to the chord.

In the case of the sector of a polygon or a circle, we have to find the c. of g. of a series of triangles, instead of their bases, which, as we have shown before, is situated at a distance equal to two-thirds of their height from the apex; hence the c. of g. of a sector is situated at a distance $= \frac{2}{3}\mathrm{Y}$ from the centre of the inscribed circle.

From the properties of the parabola, we have—

$$\frac{h}{\mathrm{H}} = \frac{b^2}{\mathrm{B}^2}$$

$$b = \frac{\mathrm{B}h^{\frac{1}{2}}}{\mathrm{H}^{\frac{1}{2}}}$$

$$\text{Area of strip} = b \,.\, dh = \frac{\mathrm{B}h^{\frac{1}{2}}}{\mathrm{H}^{\frac{1}{2}}}dh$$

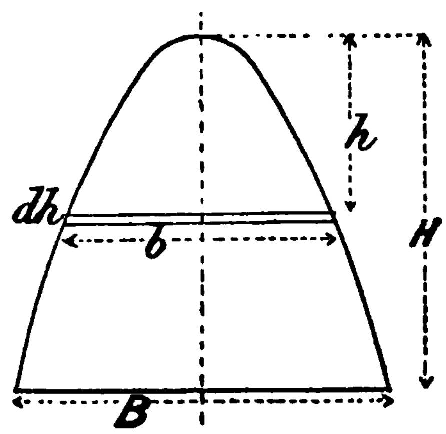
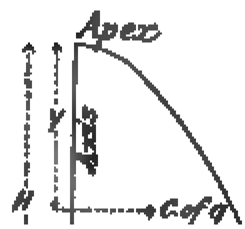


FIG. 84*a*.

(*Continued on p.* 69.)

POSITION OF CENTRE OF GRAVITY OR CENTROID.

Parabolic segment.



$$Y = \tfrac{3}{5}H \text{ (see above)}$$
$$Y_1 = \tfrac{3}{8}B$$

$$\text{moment of strip about apex} = h \,.\, b \,.\, dh = \frac{Bh^{\frac{3}{2}}}{H^{\frac{1}{2}}}dh$$

$$\text{moment of whole figure about apex} = \frac{B}{H^{\frac{1}{2}}}\int_{0}^{H} h^{\frac{3}{2}}dh = \frac{B}{H^{\frac{1}{2}}} \times \frac{H^{\frac{5}{2}}}{\frac{5}{2}}$$

$$= \tfrac{2}{5}\,BH^2$$

the area of the figure $= \frac{2}{3}\,BH$ (see p. 30)

$$\text{the dist. of the c. of g. from the apex} = \frac{\frac{2}{5}BH^2}{\frac{2}{3}BH} = \tfrac{3}{5}H$$

From the properties of the parabola, we have—

$$\frac{h}{H} = \frac{b^2}{B^2}$$

$$\text{or } \frac{H - h_0}{H} = \frac{b^2}{B^2}$$

$$B^2(H - h_0) = b^2H$$

$$B^2h_0 = B^2H - b^2H$$

$$h_0 = \frac{H}{B^2}(B^2 - b^2)$$

$$\text{area of strip} = h_0 db$$

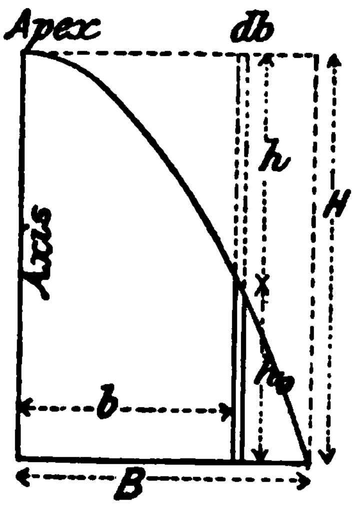


FIG. 85*a*.

$$\text{moment of strip about axis} = b \,.\, h_0 db = \frac{H}{B^2}(B^2b - b^3)db$$

$$\left.\begin{array}{c}\text{moment of whole area}\\ \text{about axis}\end{array}\right\} = \frac{H}{B^2}\int_{0}^{B}(B^2b - b^3)\,db$$

$$= \frac{H}{B^2}\left[\frac{B^2b^2}{2} - \frac{b^4}{4}\right]_{0}^{B}$$

$$= \frac{H}{B^2}\left(\frac{B^4}{2} - \frac{B^4}{4}\right) = \frac{HB^2}{4}$$

$$\text{the area of the figure} = \tfrac{2}{3}BH$$

$$\left.\begin{array}{c}\text{the distance of the c. of g.}\\ \text{from the axis}\end{array}\right\} = \frac{\frac{HB^2}{4}}{\frac{2}{3}BH} = \tfrac{3}{8}B$$

Position of Centre of Gravity or Centroid.

Ex-parabolic segment.

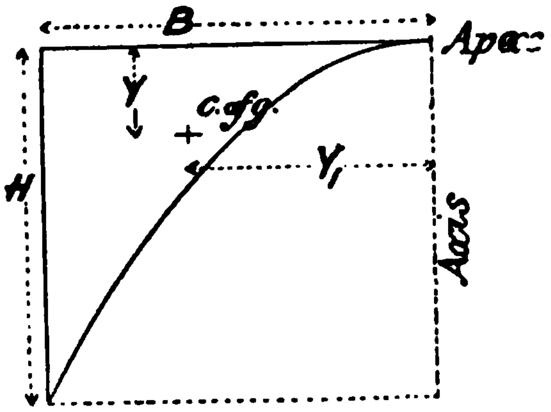
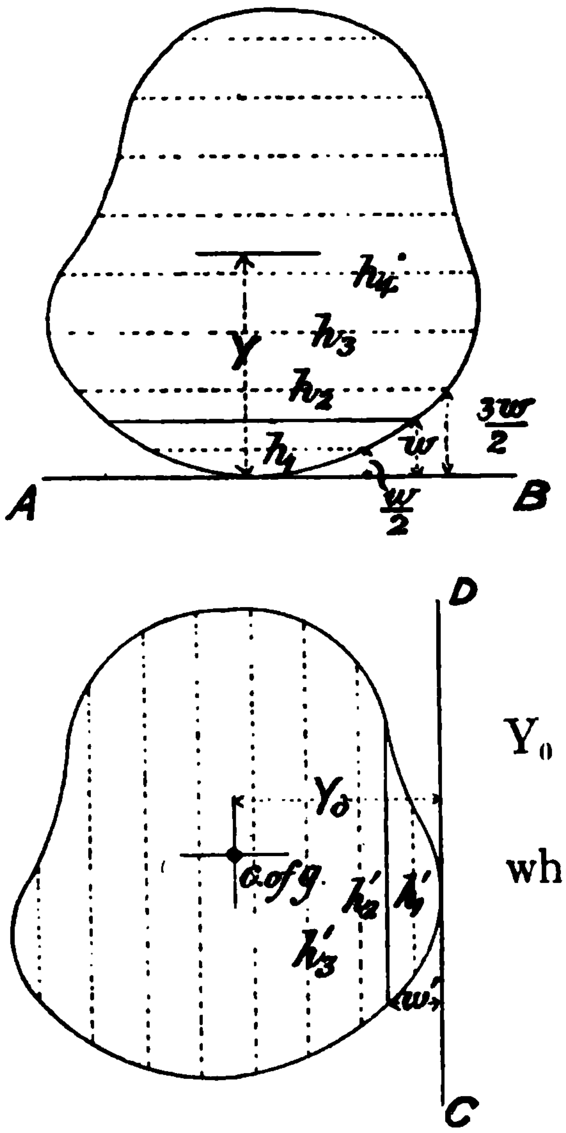


Fig. 86.

$$Y_1 = \tfrac{3}{4}B$$
$$Y = \tfrac{3}{10}H$$

where Y_1 = distance of c. of g. from axis;
Y = distance of c. of g. from apex.

Irregular figure.



$$Y = \frac{w(h_1 + 3h_2 + 5h_3 + 7h_4 +, \text{etc.})}{2(h_1 + h_2 + h_3 + h_4 +, \text{etc.})}$$

where Y = distance of c. of g. from line AB;
w = width of strips;
h = mean height of the strips.

$$Y_0 = \frac{w'}{2}\left(\frac{h'_1 + 3h'_2 + 5h'_3 + 7h'_4 +, \text{etc.}}{h'_1 + h'_2 + h'_3 + h'_4 +, \text{etc.}}\right)$$

where Y_0 = distance of c. of g. from line CD;
w' = width of strips;
h' = mean height of the strips.

Fig. 87.

By the principle of moments, we have—

Distance of c. of g. of figure from axis

$$= \frac{\left(\text{area of rect.} \times \begin{Bmatrix}\text{dist. of its c. of g.}\\ \text{from axis}\end{Bmatrix} - \begin{Bmatrix}\text{area of para.}\\ \text{segment}\end{Bmatrix} \times \begin{Bmatrix}\text{dist. of its c. of g.}\\ \text{from axis}\end{Bmatrix}\right)}{\text{area of figure}}$$

$$Y_1 = \frac{BH \times \frac{B}{2} - \frac{2}{3}BH \times \frac{3}{8}B}{\frac{1}{3}BH} = \frac{3}{4}B$$

Likewise—

$$Y = \frac{BH \times \frac{H}{2} - \frac{2}{3}BH \times \frac{3}{5}H}{\frac{1}{3}BH} = \frac{3}{10}H$$

This is a simple case of moments, in which we have—

$$\left.\begin{matrix}\text{Distance of c. of g.}\\ \text{from line AB}\end{matrix}\right\} = \frac{\text{moment of each strip about AB}}{\text{area of whole figure}}$$

The area of the first strip $= wh_1$	Moment of first strip $= wh_1 \times \frac{w}{2}$
„ „ second „ $= wh_2$	„ second „ $= wh_2 \times \frac{3w}{2}$
„ „ third „ $= wh_3$	„ third „ $= wh_3 \times \frac{5w}{2}$

and so on.

Area of whole figure $= wh_1 + wh_2 + wh_3 +$, etc.

$$\left.\begin{matrix}\text{Distance of c.}\\ \text{of g. from}\\ \text{line AB}\end{matrix}\right\} = Y = \frac{wh_1 \times \frac{w}{2} + wh_2 \times \frac{3w}{2} + wh_3 \times \frac{5w}{2} +, \text{etc.}}{wh_1 + wh_2 + wh_3 +, \text{etc.}}$$

one w cancels out top and bottom, and we have—

$$Y = \frac{w}{2}\left(\frac{h_1 + 3h_2 + 5h_3 + 7h_4 +, \text{etc.}}{h_1 + h_2 + h_3 + h_4 +, \text{etc.}}\right)$$

and similarly with Y_0.

The division of the figure may be done thus: Draw a line, xy, at any angle, and set off equal parts as shown; project the first, third, fifth, etc., on to xz drawn normal to AB.

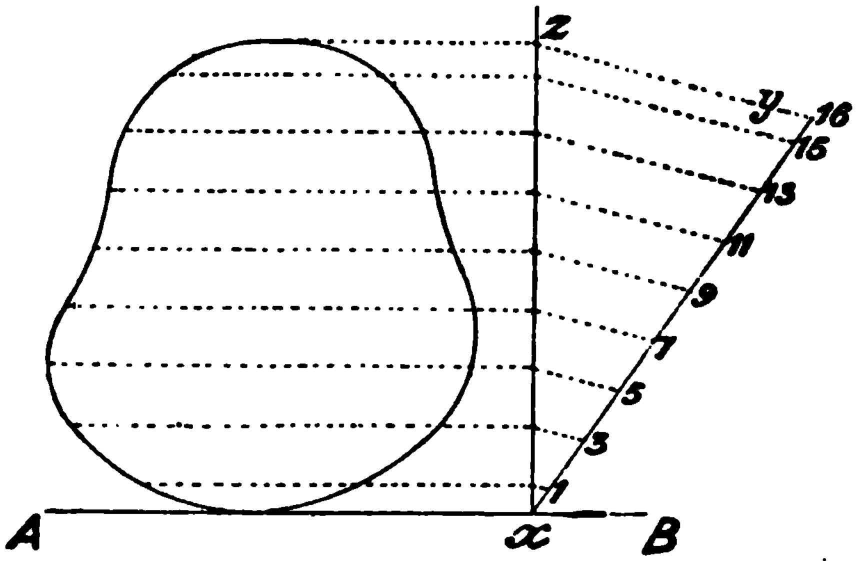


FIG. 87*a*.

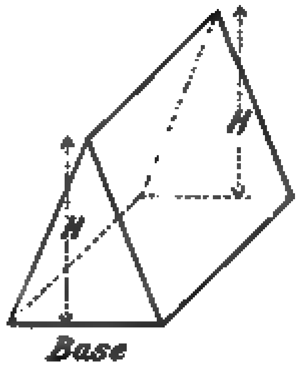


FIG. 88.

POSITION OF CENTRE OF GRAVITY OR CENTROID.

Wedge.

On a plane midway between the ends, and at a height $\frac{H}{3}$ from base.

For frustum of wedge, see Trapezium.

Pyramid.

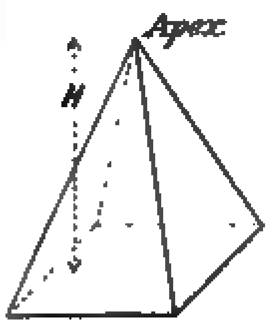


FIG. 89.

On a line drawn from the middle point of the base to the apex, and at a distance $\frac{3}{4}$H from the apex.

a line drawn from the middle point base to the apex, and at a height $\left. \frac{- n^4}{- n^3} \right)$ from the apex, where $n = \frac{H_1}{H}$

A wedge may be considered as a large number of triangular laminæ placed side by side, the c. of g. of each being situated at a height $\frac{H}{3}$ from the base.

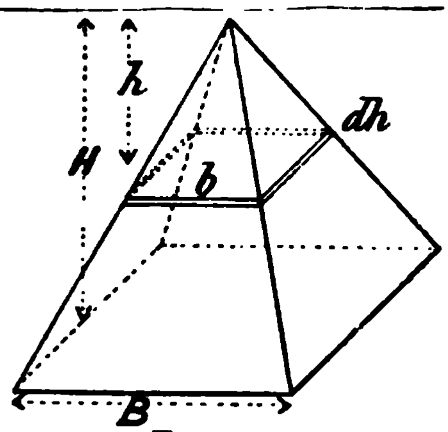


FIG. 89*a*.

$$\text{Volume of layer} = b^2 \,.\, dh$$

$$\text{moment of layer about apex} = b^2 \,.\, h \,.\, dh$$

$$\text{but } \frac{b}{h} = \frac{B}{H}$$

$$b = \frac{h.\,B}{H}$$

$$\text{moment of layer about apex} = \frac{B^2 h^3}{H^2} dh$$

$$\left.\begin{array}{l}\text{moment of the whole pyramid}\\ \text{about apex}\end{array}\right\} = \frac{B^2}{H^2}\int_0^H h^3 \,.\, dh = \frac{B^2 H^4}{4H^2} = \frac{B^2 H^2}{4}$$

$$\text{volume of pyramid} = \frac{B^2 H}{3}$$

$$\text{distance of c. of g. from apex} = \frac{\frac{B^2 H^2}{4}}{\frac{B^2 H}{3}} = \tfrac{3}{4}H$$

In the case above, instead of integrating between the limits of H and o for the moment about the apex, we must integrate between the limits H and H_1; thus—

$$\left.\begin{array}{l}\text{Moment of frustum of pyramid}\\ \text{about the (imaginary) apex}\end{array}\right\} = \frac{B^2}{H^2}\int_{H_1}^{H} h^3 \,.\, dh$$

$$= \frac{B^2}{H^2}\left(\frac{H^4}{4} - \frac{H_1^4}{4}\right) \quad . \quad \text{(i.)}$$

$$\text{volume of frustum} = \frac{B^2 H}{3} - \frac{B_1^2 H_1}{3} \quad . \quad . \quad \text{(ii.)}$$

$$\text{substituting the value } \frac{B_1}{B} = \frac{H_1}{H} = n$$

$$\left.\begin{array}{l}\text{then the distance of the c. of g.}\\ \text{from the apex}\end{array}\right\} = \frac{\text{(i.)}}{\text{(ii.)}} = \tfrac{3}{4}H\left(\frac{1 - n^4}{1 - n^3}\right)$$

POSITION OF CENTRE OF GRAVITY OR CENTROID.

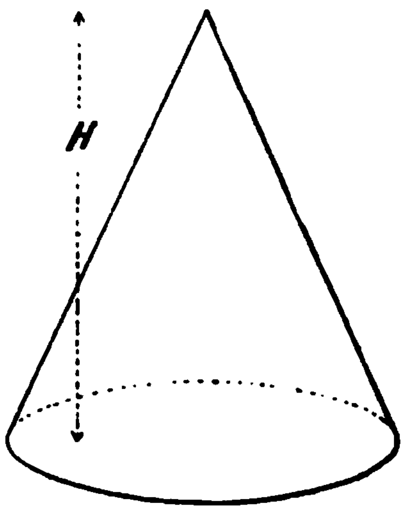


FIG. 91.

Cone.

On the axis of revolution, and at a distance $\frac{3}{4}$H from the apex.

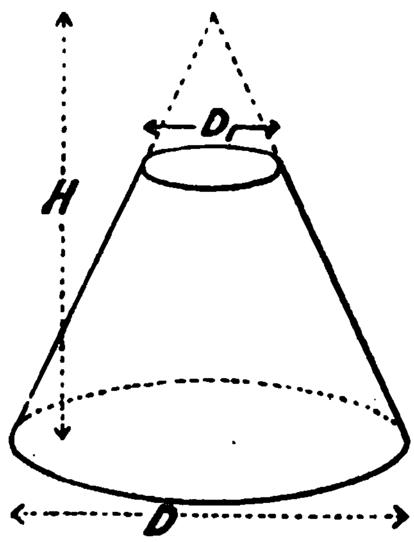


FIG. 92

Frustum of cone.

On the axis of revolution, and at a height $\frac{3}{4}H\left(\frac{1 - n^4}{1 - n^3}\right)$ from the apex, where $n = \frac{D_1}{D}$

Irregular surfaces.—

Also see Barker's "Graphical Calculus," p. 179, for a graphical integration of irregular surfaces.

are found in precisely the same way as in the pyramid and frustum of pyramid.

The c. of g. is easily found by balancing methods; thus, if the c. of g. of an irregular surface be required, cut out the required figure in thin sheet metal or cardboard, and balance on the edge of a steel straight-edge, thus: The points *a*, *a* and *b*, *b* are marked and afterwards joined: the point where they cut is the c. of g. As a check on the result, it is well to balance about a third line *cc*; the three lines should intersect at one point, and not form a small triangle.

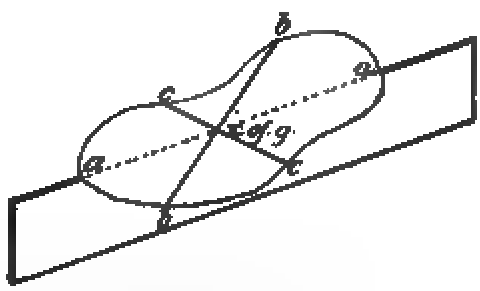


FIG. 93.

The c. of g. of many solids can also be found in a similar manner, or by suspending them by means of a wire, and dropping a perpendicular through the points of suspension.

Second Moments—Moments of Inertia.

A definition of a second moment has been given on p. 52. In every case we shall find the second moment by summing up

or integrating the product of every element of the body or surface by the square of its distance from the axis in question. In some cases we shall find it convenient to make use of the following theorems :—

Let I_0 = the second moment, or moment of inertia, of any surface (treated as a thin lamina) or body about a given axis;
I = the second moment, or moment of inertia, of any surface (treated as a thin lamina) or body about a parallel axis passing through the c. of g.;
M = mass of the body;
A = area of the surface;
R_0 = the perpendicular distance between the two axes.

$$\text{Then } I_0 = I + MR_0^2, \text{ or } I + AR_0^2$$

Let xy be the axis passing through the c. of g.

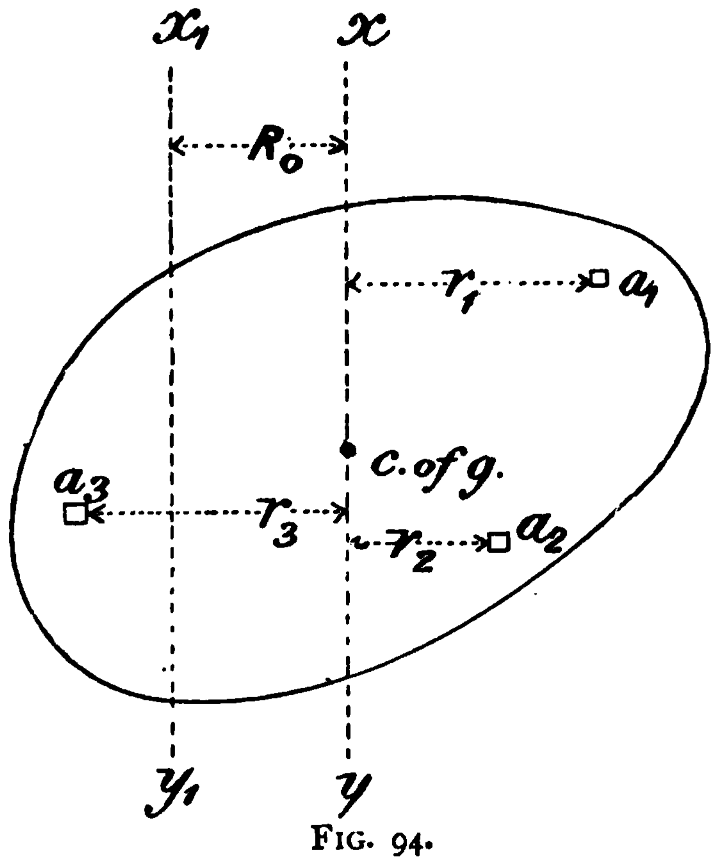


FIG. 94.

Let x_1y_1 be the axis of revolution, parallel to xy and in the plane of the surface or lamina.

Let the elemental areas, a_1, a_2, a_3, etc., be situated at distances r_1, r_2, r_3, etc., from xy. Then we have—

$$\begin{aligned} I_0 &= a_1(R_0 + r_1)^2 + a_2(R_0 + r_2)^2 +, \text{ etc.} \\ &= a_1(R_0^2 + r_1^2 + 2R_0r_1) + a_2(R_0^2 + r_2^2 + 2R_0r_2) +, \text{ etc.} \\ &= a_1r_1^2 + a_2r_2^2 +, \text{ etc.} + R_0^2(a_1 + a_2 +, \text{ etc.}) + 2R_0(a_1r_1 + a_2r_2 +, \text{ etc.}) \end{aligned}$$

But as xy passes through the c. of g. of the section, we have $a_1r_1 + a_2r_2 +$, etc. = 0 (see p. 58), for some r's are positive and some negative; hence the latter term vanishes.

The second term, $a_1 + a_2 +$, etc. = the whole area (A); whence it becomes R_0^2A.

In the first term, we have simply the second moment, or moment of inertia, about the axis passing through the c. of g. = I; hence we get—

$$I_0 = I + R_0^2A$$

We may, of course, substitute m_1, m_2, etc., for the elemental masses, and M for the mass of the body instead of A.

When a body or surface (treated as a thin lamina) revolves about an axis or pole perpendicular to its plane of revolution, the second moment, or moment of inertia, is termed the second polar moment, or polar moment of inertia.

The second polar moment of any surface is the sum of the second moments about any two rectangular axes in its own plane passing through the axis of revolution, or

$$I_p = I_x + I_y$$

Consider any elemental area a, distant r from the pole.

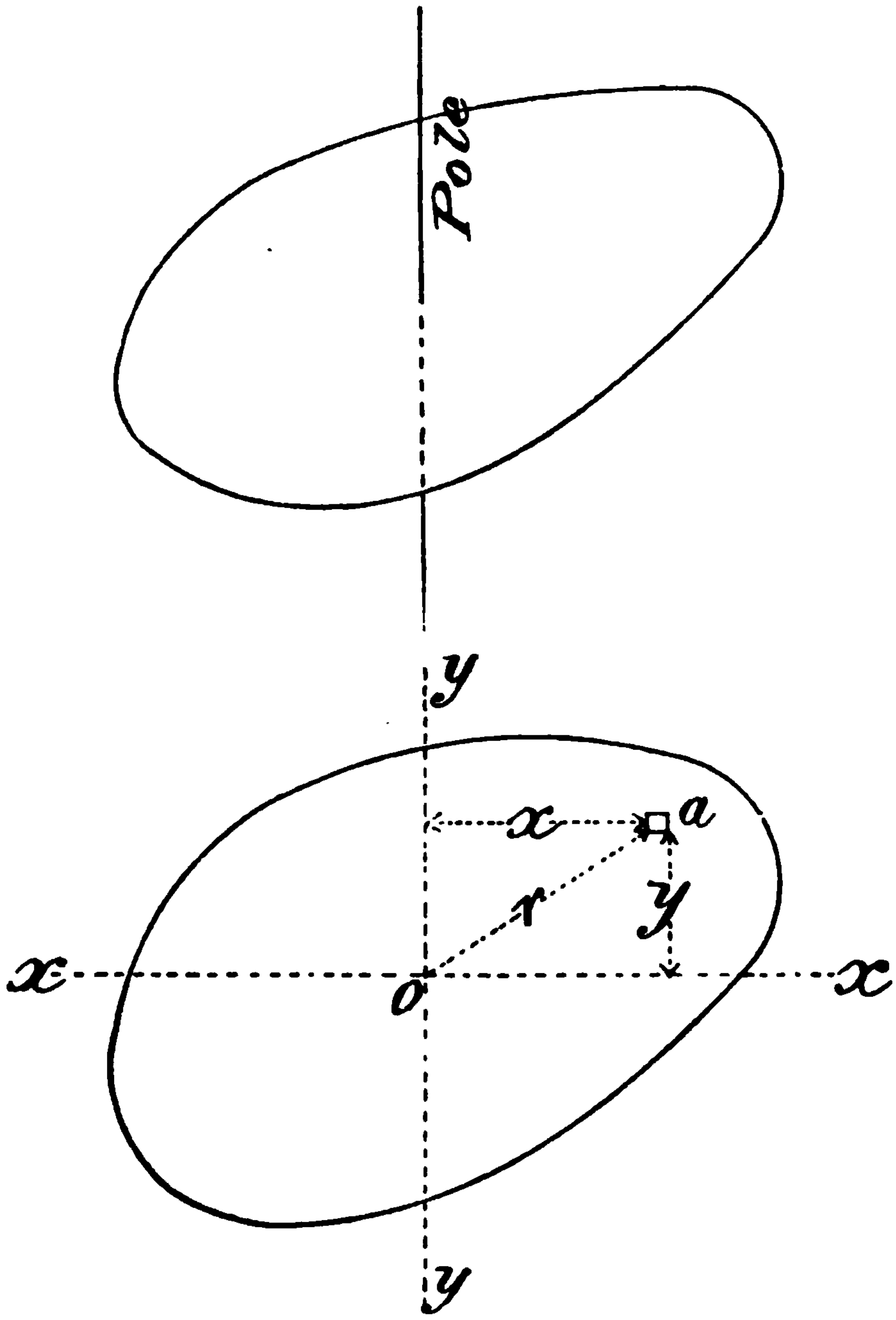


FIG. 95.

I_x about $ox = ay^2$
I_y „ $oy = ax^2$
I_p „ pole $= ar^2$
But $r^2 = x^2 + y^2$
and $ar^2 = ax^2 + ay^2$
hence $I_p = I_x + I_y$

In a similar way, it may be proved for every element of the surface.

When finding the position of the c. of g., we had the following relation :—

$$\left.\begin{array}{l}\text{Distance of c. of g. from}\\ \quad\text{the axis } xy\ (\kappa)\end{array}\right\} = \frac{\text{first moment of surface about } xy}{\text{area of surface}}$$

$$\text{or} = \frac{\text{first moment of body about } xy}{\text{volume of body}}$$

$$\kappa = \frac{\Sigma ar}{A}$$

Now, when dealing with the second moment, we have a corresponding centre, termed the centre of gyration, at which the whole of a moving body or surface may be considered to be concentrated; the distance of the centre of gyration from the axis of revolution is termed the "radius of gyration." When finding its value, we have the following relation :—

$$\left.\begin{array}{l}\text{Radius of gyration}\\ \text{about the axis } xy\ (\kappa^2)\end{array}\right\}^2 = \frac{\text{second moment of surface about } xy}{\text{area of surface}}$$

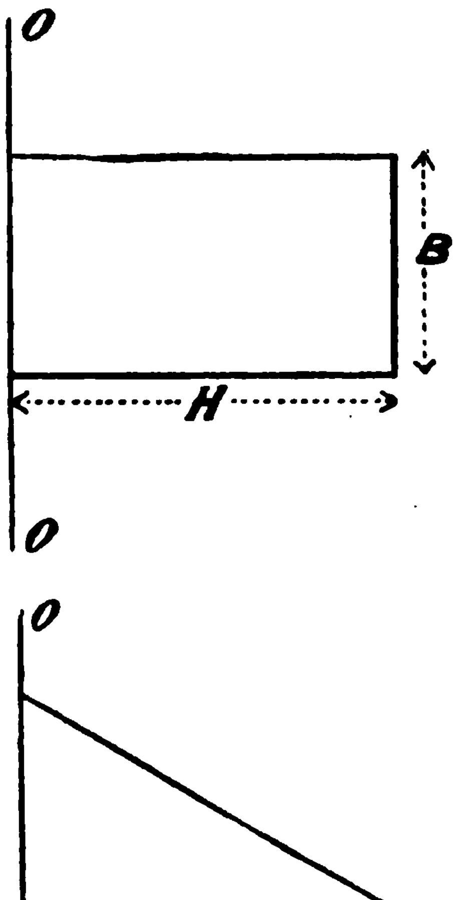
$$\text{or} = \frac{\text{second moment of body about } xy}{\text{volume of body}}$$

$$\kappa^2 = \frac{\Sigma ar^2}{A} = \frac{I}{A}, \text{ or } I = A\kappa^2$$

SECOND MOMENT, OR MOMENT OF INERTIA (I).

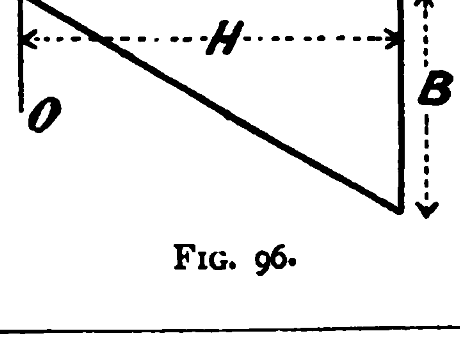
Parallelogram treated as a thin lamina about its extreme end.

Radius of gyration κ.



$$I_0 = \frac{BH^3}{3}$$

$$\frac{H}{\sqrt{3}}$$



If the figure be a square—

$$B = H = S$$

we have $I_0 = \dfrac{S^4}{3}$

FIG. 96.

Area of elemental strip = $B \cdot dh$
second moment of strip = $B \cdot h^2 \cdot dh$

$$\left.\begin{array}{r}\text{second moment}\\ \text{of whole surface}\end{array}\right\} = B\int_0^H h^2 \cdot dh = \frac{BH^3}{3}$$

$$\left.\begin{array}{r}\text{area of whole}\\ \text{surface}\end{array}\right\} = BH$$

$$\left.\begin{array}{r}\text{square of radius}\\ \text{of gyration}\end{array}\right\} = \frac{BH^3}{3BH} = \frac{H^2}{3}$$

$$\text{radius of gyration} = \frac{H}{\sqrt{3}}$$

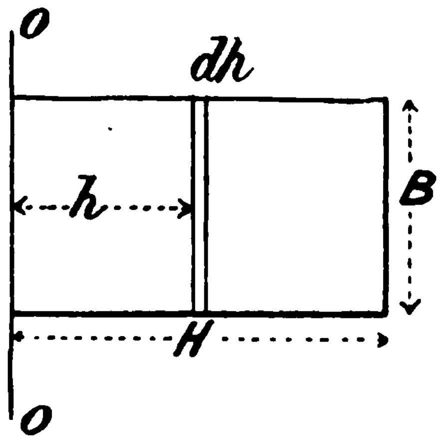


FIG. 96*a*.

It will be seen that the above reasoning holds, however the parallelogram may be distorted sideways, as shown.

Second Moment, or Moment of Inertia (I).

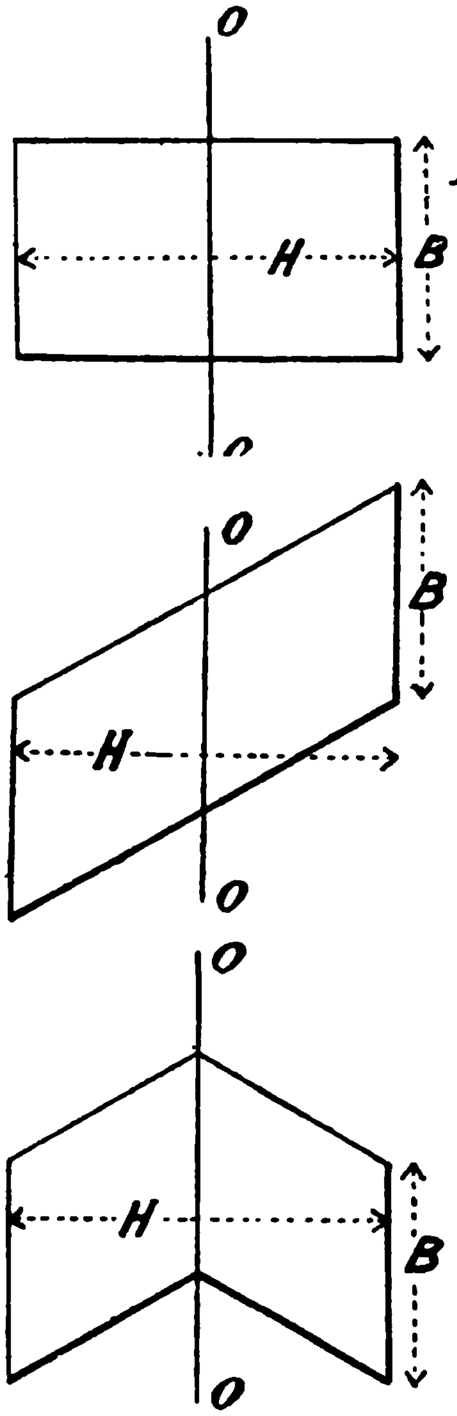


Fig. 97.

Parallelogram treated as a thin lamina about its central axis.

$$I = \frac{BH^3}{12} \text{ or } \frac{S^4}{12}$$

Radius of gyration κ.

$$\frac{H}{\sqrt{12}}$$

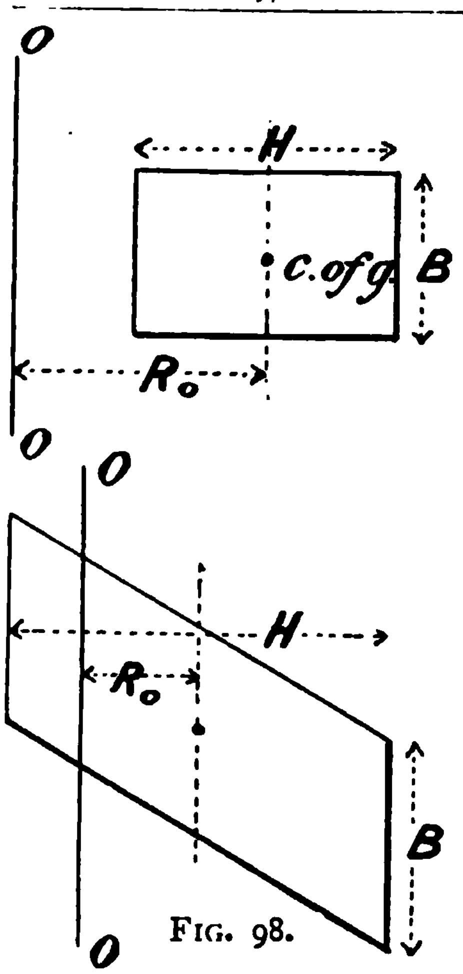


Fig. 98.

Parallelogram treated as a thin lamina about an axis distant R_0 *from its c. of g.*

$$I_0 = BH\left(\frac{H^2}{12} + R_0^2\right)$$

$$\sqrt{\frac{H^2}{12} + R_0^2}$$

This is simply a case of two parallelograms such as the above put together axis to axis, each of length $\frac{H}{2}$;

$$\text{Then the second moment of each} = \frac{B\left(\frac{H}{2}\right)^3}{3} = \frac{BH^3}{8 \times 3}$$

$$\left.\begin{array}{r}\text{then the second moment}\\ \text{of the two together}\end{array}\right\} = 2\left(\frac{BH^3}{24}\right) = \frac{BH^3}{12}$$

$$\text{area of whole surface} = BH$$

$$\text{radius of gyration} = \sqrt{\frac{BH^3}{12BH}} = \frac{H}{\sqrt{12}}$$

From the theorem given above (p. 76), we have—

$$I_0 = I + R_0^2A \qquad I = \frac{BH^3}{12}$$

$$= \frac{BH^3}{12} + R_0^2BH \qquad A = BH$$

$$= BH\left(\frac{H^2}{12} + R_0^2\right)$$

when $R = 0$, $I_0 = I$

SECOND MOMENT, OR MOMENT OF INERTIA (I).

	Radius of gyration κ.
Hollow parallelogram. Let I_e = second moment of external figure; I_i = second moment of internal figure; I_0 = second moment of hollow figure; $I_0 = I_e - I_i$.	
Triangle about an axis parallel to the base passing through the apex. $I_0 = \frac{BH^3}{4}$	$\frac{H}{\sqrt{2}}$
Triangle about an axis parallel to the base passing through the c. of g. $I = \frac{BH^3}{36}$	$\frac{H}{\sqrt{18}}$

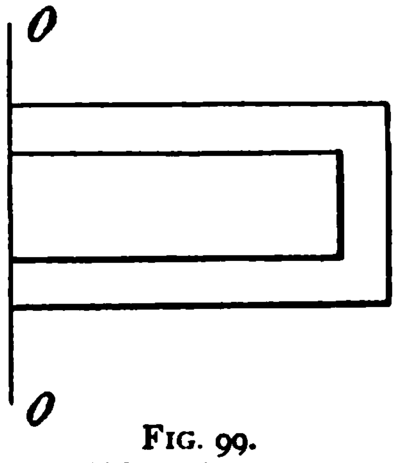


FIG. 99.

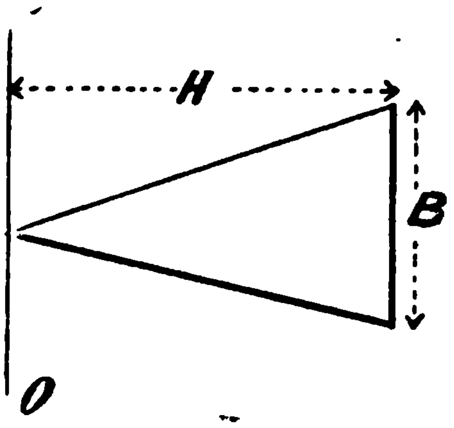


FIG. 100.

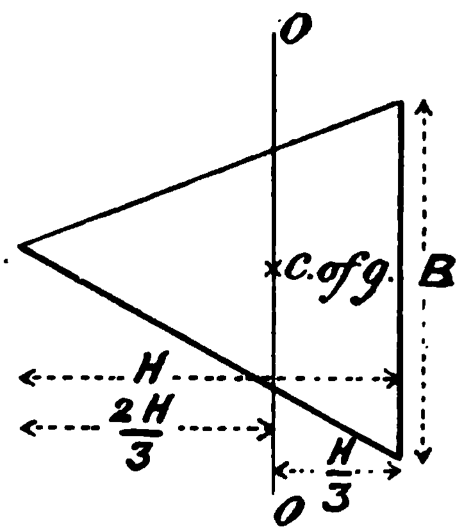


FIG. 101.

The I_0 for the hollow parallelogram is simply the difference between the I_e for the external, and the I_i for the internal parallelogram.

Area of strip $= b \,.\, dh$; but $b = \dfrac{hB}{H}$

" " $= \dfrac{B}{H} h \,.\, dh$

second moment of strip $= \dfrac{B}{H} h^3 \,.\, dh$

" " triangle $= \dfrac{B}{H}\displaystyle\int_0^H h^3 \,.\, dh$

" " " $= \dfrac{BH^4}{4H} = \dfrac{BH^3}{4}$

area of triangle $= \dfrac{BH}{2}$

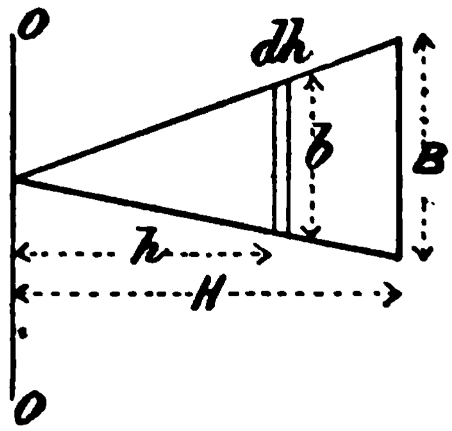


FIG. 100*a*.

radius of gyration $= \sqrt{\dfrac{\frac{BH^3}{4}}{\frac{BH}{2}}} = \sqrt{\dfrac{H^2}{2}} = \dfrac{H}{\sqrt{2}}$

From the theorem on p. 76, we have—

$I_0 = I + R_0^2 A$

$I = I_0 - R_0^2 A$

$I = \dfrac{BH^3}{4} - \dfrac{4H^2}{9} \times \dfrac{BH}{2}$

$I = \dfrac{BH^3}{36}$

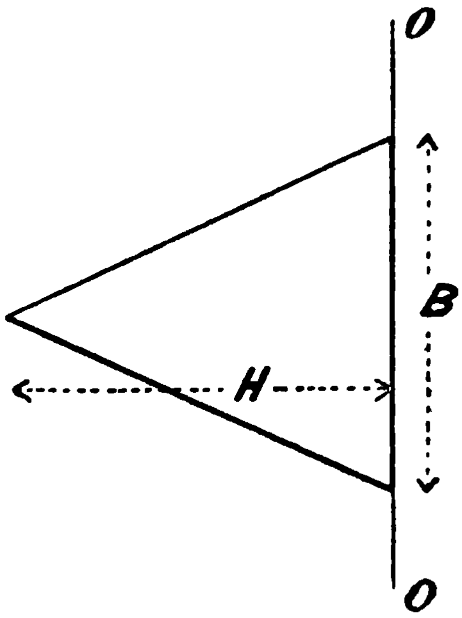
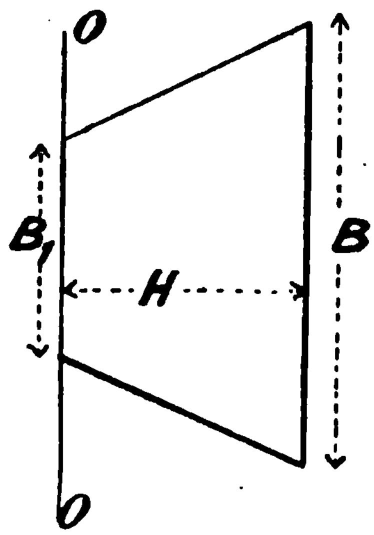
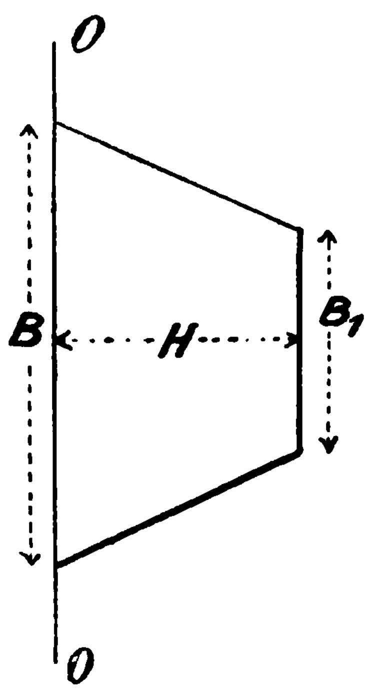
$I_0 = \dfrac{BH^3}{4}$

$R_0^2 = \left(\dfrac{2H}{3}\right)^2 = \dfrac{4H^2}{9}$

$A = \dfrac{BH}{2}$

radius of gyration $= \sqrt{\dfrac{\frac{BH^3}{36}}{\frac{BH}{2}}} = \sqrt{\dfrac{H^2}{18}} = \dfrac{H}{\sqrt{18}}$

SECOND MOMENT, OR MOMENT OF INERTIA (I).

		Radius of gyration κ.
 FIG. 102.	*Triangle about an axis at the base.* $I_0 = \frac{BH^3}{12}$	$\frac{H}{\sqrt{6}}$
Trapezium about an axis coinciding with its short base.  FIG. 103.	$I_0 = \frac{(3B + B_1)H^3}{12}$ Let $B_1 = nB$. $I_0 = \frac{BH^3}{12}(3 + n)$	$H\sqrt{\frac{n + 3}{6(n + 1)}}$
Trapezium about an axis coinciding with its long base.  FIG. 104.	$I_0 = \frac{(3B_1 + B)H^3}{12}$ or $\frac{BH^3}{12}(3n + 1)$ $n = \frac{B^1}{B}$	$H\sqrt{\frac{3n + 1}{6(n + 1)}}$

From the theorem quoted above, we have—

$$I_0 = I + R_0^2 A \qquad R_0^2 = \frac{H^2}{9}$$

$$I_0 = \frac{BH^3}{36} + \frac{H^2}{9} \times \frac{BH}{2}$$

$$I_0 = \frac{BH^3}{12}$$

Radius of gyration obtained as in the last case.

This figure may be treated as a parallelogram and a triangle about an axis passing through the apex.

For parallelogram, $I = \dfrac{B_1H^3}{3}$

for triangle, $I = \dfrac{(B - B_1)H^3}{4}$

for trapezium, $I = \dfrac{B_1H^3}{3} + \dfrac{(B - B_1)H^3}{4}$

" " $I = \dfrac{(3B + B_1)H^3}{12}$

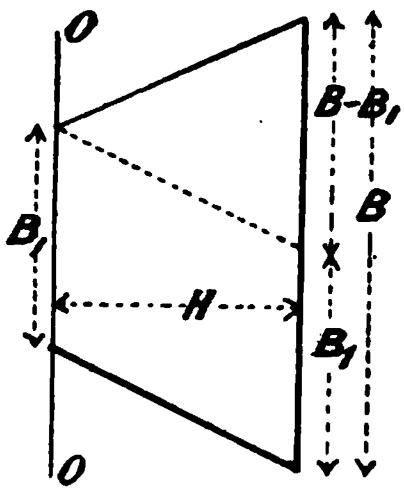


FIG. 103*a*.

When the axis coincides with the long base, the I for the triangle $= \dfrac{(B - B_1)H^3}{12}$; then, adding the I for the parallelogram as above, we get the result as given.

When $n = 1$, the figures become parallelograms, and $I = \dfrac{BH^3}{3}$, as found above.

When $n = 0$, the figures become triangles, and the $I = \dfrac{BH^3}{4}$ for the first case, as found for the triangle about its apex; and $I = \dfrac{BH^3}{12}$ for the second case, as found for the triangle about its base.

Second Moment, or Moment of Inertia (I).

Radius of gyration κ.

Trapezium about an axis passing through its c. of g. and parallel with the base.

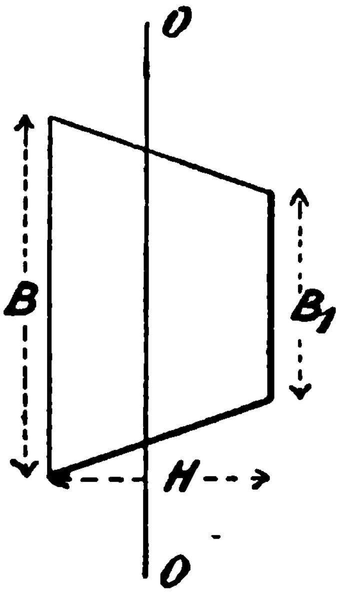


FIG. 105

$$I = \frac{(B_1^2 + 4B_1B + B^2)H^3}{36(B_1 + B)}$$

$$\text{or } \frac{BH^3}{36}\left(\frac{n^2 + 4n + 1}{n + 1}\right)$$

$$n = \frac{B_1}{B}$$

For a close approximation, see next figure.

Approximate method for trapezium about axis passing through c. of g.

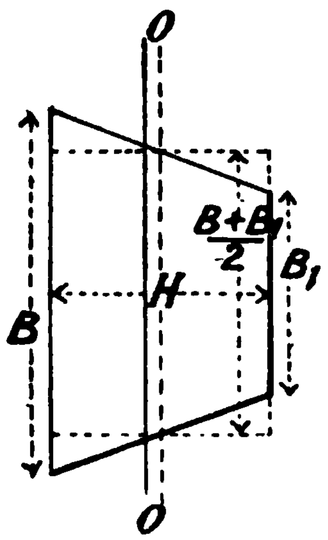


FIG. 106.

The I for dotted rectangle about an axis passing through its c. of g., is approximately the same as the I for trapezium.

For dotted rectangle—

$$I = \frac{(B + B_1)H^3}{24}$$

or if $B_1 = nB$

$$I = \frac{BH^3}{24}(n + 1)$$

From the theorem on p. 76, we have—

$$I = I_0 - R_0^2 A \qquad I_0 = \frac{(3B_1 + B)H^3}{12} \text{ (about long base)}$$

$$R_0 = \frac{H(2B_1 + B)}{3(B_1 + B)} \text{ (see p. 60)}$$

$$A = \left(\frac{B_1 + B}{2}\right)H$$

Substituting the values in the above equation and simplifying, we get the result as given. The working out is simple algebra, but too lengthy to give here.

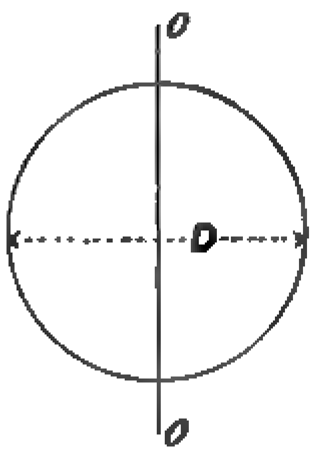
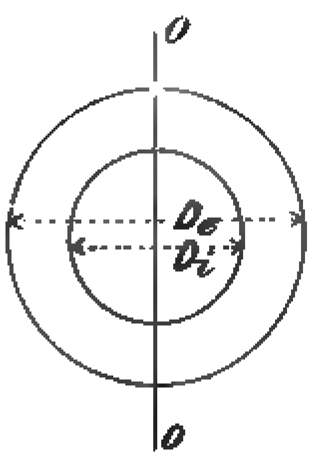
The I for a rectangle is $\frac{B'H^3}{12}$ (see p. 80). Putting in the value $\frac{B + B_1}{2} = B'$, we get—

$$I = \frac{B + B_1}{2} \times \frac{H^3}{12} = \frac{(B + B_1)H^3}{24}$$

The following table shows the error involved in the above assumption; it will be seen that the error becomes serious when $n > 0{\cdot}5$:—

Value of n.	Approx. method, the correct value being 1.
0·9	1·001
0·8	1·005
0·7	1·011
0·6	1·021
0·5	1·039
0·4	1·065
0·3	1·107
0·2	1·174

The approximate method always gives too high results.

 FIG. 108.	*Circle about a diameter.* $I = \frac{\pi D^4}{64}$	$\frac{D}{4}$
 FIG. 109.	*Hollow circle about a diameter.* $\frac{\pi}{64}(D_e^4 - D_i^4)$	$\frac{\sqrt{D_e^2 + D_i^2}}{4}$

This may be taken as two triangles about their bases (see p. 84).

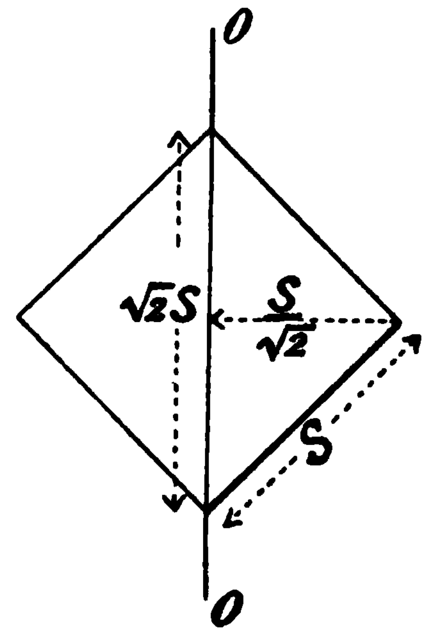


FIG 107*a*.

In this case, $B = \sqrt{2}S$

$$H = \frac{S}{\sqrt{2}}$$

$$I = 2\left(\frac{BH^3}{12}\right)$$

$$= 2\left(\frac{\sqrt{2}S\left(\frac{S}{\sqrt{2}}\right)^3}{12}\right)$$

$$= \frac{S^4}{12}$$

area of figure $= S^2$

radius of gyration $= \sqrt{\frac{S^4}{12S^2}} = \frac{S}{\sqrt{12}}$

From the theorem on p. 77, we have $I_p = I_x + I_y$; in the circle, $I_x = I_y$.

$$\text{Then } I_p = 2I_x$$

$$\text{and } I_x = \frac{I_p}{2} = \frac{\pi D^4}{2 \times 32} = \frac{\pi D^4}{64} \text{ (see Fig. 118).}$$

The I for the hollow circle is simply the difference between the I for the outer and inner circles.

Second Moment, or Moment of Inertia (I).

	Radius of gyration κ.
Hollow eccentric circle about a line normal to the line joining the two centres, and passing through the c. of g. of the figure.	

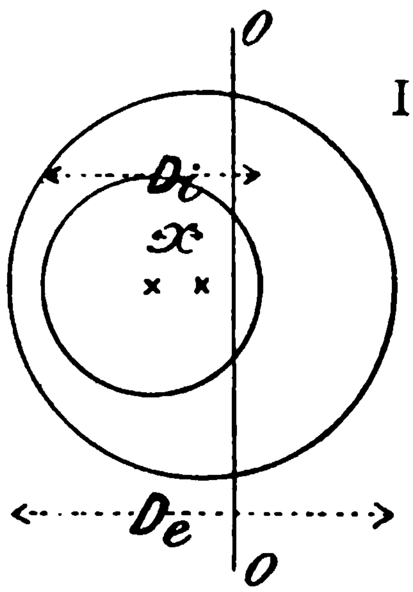


Fig. 110.

$$I = \frac{\pi}{64}\left(D_e^4 - D_i^4 - \frac{16D_e^2D_i^2x^2}{D_e^2 - D_i^2}\right)$$

where x is the eccentricity.

Note.—When the eccentricity is zero, *i.e.* when the outer and inner circles are concentric, the latter term in the above expression vanishes, and the value of I is the same as in the case given above for the hollow circle.

Ellipse about minor axis.

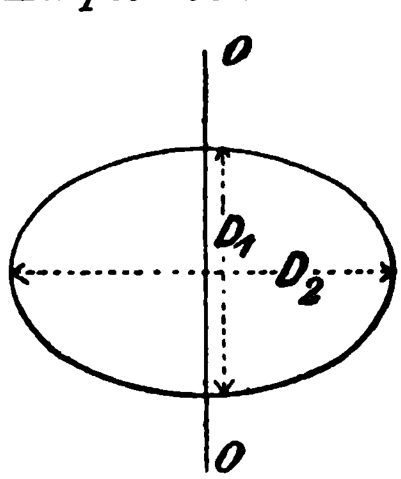


Fig. 111.

$$I = \frac{\pi D_2^3 D_1}{64}$$

Radius of gyration: $\dfrac{D_2}{4}$

The axis OO passes through the c. of g. of the figure, and is at a distance b from the centre of the outer circle, and a from the centre of the inner circle.

From the principle of moments, we have—

$$\frac{\pi}{4}D_i^2(b + x) = \frac{\pi}{4}D_e^2 b$$

$$\text{whence } b = \frac{D_i^2 x}{D_e^2 - D_i^2}$$

$$\text{also } \frac{\pi}{4}D_e^2(a - x) = \frac{\pi}{4}D_i^2 a$$

$$\text{whence } a = \frac{D_e^2 x}{D_e^2 - D_i^2}$$

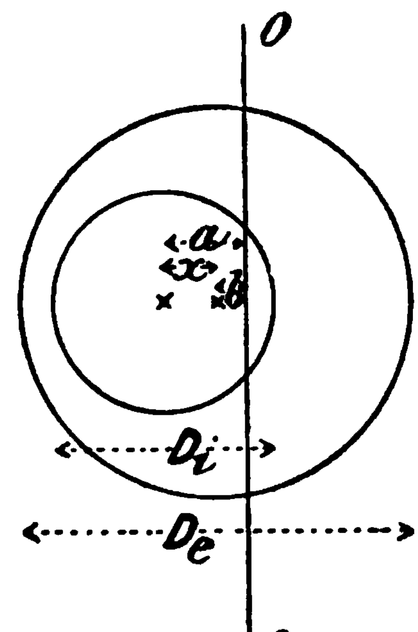


FIG. 110*a*.

From the theorem on p. 76, we have—

$I'_e = I_e + A_e b^2$ for the outer circle about the c. of g. of figure

$$= \frac{\pi D_e^4}{64} + \frac{\pi D_e^2}{4} \times b^2$$

also $I'_i = I_i + A_i a^2$ for the inner circle about the c. of g. of figure

$$= \frac{\pi D_i^4}{64} + \frac{\pi D_i^2}{4} \times a^2$$

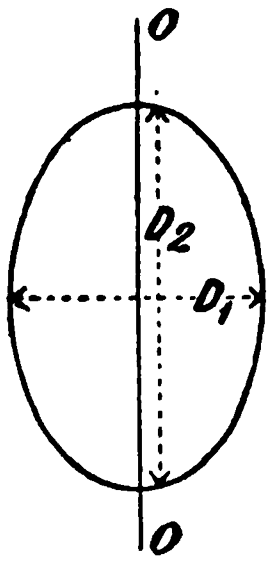
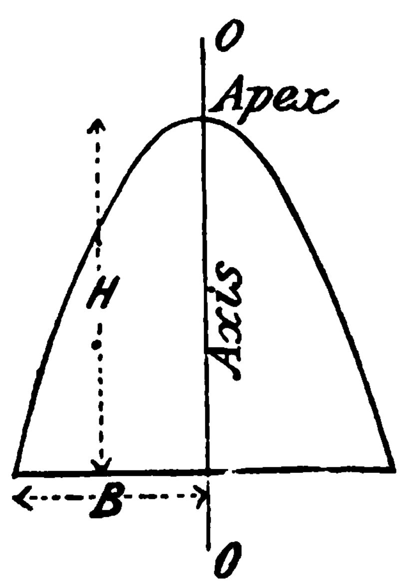
and $I = I'_e - I'_i$ for the whole figure.

Substituting the values given above, and reducing, we get the expression given on the opposite page.

The second moment, or moment of inertia, of a figure varies directly as its breadth taken parallel to the axis of revolution; hence the I for an ellipse about its minor axis is simply the I for a circle of diameter D_2 reduced in the ratio $\frac{D_1}{D_2}$

$$\text{or } \frac{\pi D_2^4}{64} \times \frac{D_1}{D_2} = \frac{\pi D_2^3 D_1}{64}$$

SECOND MOMENT, OR MOMENT OF INERTIA (I).

		Radius of gyration κ.
Ellipse about major axis.  FIG. 112.	$I = \frac{\pi D_1^3 D_2}{64}$	$\frac{D_1}{4}$
Parabola about its axis.  FIG. 113.	$I = \frac{4}{15} HB^3$	$\frac{B}{\sqrt{5}}$

And for an ellipse about its major axis, the I is that for a circle of diameter D_1 increased in the ratio $\dfrac{D}{D_1}$

$$\text{or } \frac{\pi D_1^4}{64} \times \frac{D_2}{D_1} = \frac{\pi D_1^3 D_2}{64}$$

$$h = H\left(1 - \frac{b^2}{B^2}\right) \text{ (see p. 69).}$$

$$\text{area of strip} = h \, . \, db$$

$$\text{second moment of strip} = b^2 \, . \, h \, . \, db$$

$$\text{,,} \quad \text{,,} \quad = b^2 H\left(1 - \frac{b^2}{B^2}\right) db$$

$$\left.\begin{array}{l}\text{second moment of whole}\\ \text{figure}\end{array}\right\} = H \int_0^B \left(b^2 - \frac{b^4}{B^2}\right) db$$

$$\text{,,} \quad \text{,,} \quad = H\left[\frac{b^3}{3} - \frac{b^5}{5B^2}\right]_0^B$$

$$\text{,,} \quad \text{,,} \quad = H\left[\frac{B^3}{3} - \frac{B^3}{5}\right]$$

$$\text{,,} \quad \text{,,} \quad = \frac{2HB^3}{15}$$

$$\left.\begin{array}{l}\text{second moment for double}\\ \text{figure shown on opposite}\\ \text{page}\end{array}\right\} = \frac{4}{15} HB^3$$

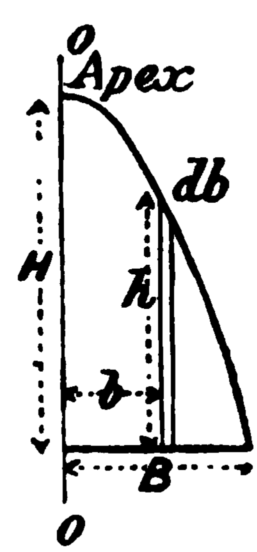
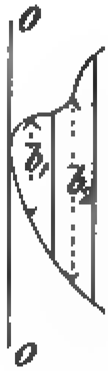


FIG. 113*a*.

Ir



$$b = \frac{B \, . \, h^{\frac{1}{2}}}{H^{\frac{1}{2}}}$$

$$\text{area of strip} = b \, . \, dh$$

$$\left.\begin{array}{c}\text{second moment}\\ \text{of strip}\end{array}\right\} = (H - h)^2 b \, . \, dh$$

$$\text{,, } \quad \text{,, } \quad = \frac{(H - h)^2 B h^{\frac{1}{2}} \, . dh}{H^{\frac{1}{2}}}$$

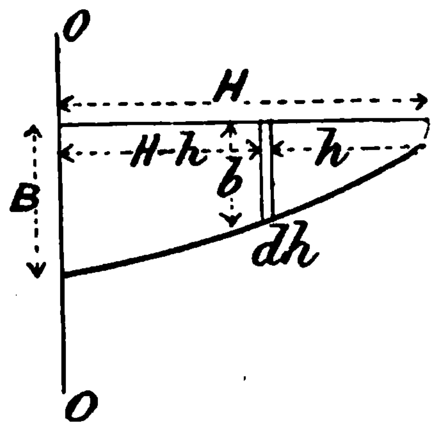


FIG. 114*a*.

$$\left.\begin{array}{l}\text{second moment of whole}\\ \text{figure}\end{array}\right\} = \frac{B}{H^{\frac{1}{2}}} \int_0^H (H^2 h^{\frac{1}{2}} + h^{\frac{5}{2}} - 2h^{\frac{3}{2}} H) dh$$

$$= \frac{B}{H^{\frac{1}{2}}} \left[\frac{2H^2 h^{\frac{3}{2}}}{3} + \frac{2h^{\frac{7}{2}}}{7} - \frac{4h^{\frac{5}{2}} H}{5} \right]_0^H$$

$$= \frac{B}{H^{\frac{1}{2}}} \left(\tfrac{2}{3} H^{\frac{7}{2}} + \tfrac{2}{7} H^{\frac{7}{2}} - \tfrac{4}{5} H^{\frac{7}{2}} \right)$$

$$= B \left(\tfrac{2}{3} H^3 + \tfrac{2}{7} H^3 - \tfrac{4}{5} H^3 \right)$$

$$= \tfrac{16}{105} BH^3$$

$$\left.\begin{array}{l}\text{for the double figure shown}\\ \text{on opposite page}\end{array}\right\} = \tfrac{32}{105} BH^3$$

Divide the figure up as shown in Fig. 87*a*.

Let the areas of the strips be a_1, a_2, a_3, a_4, etc., respectively; and their mean distances from the axis be r_1, r_2, r_3, r_4, etc., respectively.

Then $I_0 = a_1 r_1^2 + a_2 r_2^2 + a_3 r_3^2 +$, etc.

But $a_1 = wb_1$, and $a_2 = wb_2$, and so on

and $r_1 = \frac{w}{2}$, $r_2 = \frac{3w}{2}$, $r_3 = \frac{5w}{2}$, and so on

$$\text{hence } I_0 = w \left\{ b_1 \left(\frac{w}{2} \right)^2 + b_2 \left(\frac{3w}{2} \right)^2 + b_3 \left(\frac{5w}{2} \right)^2 +, \text{ etc.} \right\}$$

$$I_0 = \frac{w^3}{4} (b_1 + 9b_2 + 25b_3 + 49b_4 +, \text{ etc.})$$

$$\text{Also } \kappa^2 = \frac{\frac{w^3}{4}(b_1 + 9b_2 + 25b_3 + 49b_4 +, \text{ etc.})}{w(b_1 + b_2 + b_3 + b_4 +, \text{ etc.}}$$

$$\kappa = \frac{w}{2} \sqrt{\frac{b_1 + 9b_2 + 25b_3 +, \text{ etc.}}{b_1 + b_2 + b_3 +, \text{ etc.}}}$$

This expression should be compared with that obtained for finding the position of the c. of g. on p. 71. The comparison helps one to realize the relation between the first and second moments.

Second Moment, or Moment of Inertia (I).

Graphic method. — Radius of gyration κ.

o

Let A = shaded area;
Y = distance of c. of g. of shaded area from *oo*;
H = extreme dimension of figure measured normal to *oo*;

o

Then $I_o = AYH$

Fig. 116.

Second Polar Moments of Surfaces or Thin Laminæ.

Parallelogram about a pole passing through its c. of g.

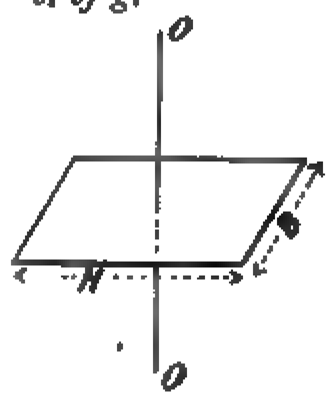


Fig. 117.

$$I_p = \frac{BH}{12}(H^2 + B^2)$$ — Radius of gyration: $\sqrt{\frac{H^2 + B^2}{12}}$

square of side S—

$$I_p = \frac{S^4}{6}$$ — Radius of gyration: $\frac{S}{\sqrt{6}}$

Circle about a pole passing through the c. of g.

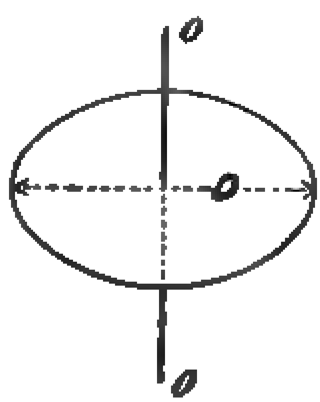


Fig. 118.

$$I_p = \frac{\pi D^4}{32}, \text{ or } \frac{\pi R^4}{2}$$ — Radius of gyration: $\frac{D}{\sqrt{8}}$ or $\frac{R}{\sqrt{2}}$

Divide the figure up into a number of strips, as shown in Fig. 116; project each on to the base-line, *e.g.* ab projected to a_1b_1; join a_1 and b_1 to c, some convenient point on oo, cutting ab in a_0b_0, and so on with the other lines, which when joined up give the boundary of the shaded figure. Find the c. of g. of shaded figure (by cutting out in cardboard and balancing). The principle of this construction is fully explained in Chap. IX., p. 296.

See also Barker's "Graphical Calculus," p. 184.

From the theorem on p. 77, we have—

$$I_p = I_y + I_x = \frac{BH^3}{12} + \frac{HB^3}{12}$$
$$= \frac{BH}{12}(H^2 + B^2)$$

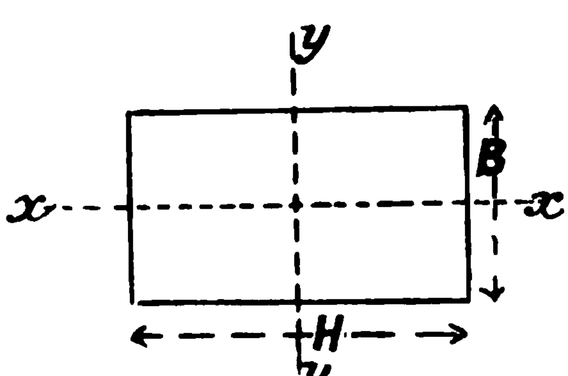


FIG. 117*a*.

Thickness of ring $= dr$

area of ring $= 2\pi r \, . \, dr$

second moment of ring $= 2\pi r \, . \, r^2 \, . \, dr$

" " circle $= 2\pi \int_0^R r^3 \, . \, dr$

" " " $= \frac{2\pi R^4}{4} = \frac{\pi R^4}{2}$

$\left(\text{but } R = \frac{D}{2}\right) \quad = \frac{\pi D^4}{2 \times 16} = \frac{\pi D^4}{32}$

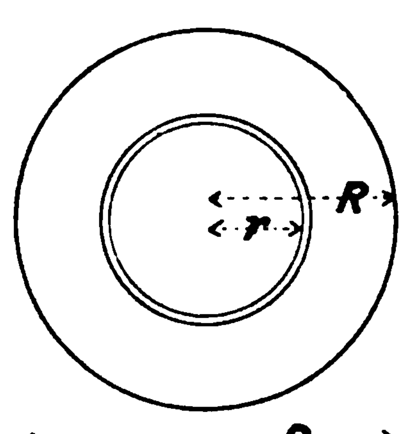


FIG. 118*a*.

SECOND POLAR MOMENT, OR POLAR MOMENT OF INERTIA.

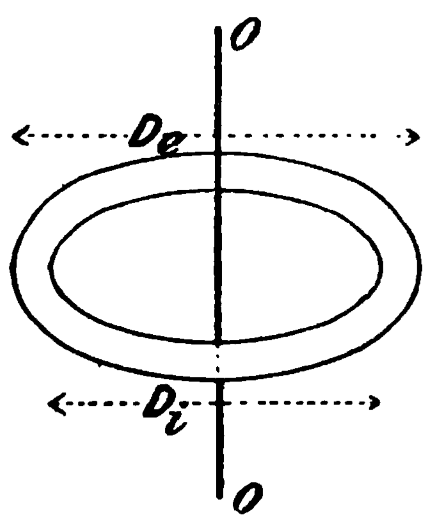


FIG. 119.

Hollow circle about a pole passing through the c. of g.

$$I_p = \frac{\pi}{32}(D_e^4 - D_i^4)$$

Radius of gyration κ.

$$\sqrt{\frac{D_e^2 + D_i^2}{8}}$$

or

$$\sqrt{\frac{R_e^2 + R_i^2}{2}}$$

Second Polar Moments of Solids.

Bar of rectangular section about a pole passing through its c. of g.

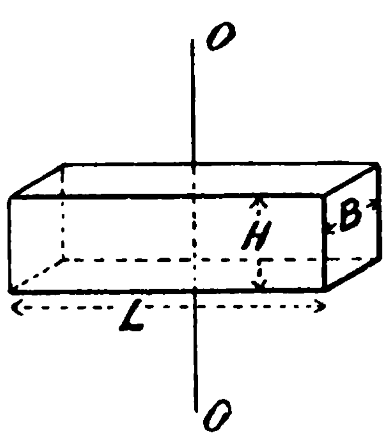


FIG. 120.

$$I_p = \frac{LBH}{12}(L^2 + B^2)$$

$$\sqrt{\frac{L^2 + B^2}{12}}$$

Cylinder about a pole passing through its centre.

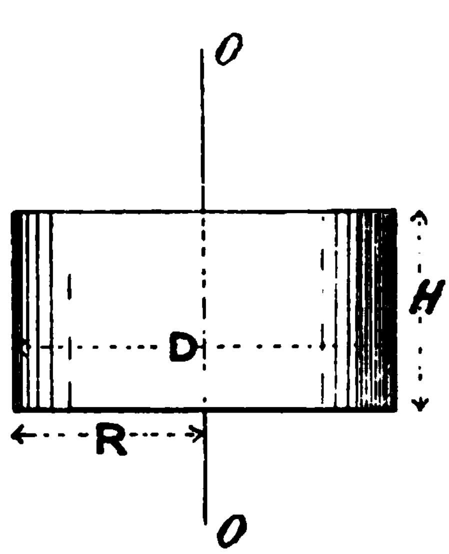


FIG. 121.

$$I_p = \frac{\pi D^4 H}{32}, \text{ or } \frac{\pi R^4 H}{2}$$

$$\frac{D}{\sqrt{8}} \text{ or } \frac{R}{\sqrt{2}}$$

The I_p for the hollow circle is simply the difference between the I_p for the outer and the I_p for the inner circles.

The bar may be regarded as being made up of a great number of thin laminæ of rectangular form, of length L and breadth B, revolving about their polar axis, the radius of gyration of each being $K = \sqrt{\frac{L^2 + B^2}{12}}$ (see Fig. 117), which is the radius of gyration of the bar. The second moment of the bar will then be K^2 (volume of bar), or $\frac{L^2 + B^2}{12} LBH$, or $\frac{LBH}{12}(L^2 + B^2)$

The cylinder may be regarded as being made up of a great number of thin circular laminæ revolving about a pole passing through their centre, the radius of gyration of each being $K = \sqrt{\frac{D^2}{8}}$

The second moment of cylinder $= K^2$ (volume of cylinder)

$$= \frac{D^2}{8} \times \frac{\pi D^2 H}{4} = \frac{\pi D^4 H}{32}$$

Second Polar Moment, or Polar Moment of Inertia.

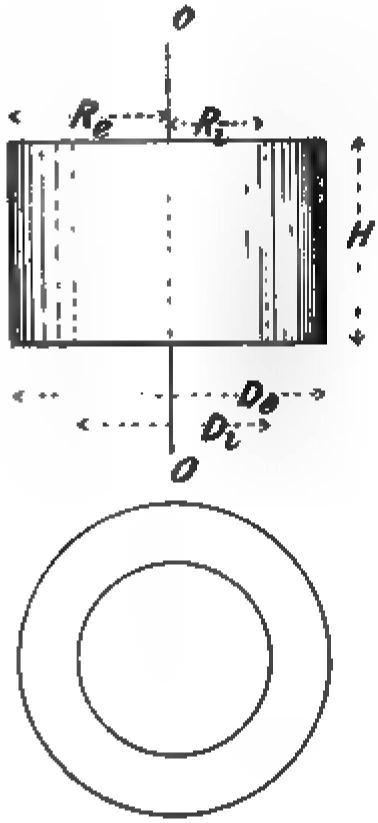


Fig. 122.

Hollow cylinder about a pole passing through its axis.

$$I_p = \frac{\pi H}{32}(D_e^4 - D_i^4),$$

$$\text{or } \frac{\pi H}{2}(R_e^4 - R_i^4)$$

Radius of gyration κ.

$$\sqrt{\frac{D_e^2 + D_i^2}{8}}$$

or

$$\sqrt{\frac{R_e^2 + R_i^2}{2}}$$

between the I_p for the outer and the inner cylinders.

It must be particularly noticed that the radius of gyration of a solid body, such as a cylinder, flywheel, etc., is not the radius of gyration of a plane section; the radius of gyration of a plane section is that of a thin lamina of uniform thickness, while the radius of gyration of a solid is that of a thin wedge. The radius of gyration of a solid may be found by correcting the section in this manner, and finding the I for the shaded figure treated as a plane surface.

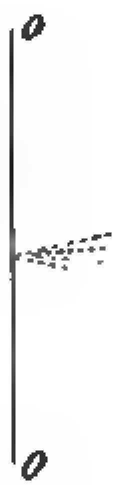


FIG. 1234.

The construction simply reduces the width of the solid section at each point proportional to its distance from *oo*; it is, in fact, the "modulus figure" (see Chap. IX) of the section.

SECOND POLAR MOMENT, OR MOMENT OF INERTIA.

	Radius of gyration κ.
Flywheel with arms.	

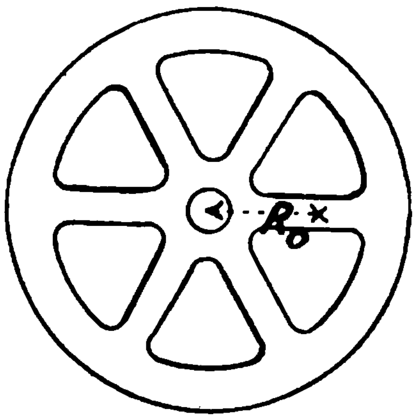


FIG. 124.

Treat the rim and boss separately as hollow cylinders, and each arm thus (assumed parallel)—

For each arm (see Fig. 120)—

$$\frac{\text{L (sectional area)}}{12}(L^2 + B^2 + 12R_0^2)$$

where R_0 = the radius of the c. of g. of the arm.

For most practical purposes the rim only is considered, and the arms and boss neglected.

Sphere about its diameter.

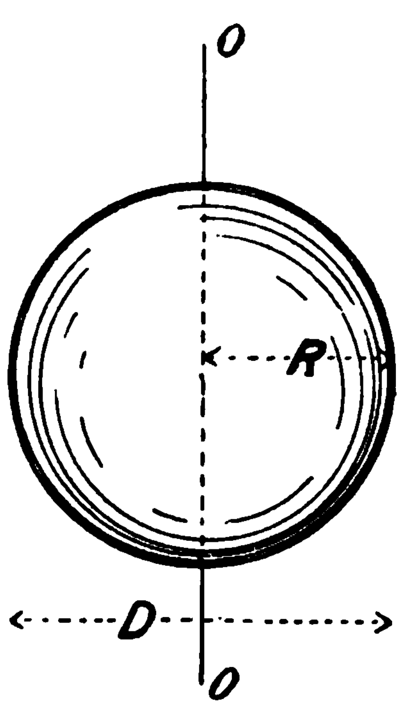


FIG. 125.

$$I_p = \frac{8}{15}\pi R^5, \text{ or } \frac{1}{60}\pi D^5$$

Radius of gyration: $R\sqrt{\frac{2}{5}}$ or $\frac{D}{\sqrt{10}}$

The arms are assumed to be of rectangular section; if they are not, the error involved will be exceedingly small.

The sphere may be regarded as being made up of a great number of thin circular layers of radius r_1, and radius of gyration $\frac{r_1}{\sqrt{2}}$ (see p. 96).

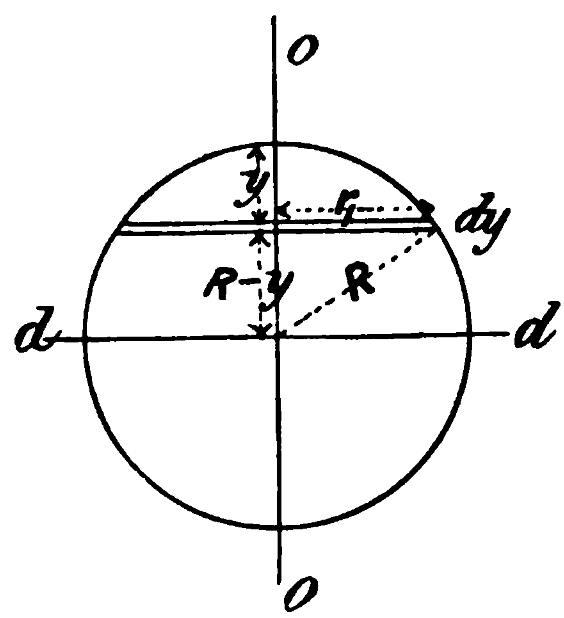


FIG. 125*a*.

$$r_1^2 = R^2 - (R-y)^2 = R^2 - R^2 - y^2 + 2Ry = 2Ry - y^2$$

volume of thin layer $= \pi r_1^2 dy$

second moment of layer about *oo* $\Big\} = \pi r_1^2 dy \times \frac{r_1^2}{2}$

$$= \frac{\pi r_1^4}{2} dy$$

$$= \frac{\pi}{2}(2Ry - y^2)^2 dy$$

$$= \frac{\pi}{2}(4R^2y^2 + y^4 - 4Ry^3)dy$$

second moment of hemisphere, *i.e.* of all layers on one side of the diameter *dd* $\Bigg\} = \frac{\pi}{2}\int_{y=0}^{y=R}(4R^2y^2 + y^4 - 4Ry^3)dy$

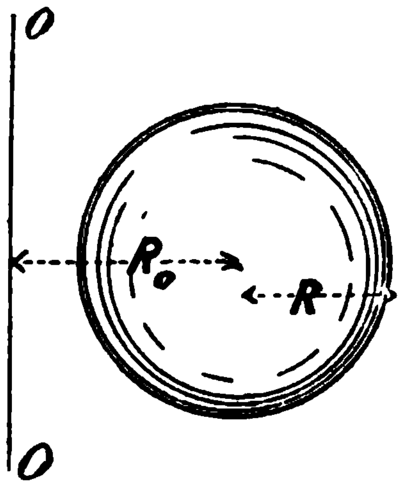
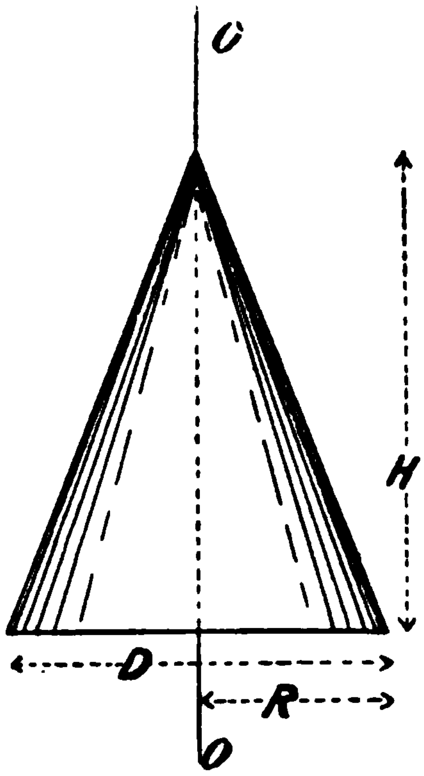
$$= \frac{\pi}{2}\left[\frac{4R^2y^3}{3} + \frac{y^5}{5} - \frac{4Ry^4}{4}\right]_{y=0}^{y=R}$$

$$= \frac{\pi}{2}\left[\frac{4R^5}{3} + \frac{R^5}{5} - \frac{4R^5}{4}\right]$$

$$= \frac{\pi}{2}\left[\frac{8R^5}{15}\right] = \tfrac{4}{15}\pi R^5$$

second moment of sphere $= \tfrac{8}{15}\pi R^5$

SECOND POLAR MOMENT, OR POLAR MOMENT OF INERTIA.

	Radius of gyration κ.
Sphere about an external axis.  FIG. 126. $I_{op} = \frac{4}{3}\pi R^3(\frac{2}{5}R^2 + R_0^2)$ When the axis becomes a tangent, $R_0 = R$; $I_{op} = \frac{28}{15}\pi R^5$	
Cone about its axis.  FIG. 127. $I_p = \frac{\pi}{10}R^4H$, or $\frac{\pi}{160}D^4H$ *i.e.* $\frac{1}{5}$ of the I for the circumscribing cylinder.	$R\sqrt{\frac{3}{10}}$ $D\sqrt{\frac{3}{40}}$

From the theorem on p. 76, we have—

$$I_{op} = I_p + R_0^2V \qquad I_p = \tfrac{8}{15}\pi R^5$$
$$= (\tfrac{8}{15}\pi R^5 + \tfrac{4}{3}\pi R^3R_0^2) \qquad V = \tfrac{4}{3}\pi R^3$$
$$= \tfrac{4}{3}\pi R^3(\tfrac{2}{5}R^2 + R_0^2)$$

The cone may also be regarded as being made up of a great number of thin layers.

Volume of thin layer $= \pi r^2 dh$

$$\left.\begin{array}{l}\text{second moment of}\\ \text{layer about axis } oo\end{array}\right\} = \pi r^2 dh \times \frac{r^2}{2} = \frac{\pi r^4}{2}dh$$

$$\left(\text{but } r = \frac{Rh}{H}\right) \qquad = \frac{\pi R^4 h^4}{2H^4}dh$$

$$\left.\begin{array}{l}\text{second mo-}\\ \text{ment of cone}\end{array}\right\} = \frac{\pi R^4}{2H^4}\int_0^H h^4 . dh = \frac{\pi R^4 H^5}{10H^4}$$

$$= \frac{\pi R^4 H}{10}$$

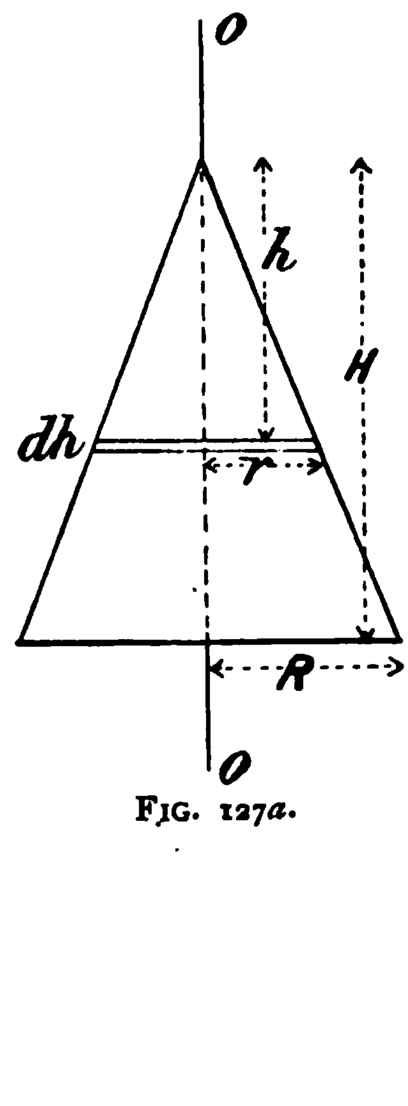


FIG. 127*a*.

CHAPTER IV.

RESOLUTION OF FORCES.

WE have already explained how two forces acting on a point may be replaced by one which will have precisely the same effect on the point as the two. We must now see how to apply the principle involved to more complex systems of forces.

Polygon of Forces.—If we require to find the resultant of more than two forces which act on a point, we can do so by finding the resultant of any two by means of the parallelogram of forces, and then take the resultant of this resultant and the next force, and so on, as shown in the diagram. The resultant of 1 and 2 is marked R1.2., and so on. Then we finally get the resultant R1.2.3.4. for the whole system.

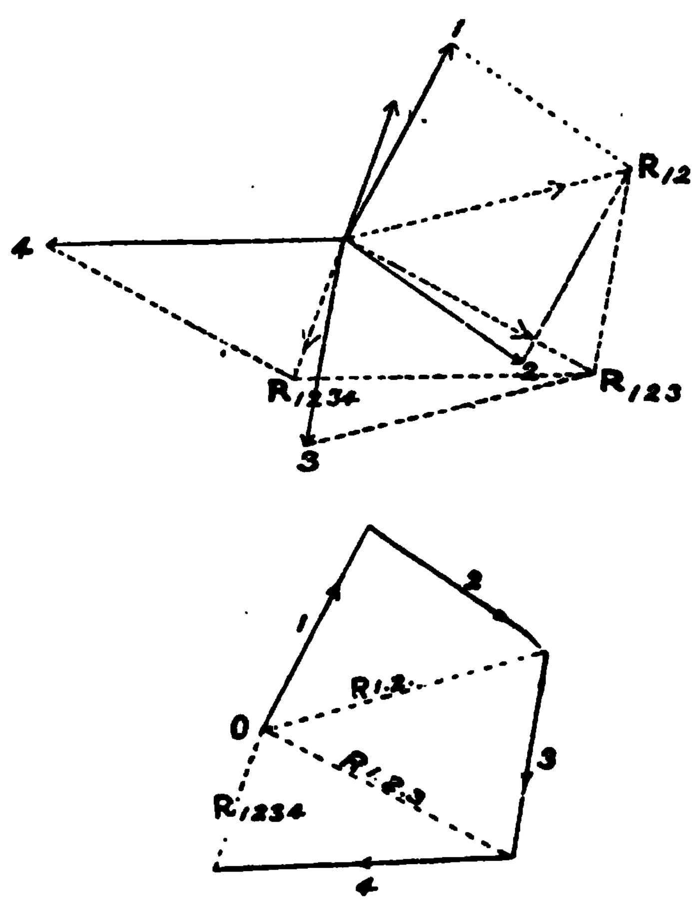


FIG. 128.

Such a method is, however, clumsy. The following will be found much more direct and convenient: Start from any point *o*, and draw the line 1 parallel and equal on a given scale to the force 1; from the extremity of 1 draw the line 2 equal and parallel to the force 2; then, by the triangle of forces, it will be seen that the line R1.2. is the resultant of the forces 1 and 2. From the extremity of 2 draw 3 in a similar manner, and so on with all the forces; then it will be seen that the line R1.2.3.4. represents the resultant of the forces. In using this construction, there is no need to put in the lines R1.2., etc.; in the figure they have been inserted in order to make it

clear. Hence it follows that, if any number of forces act upon a point in such a manner that, if lines be drawn parallel to them, when taken in order they form a closed polygon, the point is in equilibrium under the action of those forces. This is known as the theorem of the polygon of forces.

Method of lettering Force Diagrams.—In order to keep force diagrams clear, it is essential that the forces be lettered in each diagram to prevent confusion. Instead of lettering the force itself, it is very much better to letter the spaces, and to designate the force by the letters corresponding to the spaces on each side, thus: The force separating *a* from *b* is termed the force *ab*; likewise the force separating *d* from *b*, *db*.

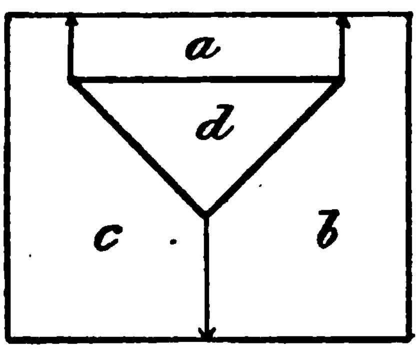


FIG. 129.

This method of notation is usually attributed to Bow; several writers, however, claim to have been the first to use it.

Funicular or Link Polygons.—When forces in equilibrium act at the corners of a series of links jointed together at their extremities, the force acting along each link can be readily found by a special application of the triangle of forces.

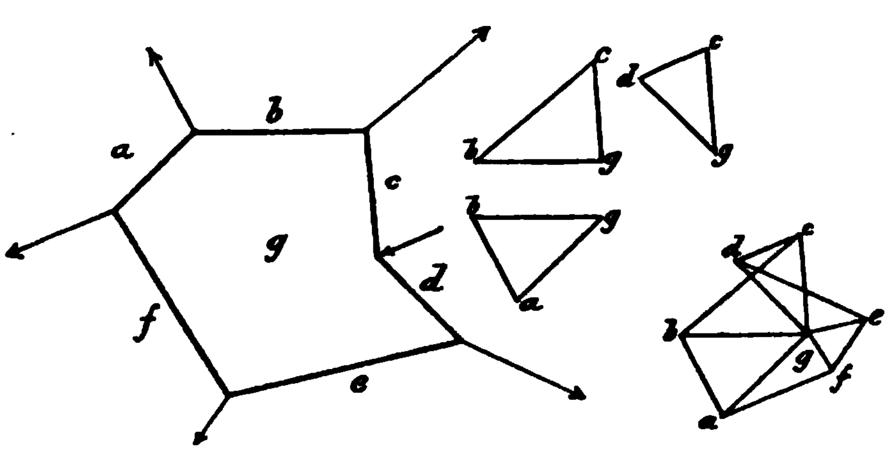


FIG. 130.

Consider the links *ag* and *bg*. There are three forces in equilibrium, viz. *ab*, *ag*, *bg*, acting at the joint. The magnitude of *ab* is known, therefore the magnitude of the other two acting on the links may be obtained from the triangle of forces shown on the right-hand side, viz. *abg*. Similarly consider all the other joints. It will be found that each triangle of forces contains a line equal in every respect to a line in the preceding triangle, hence all the triangles may be brought together to form one diagram, as shown to the extreme right hand. It should be noticed that the external forces form a closed polygon, and the forces in the bars are represented by radial lines meeting in the point or pole *g*.

It will be evident that the form taken up by the polygon depends on the magnitude of the forces acting at each joint.

Suspension Bridge.—Another special application of the triangle of forces in a funicular polygon is that of finding the forces in the chain of a suspension bridge. The platform on which the roadway is carried is supported from the chain by means of vertical ties. We will assume that the weight supported by each tie is known. The force acting on each portion of the chain can be found by constructing a triangle of forces at each joint of a vertical tie to the chain, as shown in the figure above the chain. But *bo* occurs in both triangles; hence the two triangles may be fitted together, *bo* being common to each. Likewise all the triangles of forces for all the joints may be fitted together. Such a figure is shown at the side, and is known as a ray or vector polygon. Instead,

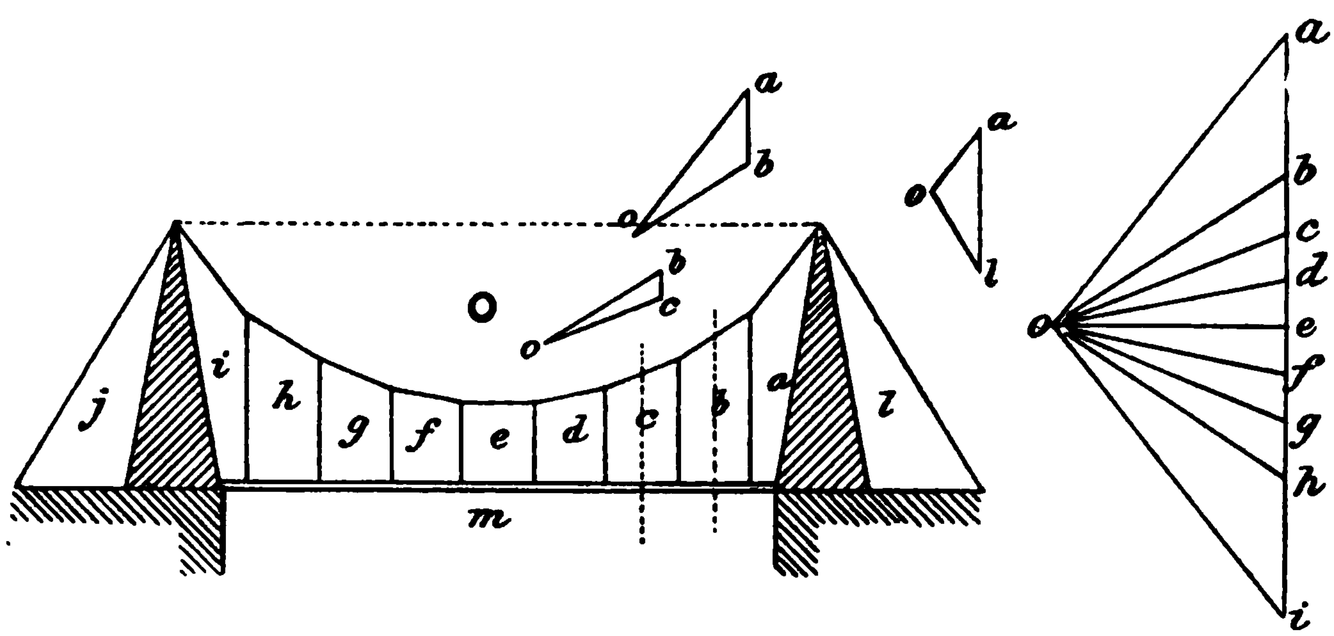


FIG. 131.

however, of constructing each triangle separately and fitting them together, we simply set off all the vertical loads *ab*, *bc*, etc., on a straight line, and from them draw lines parallel to each link of the suspension chain; if correctly drawn, all the rays will meet in a point. The force then acting on each segment of the chain is measured off the vector polygon, to the same scale as the vertical loads. In the figure the vertical loads are drawn to a scale of 1″ = 10 tons; hence, for example, the tension in the segment *ao* is 9·8 tons.

The downward pressure on the piers and the tension in the outer portion of the chain is given by the triangle *aol*.

If a chain (or rope) hangs freely without any platform suspended below, the vertical load will be simply that due to the weight of the chain itself. If the weight per foot of horizontal span were constant, it is easy to show that the curve taken up by the chain is a parabola (see p. 345). In the same chapter, the link and vector polygon construction is employed

to determine the bending moment due to an evenly distributed load. The bending moment Mx at x is there shown to be the depth D_x multiplied by the polar distance OH (Fig. 132); *i.e.*

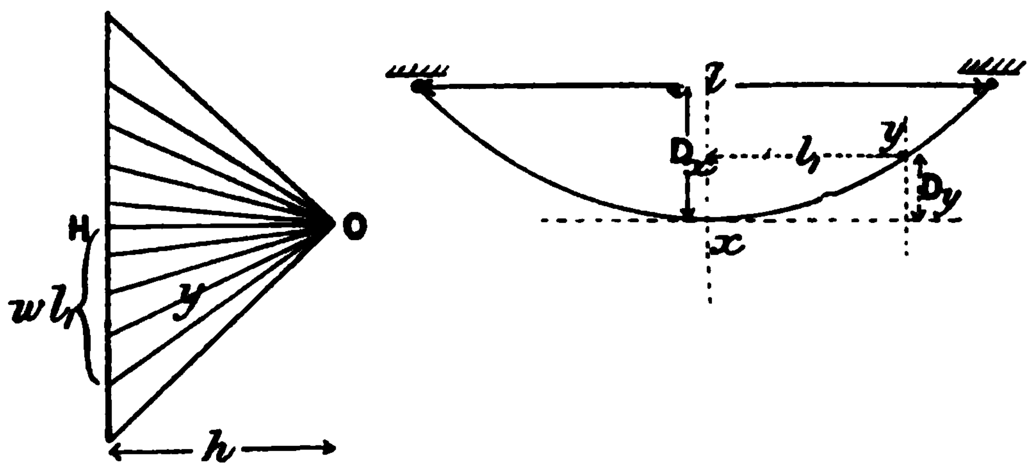


FIG. 132.

the dip of the chain at x multiplied by the horizontal component of the forces acting on the links, viz. the force acting on the link at x, or $Mx = D_x \times OH$. In the same chapter, it is also shown that with an evenly distributed load $Mx = \frac{wl^2}{8}$, where w is the load per foot run, and l is the span in feet.

$$\text{Hence } \frac{wl^2}{8} = D_x \,.\, \overline{OH} = D_x h$$

$$\text{or } h = \frac{wl^2}{8D_x}$$

where h is the tension at the bottom of the chain, viz at x.

It will, however, be seen that the load on a freely hanging chain is not evenly distributed per foot of horizontal run, because the inclination of the chain varies from point to point. Therefore the curve is not parabolic; it is, in reality, a catenary curve. For nearly all practical purposes, however, when the dip is not great compared with the span, it is sufficiently accurate to take the curve as being parabolic.

Then, assuming the curve to be parabolic, the tension at any other point, y, is given by the length of the corresponding line on the vector polygon, which is readily seen to be—

$$T_y = \sqrt{h^2 + w^2 l_1^2}$$

The true value of the tension obtained from the catenary is—

$$T_y = h + wD_y$$

(see Unwin's "Machine Design," p. 421), which will be found to agree closely with the approximate value given above.

Data for Force Polygons.—Sometimes it is impossible to construct a polygon of forces on account of the incompleteness of the data.

In the case of the triangle and polygon of forces, the following data must be given in order that the triangle or polygon can be constructed. If there are n conditions in the completed polygon, $n - 2$ conditions must be given; thus, in the triangle of forces there are six conditions, three magnitudes and three directions: then at least four must be supplied before the triangle can be constructed, such as—

3 magnitude(s) and 1 direction(s)
2 ,, ,, 2 ,,
1 ,, ,, 3 ,,

Likewise in a five-sided polygon, there are ten conditions, eight of which must be known before the polygon can be constructed. When the two unknown conditions refer to the same or adjacent sides, the construction is perfectly simple, but when the unknown conditions refer to non-adjacent sides, a special construction is necessary. Thus, for example, suppose, when dealing with five forces, the forces 1, 2, and 4 are completely known, but only the directions, not the magnitudes, of 3 and 5 are known. We proceed thus:

Draw lines 1 and 2 in the polygon of forces, Fig. 133, in the usual way. From the extremity of 2 draw a line of indefinite

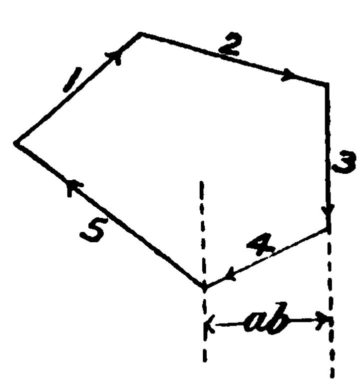


FIG. 133.

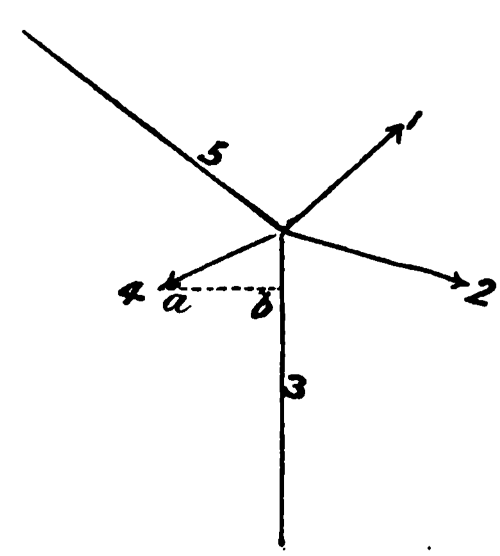


FIG. 134.

length parallel to the force 3; its length cannot yet be fixed, because we do not know its value. From the origin of 1 draw a line of indefinite length parallel to 5; its length is also not yet known. From the extremity of 4 in the diagram of forces, Fig. 134, drop a perpendicular *ab* on to 3, and in the polygon of forces, Fig. 133, draw a line parallel to 3, at a distance *ab*

from it. The point where this line cuts the line 5 is the extremity of 5. From this point draw a line parallel to 4; then by construction it will be seen that its extremity falls on the line 3, giving us the length of 3.

This method will frequently be found to be of great convenience.

Forces in the Members of a Jib Crane. CASE I. *The weight* W *simply suspended from the end of the jib.*—There is no need to construct a separate diagram of forces. Set off bc = W, or BC on some convenient scale,[1] and draw ca parallel to the tie CA; then the triangle bac is the triangle of forces acting on the point b. On measuring the force diagram, we find there is a compressive force of 15·2 tons along AB, and a tension force of 9·8 tons along AC.

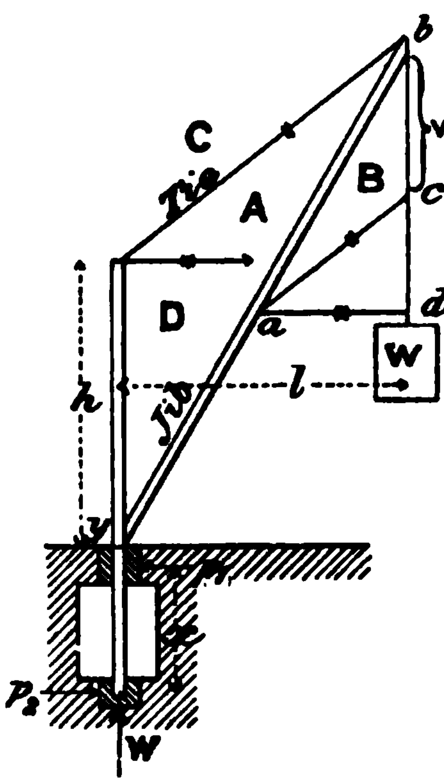


FIG. 135.

The pressure on the bottom pivot is W (neglecting the weight of the crane itself). The horizontal pull at the top of the crane-post is ad, or 7·9 tons; and the force (tension) acting on the post between the junction of the jib and the tie is cd, or 6 tons.

The bending moment at y will be $ad \times h$, or $W \times l$. For determining the bending stress at y, see Chap. IX.

Taking moments about the pivot bearing, we have—

$$p_1 = \frac{\overline{ad}(h + x)}{x}$$

Taking moments about the upper bearing, we have—

$$p_2 = \frac{\overline{ad} \, . \, h}{x}$$

The sections of the various parts of the structure must be determined by methods to be described later on.

The weight of the structure itself should be taken into account, which can only be arrived at by a process of approximation; the dimensions and weight may be roughly arrived at by neglecting the weight of the structure in the first instance. Then, as the centre of gravity of each portion will be approximately at the middle of each length, the load W must be increased to

[1] In this case the scale is 0·1 inch = 2 tons, and W = 7 tons.

W + ½(weight of jib and tie). The downward pressure on the pivot will be W + weight of structure.

The dimensions of the structure must then be increased accordingly. In a large structure the forces should be again determined, to allow for the increased dimensions.

The bending moment on the crane-post at *y* may be very much reduced by placing a balance weight W_1 on the crane, as shown. The forces acting on the balance-weight members are found in a similar manner to that described above, and, neglecting the weight of the structure, are found to be 8·1 tons on the tie, and 4·4 tons on the horizontal strut.

The balance weight produces a compression in the upper part of the post of 6·8 tons; but, due to the tie *ac*, we had a tension of 6·0 tons, therefore there is a compression of 0·8 ton

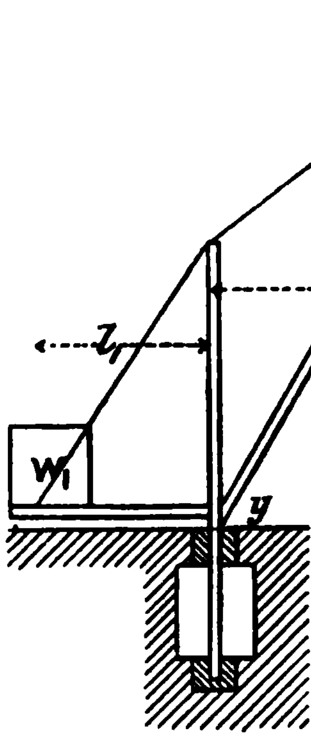


FIG. 136.

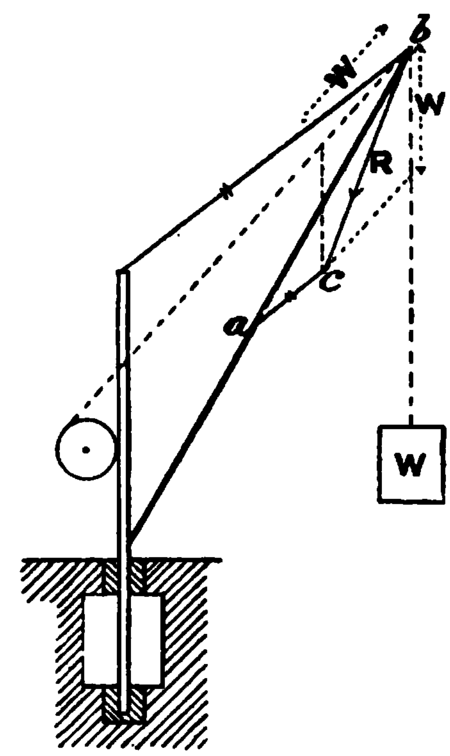


FIG. 137.

in the upper part of the crane-post. The pressure on the lower part of the crane-post and pivot is W + W_1 + weight of structure.

Then, neglecting the weight of the structure, the bending moment on the post at *y* will be—

$$Wl - W_1l_1$$

The moment W_1l_1 should be made equal to $\frac{Wl}{2}$, then the post will never be subjected to a bending moment of more than one-half that due to the lifted load, and the pressures p_1 and p_2 will be correspondingly reduced.

CASE II. *The weight* W *suspended from a chain passing to a barrel on the crane-post.*—As both portions of the chain are

subjected to a pull W, the resultant R is readily determined. From *c* a line *ac* is drawn parallel to the tie; then the force acting down the jib is $ab = 16{\cdot}4$ tons; down the tie $ac = 4{\cdot}4$ tons. The bending moments on the post, etc., are determined in precisely the same manner as in Case I.

When pulley blocks are used for lifting the load, the pull in the chain between the jib pulley and the barrel will be less than W in the proportion of the velocity ratio.

The general effect of the pull on the chain is to increase the thrust on the jib, and to reduce the tension in the tie. In designing a crane, the members should be made strong enough to resist the greater of the two, as it is quite possible that a link of the chain may catch in the jib pulley, and the conditions of Case I. be realized.

Forces in the Members of a Sheer Legs.—In the type of crane known as sheer legs the crane-post is dispensed with, and lateral stability is given by using two jibs or sheer legs spread out at the foot; the tie is usually brought down to the level of the ground, and is attached to a nut working in guides. By means of a horizontal screw, the sheer legs can be tilted or "derricked" at will: the end thrust on the screw is taken by a thrust block; the upward pull on the nut and guides is taken by bolts passing down to massive foundations below. The forces are readily determined by the triangle of forces.

The line *bc* is drawn parallel to the tie, and represents the force acting on it; then *ac* represents the force acting down the middle line of the two sheer legs. This is shown more clearly on the projected view of the sheer legs. *cd* is then drawn parallel to the sheer leg *ae*; then *dc* represents the force acting down the sheer leg *ae*; likewise *ad* down the leg *af*, and

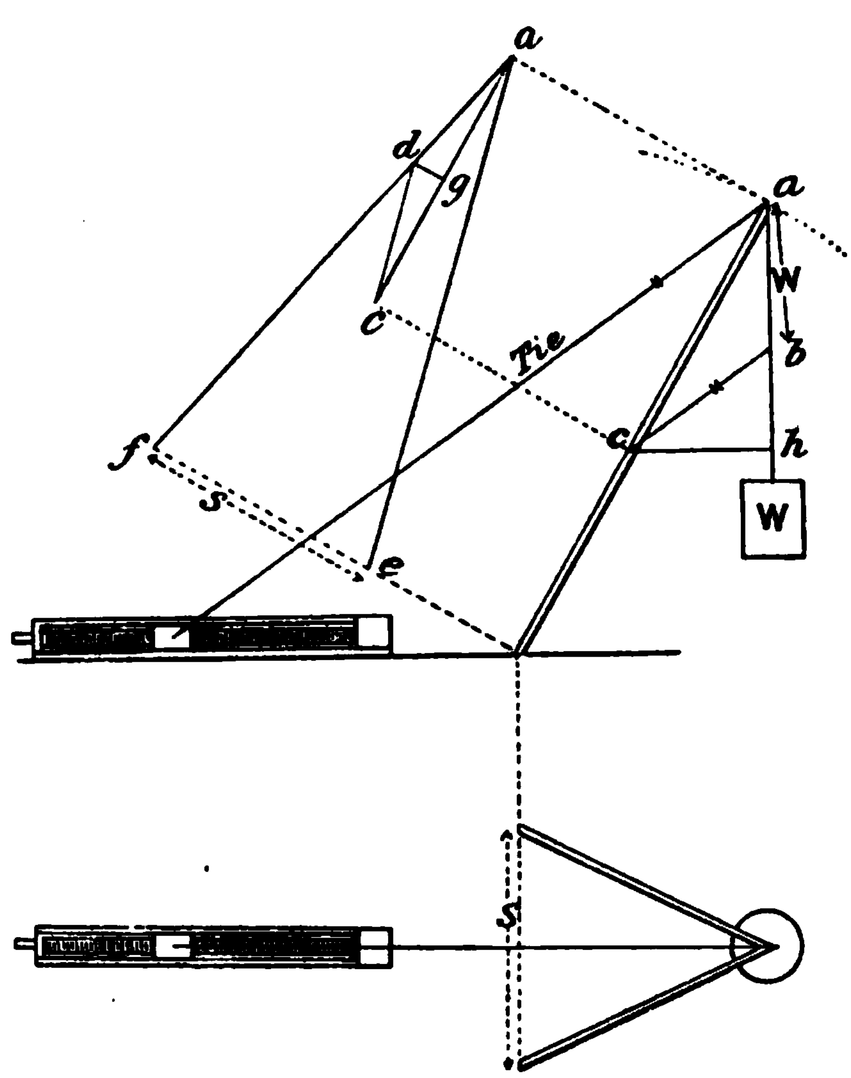


FIG. 138.

dg the force acting at the bottom of the sheer legs tending to make them spread; *ch* represents the thrust of the screw on the thrust block and the force on the screw, and *bh* the upward pull which has to be resisted by the nut guides and the foundation bolts.

The members of this type of structure are necessarily very heavy and long, consequently the bending stress due to

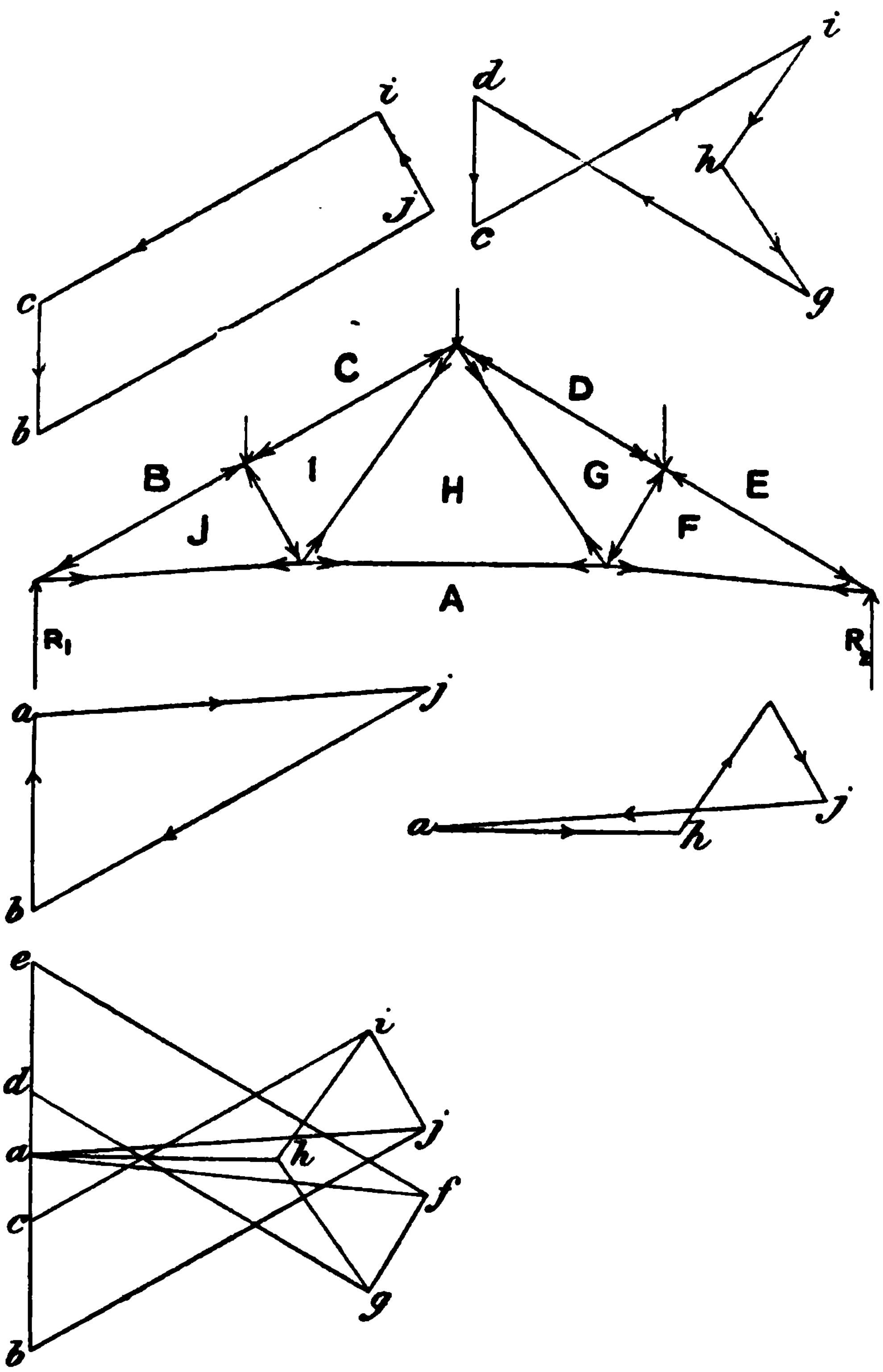


FIG. 139.

their own weight is very considerable, and has to be carefully considered in the design. The problem of combined bending and compression is dealt with in Chapter XII.

Forces in the Members of a Roof Truss.—Let the roof truss be loaded with equal weights at the joints, as shown;

the reactions at each support will be each equal to half the total load on the structure. We shall for the present neglect the weight of the structure itself.

The forces acting on each member can be readily found by a special application of the polygon of forces.

Consider the point at the left-hand support BJA or R_1. We have three forces meeting at a point; the magnitude of one, viz. R_1 or *ba*, and the direction of all are known, hence we can determine the other two magnitudes by the triangle of forces. This we have done in the triangle *ajb*.

Consider the joint BJIC. Here we have four forces meeting at a point; the magnitude of one is given, viz. *bc*, and the direction of all the others; but this is not sufficient—we must have at least six conditions known (see p. 110). On referring back to the triangle of forces just constructed, we find that the force *bj* is known; hence we can proceed to draw our polygon of forces *cbji* by taking the length of *bj* from the triangle previously constructed. By proceeding in a similar manner with every joint, we can determine all the forces acting on the structure.

On examination, we find that each polygon contains one side which has occurred in the previous polygon; hence, if these similar and equal sides be brought together, each polygon can be tacked on to the last, and so made to form one figure containing all the sides. Such a figure is shown below the structure, and is known as a "reciprocal diagram."

When determining the forces acting on the various members of a structure, we invariably use the reciprocal diagram without going through the construction of the separate polygons. We have only done so in this case in order to show that the reciprocal diagram is nothing more nor less than the polygon of forces.

We must now determine the nature of the forces, whether tensile or compressive, acting on the various members. In order to do this, we shall put arrows on the bars to indicate the *direction in which the bars resist the external forces.*

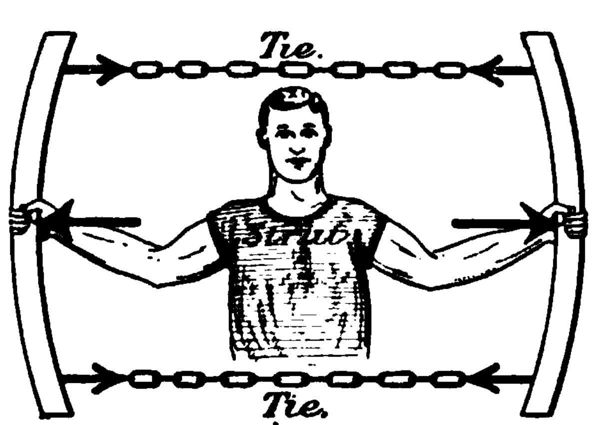


FIG. 140.

The illustration represents a man's arm stretched out, resisting certain forces. The arrows indicate the direction in which he is exerting himself, from which it will be seen that when the arrows on his arms point outwards his arms are in compression, and

when in the reverse direction, as in the chains, they are in tension; hence, when we ascertain the directions in which a bar is resisting the external forces acting on it, we can at once say whether the bar is in tension or compression, or, in other words, whether it is a tie or a strut.

We know, from the triangle and polygon of forces, that the arrows indicating the directions in which the forces act follow round in the same rotary direction; hence, knowing the direction of one of the forces in the polygon, we can immediately find the direction of the others. Thus at the joint BJA we know that the arrow points upwards from *b* to *a*; then, continuing round the triangle, we get the arrow-heads as shown. Transfer these arrows to the bars themselves *at the joint in question;* then, if an arrow points outwards at one end of a bar, the arrow at the other end must also point outwards; hence we can at once put in the arrow at the other end of the bar, and determine whether it is a strut or tie. When the arrows point outwards the bar is a strut, and when inwards a tie. Each separate polygon has been thus treated, and the arrow-heads transferred to the structure. But arrow-heads must *not* be put on the reciprocal diagram; if they are they will cause hopeless confusion. With a very little practice, however, one can run round the various sections of the reciprocal diagram by eye, and put the arrow-heads on the structure without making a single mark on the diagram. If a mistake has been made anywhere, it is certain to be detected before all the bars have been marked. If the beginner experiences any difficulty, he should make separate rough sketches for each polygon of forces, and mark the arrow-heads on each side. At some joints, where there are no external forces, the direction of the arrows will not be evident at first; they must not be taken from other polygons, but from the arrow-heads on the structure itself *at the joint in question.* For example, the arrows at the joints ABJ and BJIC are perfectly readily obtained, the direction being started by the forces AB and BC, but at the joint JIA the direction of the arrow on the bars JI and JA are known *at the joint;* either of these gives the direction for starting round the polygon *ahij.*

The following bars are struts: BJ, IC, GD, FE, JI, GF.

The following bars are ties: JA, IH, HA, HG, FA.

Some more examples of reciprocal diagrams will be given in the chapter on "Framework Structures."

CHAPTER V.

MECHANISMS.

PROFESSOR KENNEDY[1] defines a machine as "a combination of resistant bodies, whose relative motions are completely constrained, and by means of which the natural energies at our disposal may be transformed into any special form of work." Whereas a mechanism consists of a combination of simple links, arranged so as to give the same relative motions as the machine, but not necessarily possessing the resistant qualities of the machine parts; thus a mechanism may be regarded as a skeleton form of a machine.

Constrained and Free Motion.—Motion may be either constrained or free. A body which is free to move in any direction relatively to another body is said to have *free* motion, but a body which is constrained to move in a definite path is said to have *constrained* motion. Of course in both cases the body moves in the direction of the resultant of all the forces acting upon it, but in the latter case, if any of the forces do not act in the direction of the desired path, they automatically bring into play constraining forces in the shape of stresses in the machine parts. Thus, in the figure, let *ab* be a crank which revolves about *a*, and let the force *bc* in the direction of the connecting-rod act on the pin at *b*. Then, if *b* were free, it would move off in the direction of the dotted line, but as *b* must move in a circular path, a force must act along the crank in order to prevent it following the dotted line. This force acting along the crank is readily found by resolving *bc* in

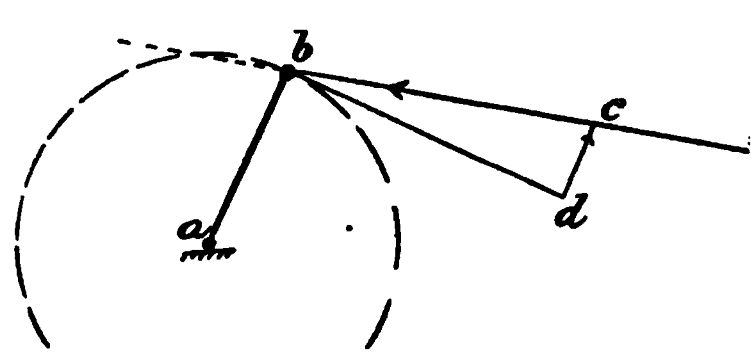


FIG. 141.

[1] "Mechanics of Machinery," p. 2.

a direction normal to the crank, viz. *bd*, *i.e.* in the direction in which *b* is moving, and along the crank, viz. *dc*, which in this instance is a compression. Hence the path of *b* is determined by the force acting along the connecting-rod and the force acting along the crank.

The constraining forces always have to be supplied by the parts of the machine itself. Machine design consists in arranging suitable materials in suitable form to supply these constraining forces.

The various forms of constrained motion we shall now consider.

Plane Motion.—When a body moves in such a manner that any point of it continues to move in one plane, such as in revolving shafts, wheels, connecting-rods, cross-heads, links, etc., such motion is known as plane motion. In plane motion a body may have either a motion of translation in any direction in a given plane or a motion of rotation about an axis.

Screw Motion.—When a body has both a motion of rotation and a translation perpendicular to the plane of rotation, a point on its surface is said to have a *screw* motion, and when the velocity of the rotation and translation are kept constant, the point is said to describe a helix, and the amount of translation corresponding to one complete rotation is termed the *pitch* of the helix or screw.

Spheric Motion.—When a body moves in such a manner that every point in it remains at a constant distance from a fixed point, such as when a body slides about on the surface of a sphere, the motion is said to be spheric. When the sphere becomes infinitely great, spheric motion becomes plane motion.

Relative Motion.—When we speak of a body being in motion, we mean that it is shifting its position relatively to some other body. This, indeed, is the only conception we can have of motion. Generally we speak of bodies as being in motion relatively to the earth, and, although the earth is going through a very complex series of movements, it in nowise affects our using it as a standard to which to refer the motions of bodies; it is evident that the relative motion of two bodies is not affected by any motions which they may have in common. Thus, when two bodies have a common motion, and at the same time are moving relatively to one another, we may treat the one as being stationary, and the other as moving relatively to it; that is to say, we may subtract their common

motion from each, and then regard the one as being at rest. Similarly, we may add a common motion to two moving bodies without affecting their relative motion. We shall find that such a treatment will be a great convenience in solving many problems in which we have two bodies, both of which are moving relatively to one another and to a third. As an example of this, suppose we are studying the action of a valve gear on a marine engine; it is a perfectly simple matter to construct a diagram showing the relative positions of the valve and piston. Precisely the same relations will hold, as regards the valve and piston, whether the ship be moving forwards or backwards, or rolling. In this case we, in effect, add or subtract the motion of the ship to both the motion of the valve and the piston.

Velocity.—Our remarks in the above paragraph, as regards relative motion, hold equally well for relative velocity.

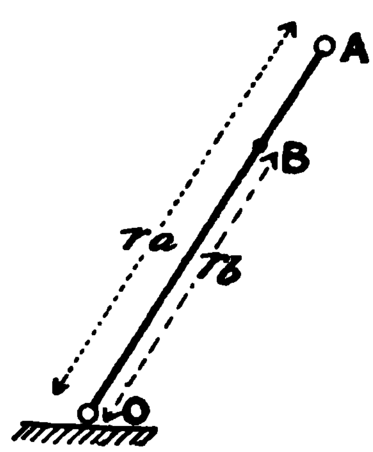


FIG. 142.

Many problems in mechanisms resolve themselves into finding the velocity of one part of a mechanism relatively to that of another. The method to be adopted will depend upon the very simple principle that the linear velocity of any point in a rotating body varies directly as the distance of that point from the axis or centre of rotation. Thus, when the link OA rotates about O, we have—

$$\frac{\text{velocity of A}}{\text{velocity of B}} = \frac{V_a}{V_b} = \frac{r_a}{r_b}$$

If the link be rotating with an angular velocity ω radians per second (see p. 4), then the linear velocity of a, viz. $V_a = \omega r_a$, and of b, $V_b = \omega r_b$, but the angular velocity of every point in the link is the same.

As the link rotates, every point in it moves at any given instant in a direction normal to the line drawn to the centre of rotation, hence at each instant the point is moving in the direction of the tangent to the path of the point, and the centre about which the point is rotating lies on a line drawn normal to the tangent of the curve at that point. This property will enable us to find the centre about which a body having plane motion is rotating. The plane motion of a body is completely known when we know the motion of any two points in the body. If the paths of the points be circular and concentric, then the centre of rotation will be the same for all positions of

the body. Such a centre is termed a "permanent" or "fixed" centre; but when the centre shifts as the body shifts, its centre at any given instant is termed its "instantaneous" or "virtual" centre.

Instantaneous or Virtual Centre.—Complex plane motions of a body can always be reduced to one very simply expressed by utilizing the principle of the virtual centre. For example, let the link *ab* be part of a mechanism having a complex motion. The paths of the two end points, *a* and *b*, are known, and are shown dotted. In order to find the relative velocities of the two points, we draw tangents to the paths at *a* and *b*, which gives us the directions in which each is moving at the instant. From each point draw a normal to the tangents *aa'* and *bb'*, then the centre about which *a* is moving at the instant lies somewhere on the line *aa'*, likewise with *bb'*; hence the centre about which both points are revolving at the instant, must be at the intersection of the two lines, viz. at O. This

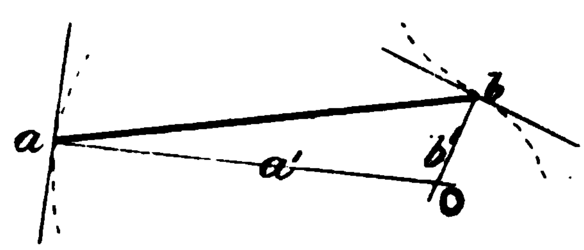


Fig. 143.

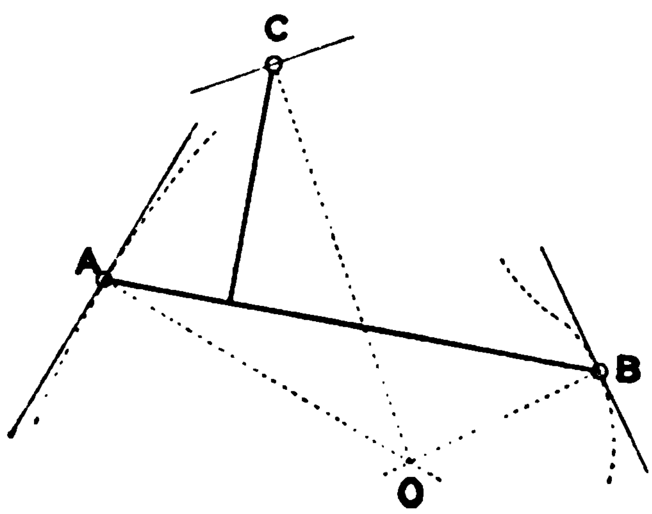


Fig. 144.

point is termed the *virtual* or *instantaneous* centre, and the whole motion of the link at the instant is the same as if it were attached by rods to the centre O. As the link has thickness normal to the plane of the paper, it would be more correct to speak of O as the plan of the virtual *axis*. If the bar had an arm projecting as shown in Fig. 144, the path of the point C could easily be determined, for every point in the body, at the instant, is describing an arc of a circle round the centre O; thus, in order to determine the path of the point C, all we have to do is to describe a small arc of a circle passing through C, struck from the centre O with the radius OC.

The radii OA, OB, OC are known as the virtual radii of the several points.

If the tangents to the point-paths at A and B had been parallel, the radii would not meet, except at infinity. In that

case, the points may be considered to be describing arcs of circles of infinite radius, *i.e.* their point-paths are straight parallel lines.

If the link AB had yet another arm projecting as shown in the figure, the end point of which coincided with the virtual centre O, it would, at the instant, have no motion at all relatively to the plane, *i.e.* it is a fixed point. Hence there is no reason why we should not regard the virtual centre as a point in the moving body itself.

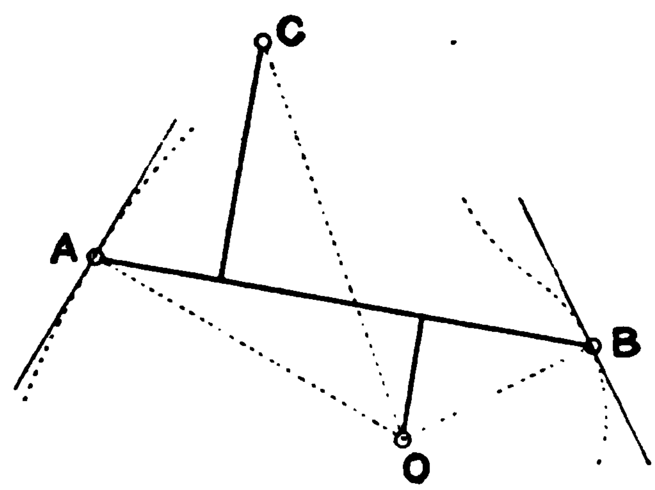


FIG. 145.

It is evident that there cannot be more than one of such fixed points, or the bar as a whole would be fixed, and then it could not rotate about the centre O.

It is clear, from what we have said on relative motion, that if we fixed the bar, which we will term *m* (Fig. 146), and move the plane, which we will term *n*, the relative motion of the two would be precisely the same. We shall term the virtual centre of the bar *m* relatively to the plane *n*, O*mn*.

Centrode and Axode.—As the link *m* moves in such a manner that its end joints *a* and *b* follow the point-paths, the virtual centre O*mn* also shifts relatively to the plane, and traces out the curve as shown in Fig. 146. This curve is simply the point-path of the virtual centre, or the virtual axis. This curve is known as the *centrode*, or *axode*.

Now if we fix the link *m*, and move the plane *n* relatively to it, we shall, at any instant, obtain the same relative motion, therefore the position of the virtual centre will be the same in both cases. The centrodes, however, will not be the same, but as they have one point in common, viz. the virtual centre, they will always touch at this point, and as the motions of the two bodies continue, the two centrodes will roll on one another.

This rolling action can be very clearly seen in the simple four-bar mechanism shown in Fig. 147. The point A moves in the arc of a circle struck from the centre D, hence AD is normal to the tangent to the point-path of A; hence the virtual centre lies somewhere on the line AD. For a similar reason, it lies somewhere on the line BC; the only point common to the two is their intersection O, which is therefore their virtual centre. If the virtual centre, *i.e.* the intersection of the two

bars, be found for several positions of the mechanism, the centrodes will be found to be ellipses.

As the mechanism revolves, the two ellipses will be found to

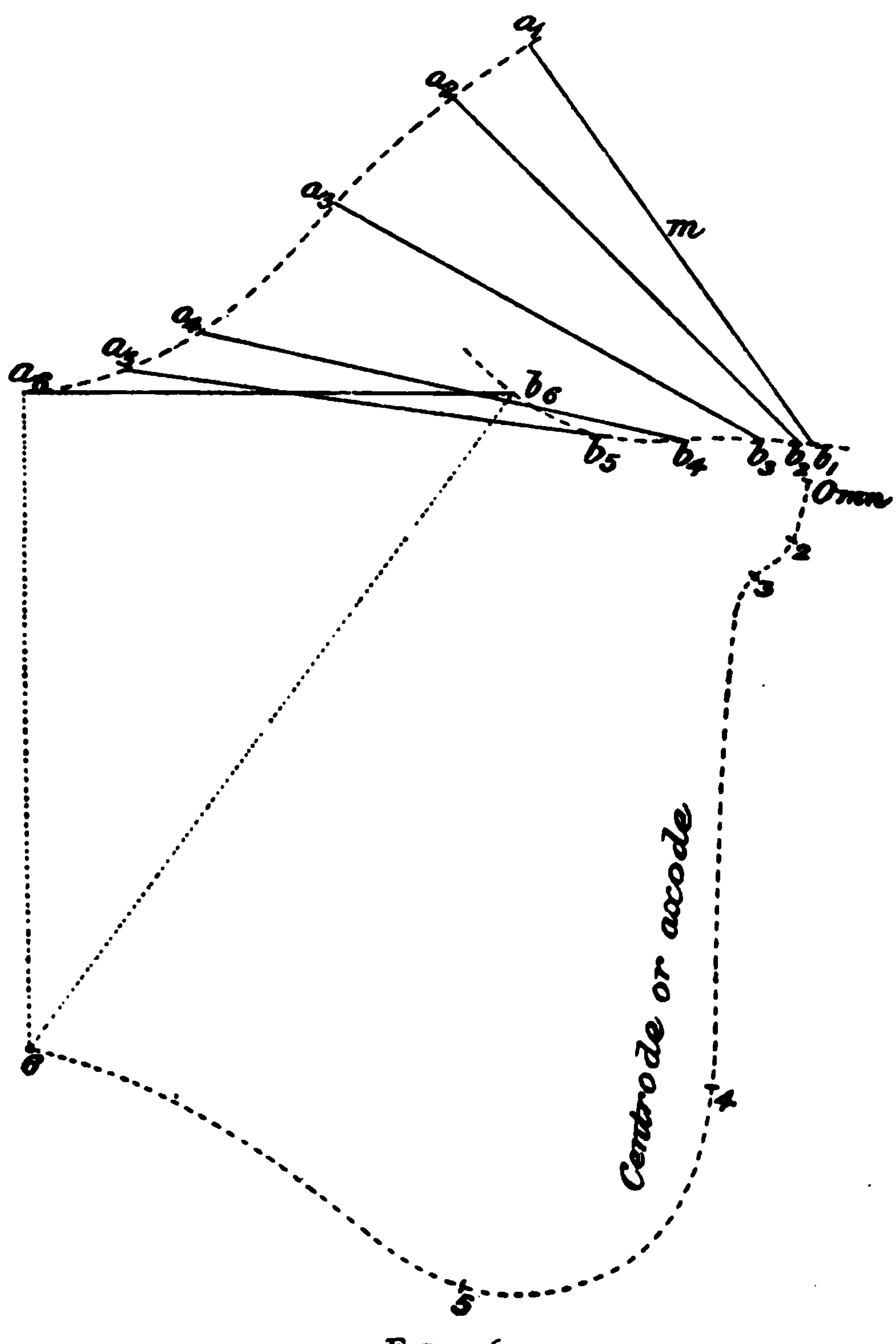


FIG. 146.

roll on one another. That such is the case can easily be proved experimentally, by a model consisting of two ellipses cut out of suitable material and joined by cross-bars AD and BC; it will be found that they will roll on one another perfectly.

Hence we see that, if we have given a pair of centrodes for two bodies, we can, by making the one centrode roll on the other, completely determine the relative motion of the two bodies.

Position of Virtual Centre.—We have shown above that when two point-paths of any body are known, we can

readily find the position of the virtual centre. In the case of most mechanisms, however, we can determine the virtual centres without first constructing the point-paths. We will show this by taking one or two simple cases. In the four-bar mechanism shown in Fig. 148, it is evident that if we consider *d* as stationary, the virtual centre O*ad* will be at the joint of *a* and *d*, and the velocity of any point in *a* relatively to any point in *d* will be proportional to the distance from this joint; likewise with O*dc*. Then, if we consider *b* as fixed, the virtual centre of *a* and *b* will also be at their joint. By similar reasoning, we have the virtual centre O*bc*. Again, let *d* be fixed, and consider the motion of *b* relatively to *d*. The point-path of one end of *b*, viz. O*ab*, describes the arc of a circle about O*ad*, therefore the virtual centre lies on *a* produced; for a similar reason, the virtual centre lies on *c* produced, hence it must be at O*bd*, the meet of the two lines.

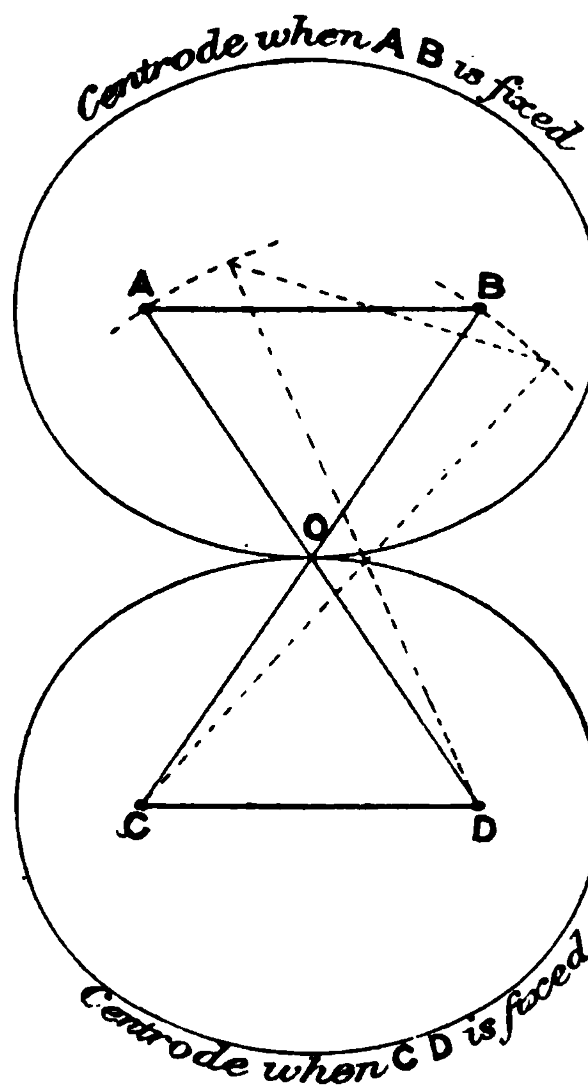


FIG. 147.

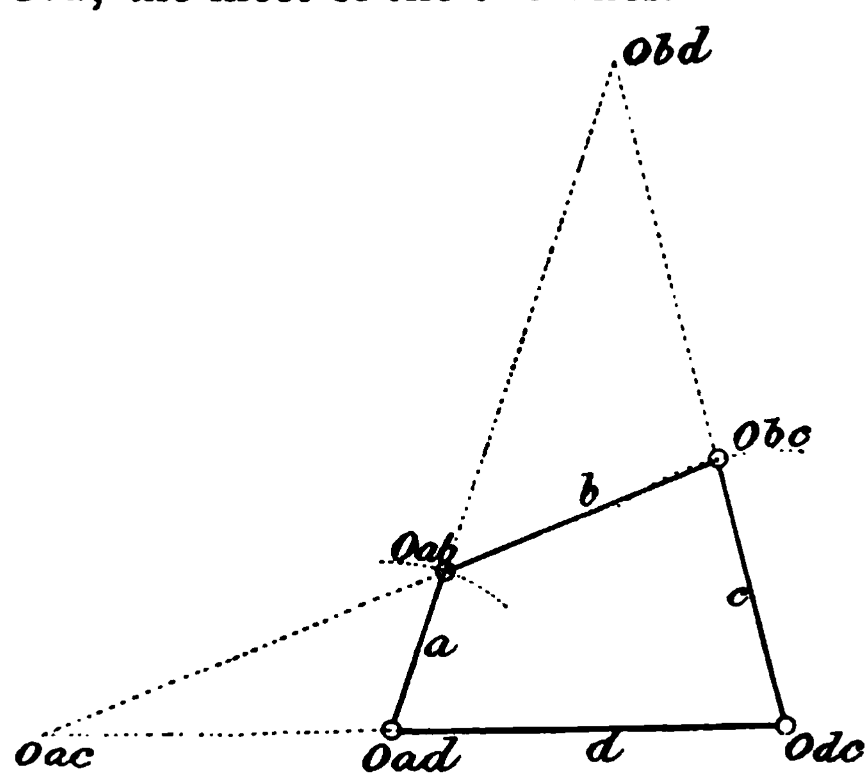


FIG. 148.

In a similar manner, consider the link *c* as fixed; then, fo

the same reason as was given above for *b* and *d*, the virtual centre of *a* and *c* lies at the meet of the two lines *b* and *d*, viz. O*ac*.

If the mechanism be slightly altered, as shown in Fig. 149, we shall get one of the virtual centres at infinity, viz. O*cd*.

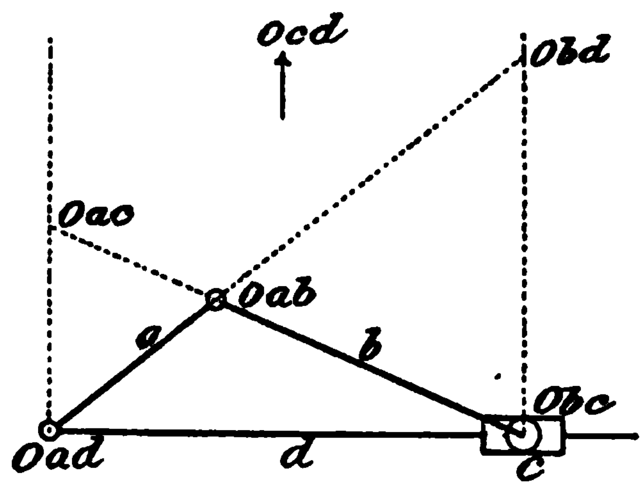


FIG. 149.

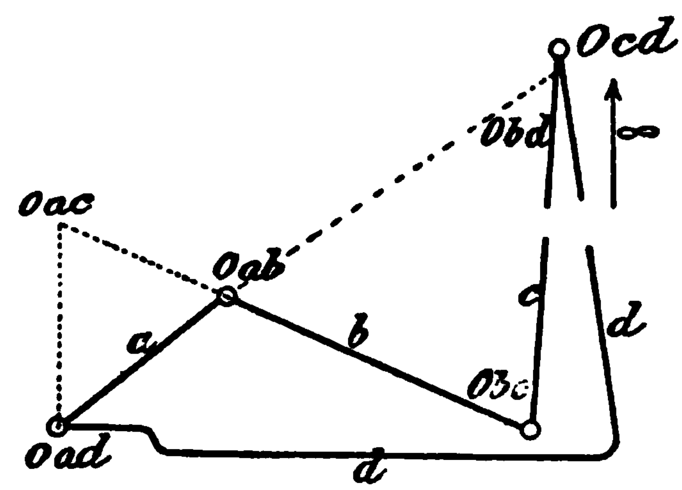


FIG. 150.

The mechanism shown in Fig. 149 is, as far as its action goes, precisely similar to the mechanism in Fig. 150. Instead of *c* sliding to and fro in guides, a link of infinite length has been substituted, and the fixed link *d* has been carried to infinity in order to provide a centre from which *c* shall swing. Then it is evident that the joint O*bc* moves in the arc of a circle of infinite radius, *i.e.* it moves in a straight line in precisely the same manner as the sliding link *c* in Fig. 149.

The only virtual centre that may present any difficulty in finding is O*ac*. Consider the link *c* as fixed, then the bar *d* swings about a centre at an infinite distance away; hence every point in it moves in a straight path at right angles to *c*, *i.e.* in the same straight line or parallel to *d* (Fig. 150). Hence the virtual centre lies on a line normal to *d*; also, for reasons given above, it lies on the prolongation of the bar *b*, viz. O*ac*.

Three Virtual Centres on a Line.—By referring to the figures above, it will be seen that there are always three virtual centres on each line. In Figs. 149, 150, it must be remembered that the three virtual centres O*ad*, O*ac*, O*cd* are on one line; also O*bc*, O*bd*, O*cd*.

The proof that the three virtual centres corresponding to the three contiguous links must lie on one line is quite simple, and as this property is of very great value in determining the positions of the virtual centres for complex mechanisms, we will give it here. Let *b* (Fig. 151) be a body moving relatively to *a*, and let the virtual centre of its motion relative to *a* be O*ab*; likewise let O*ac* be the virtual centre of *c*'s motion relative to *a*. If we

want to find the velocity of a point in *b* relatively to a point in *c*, we must find the virtual centre, O*bc*. Let it be at O: then, considering it as a point of *b*, it will move in the arc 1.1 struck from the centre O*ab*; but considering it as a point in *c*, it will move in the arc 2.2 struck from the centre O*ac*. But the tangents of these arcs intersect at O, therefore the point O has a motion in two directions at the same time, which is impossible. In the same manner, it may be shown that the virtual centre O*bc* cannot lie anywhere but on the line joining O*ab*, O*ac*, for at that point only will the tangents to the point-paths at O coincide, therefore the three virtual centres must lie on one straight line.

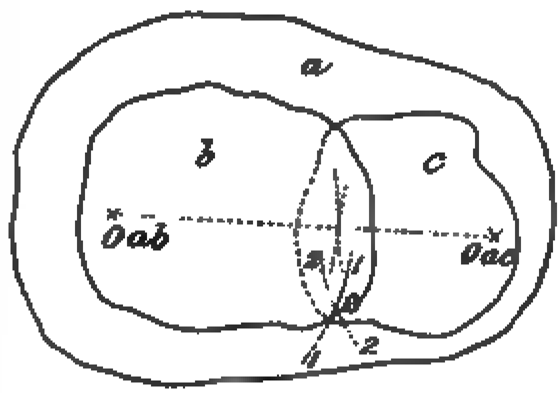


FIG. 151.

Relative Linear Velocities of Points in Mechanisms.—Once having found the virtual centre of any two bars of a mechanism, the finding of the velocity of any point in one bar relatively to any point in the other is a very simple matter, for their velocities vary directly as their virtual radii.

In the mechanism shown, let the bar *d* be fixed; to find the relative velocities of the points 1 and 2, we have—

$$\frac{\text{velocity } 1}{\text{velocity } 2} = \frac{Obd\,1}{Obd\,2} = \frac{r_1}{r_2}$$

Similarly—

$$\frac{\text{velocity } 1}{\text{velocity } 3} = \frac{r_1}{r_3}$$

$$\text{and } \frac{\text{velocity } 3}{\text{velocity } 4} = \frac{r_3}{r_4}$$

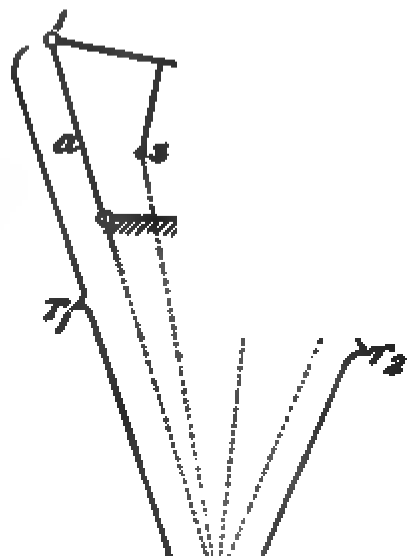


FIG. 152.

The relative velocities are not affected in the slightest degree by the *shape* of the bars. Let them be curved as shown in Fig. 153; then, as before, we have—

$$\frac{\text{velocity } 5}{\text{velocity } 6} = \frac{\text{O}ac\ 5}{\text{O}ac\ 6} = \frac{r_5}{r_6}$$

and so for any other points on the two links. But if we wished to find the velocity of the point 6 on the bar *a* relatively to the

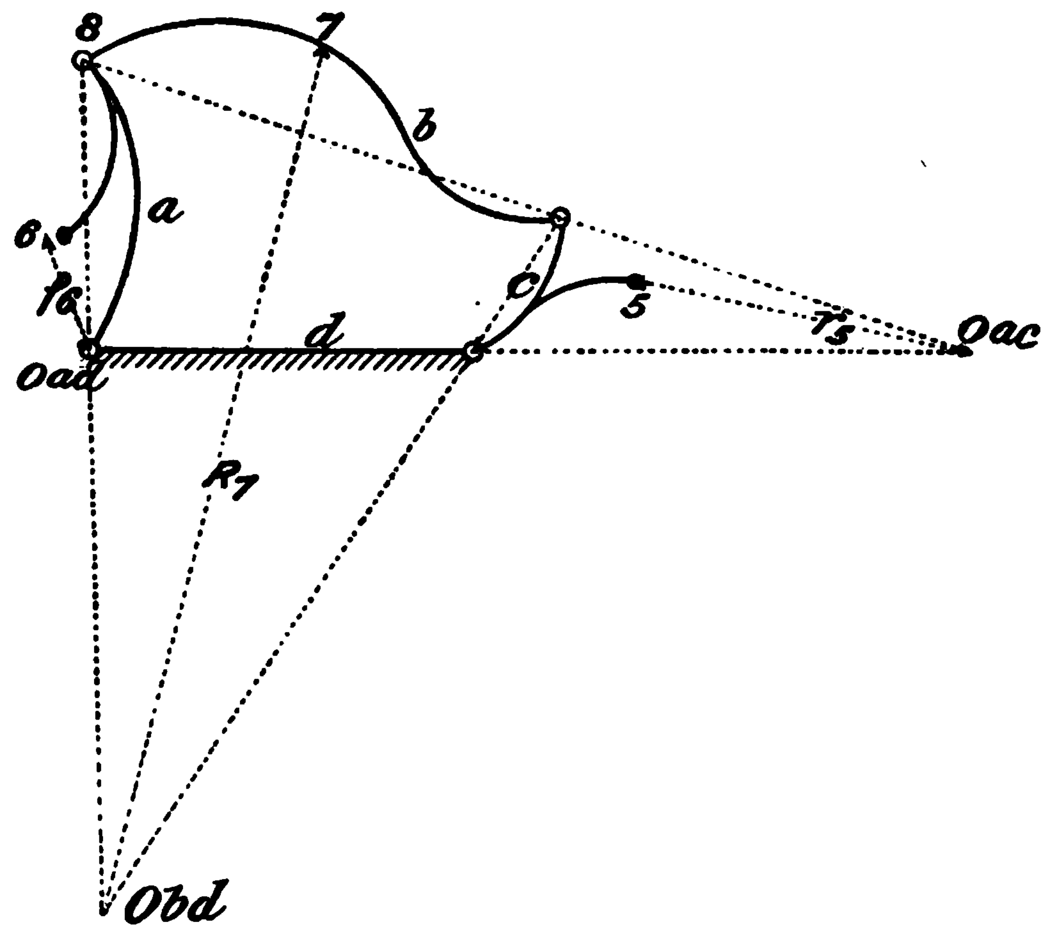


FIG. 153.

point 7 on the bar *b*, we should proceed thus: Denoting all radii from the virtual centre O*ad* as ρ_6, ρ_8, etc., and from O*bd* as R_7, R_8, etc.

$$\frac{\text{velocity } 6}{\text{velocity } 8} = \frac{\rho_6}{\rho_8}$$

Likewise—

$$\frac{\text{velocity } 7}{\text{velocity } 8} = \frac{R_7}{R_8}$$

$$\text{hence } \frac{\text{velocity } 6}{\text{velocity } 7} = \frac{\rho_6 \times \text{velocity } 8 \times R_8}{\rho_8 \times R_7 \times \text{velocity } 8}$$

$$= \frac{\rho_6 \times R_8}{\rho_8 \times R_7}$$

This can be arrived at much more readily by a graphical process, thus (Fig. 154): With O*ad* as centre, and ρ_6 as radius, set off O*ad*.*h* = ρ_6 along the line joining O*ad* to 8; set off a line *hi* to a convenient scale in any direction to represent

the velocity of 6. From O*ad* draw a line through *i*, and from 8 draw a line 8*e* parallel to *hi*; this line will then represent the

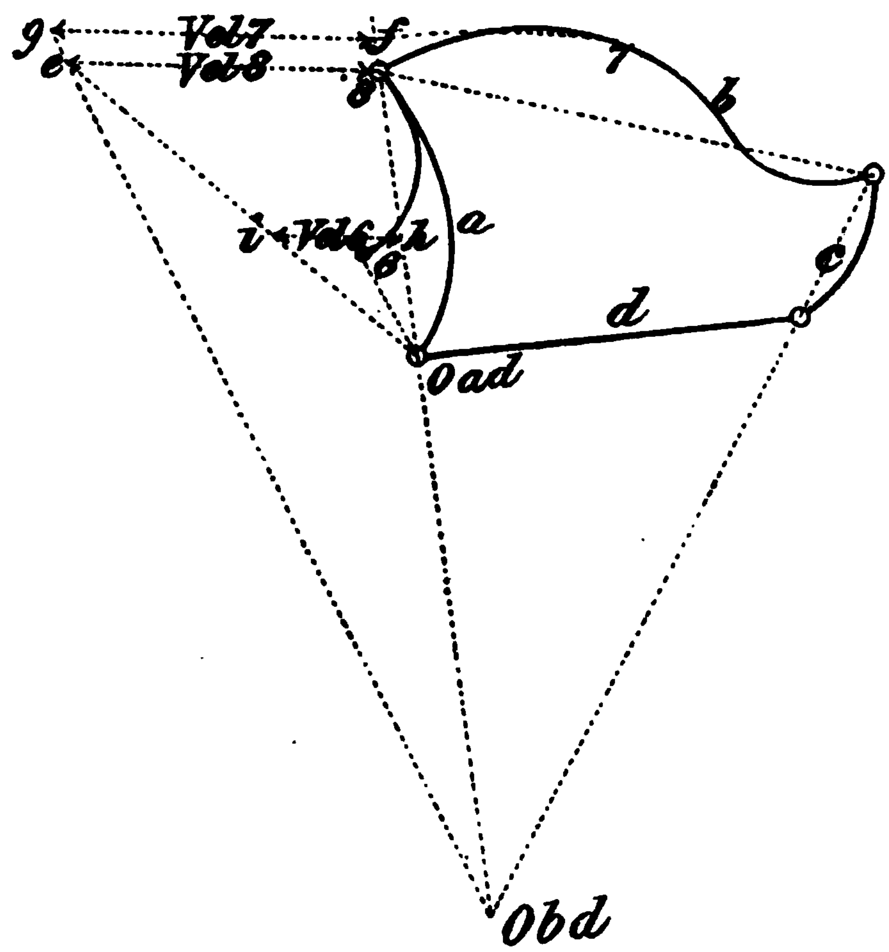


FIG. 154.

velocity of the point 8 to the same scale as 6, for the two triangles O*ad*.8.*e* and O*ad*.*h*.*i* are similar, therefore—

$$\frac{8e}{hi} = \frac{\text{O}ad.8}{\text{O}ad.h} = \frac{\rho.8}{\rho.6} = \frac{\text{velocity } 8}{\text{velocity } 6}$$

From the centre O*bd* and radius R_7, set off O*bd*.*f* = R_7; draw *f*.*g* parallel to *e*.8, and from O*bd* draw a line through *e* to meet this line in *g*; then *fg* = V_7, for the two triangles O*bd*.*f*.*g* and O*bd*.8.*e* are similar, therefore—

$$\frac{fg}{8e} = \frac{\text{O}bd.f}{\text{O}bd.8} = \frac{R_7}{R_8} = \frac{\text{velocity } 7}{\text{velocity } 8}$$

$$\text{and } \frac{\text{velocity } 6}{\text{velocity } 7} = \frac{ih}{fg}$$

The same graphical process can be readily applied to all cases of velocities in mechanisms.

Steam-engine Mechanism.—For many problems connected with the steam-engine and reciprocating pumps, we require to know the velocity of the crosshead or the piston when we know the velocity of the crank-pin; this we can very

readily obtain from the principles laid down above. We have shown that—

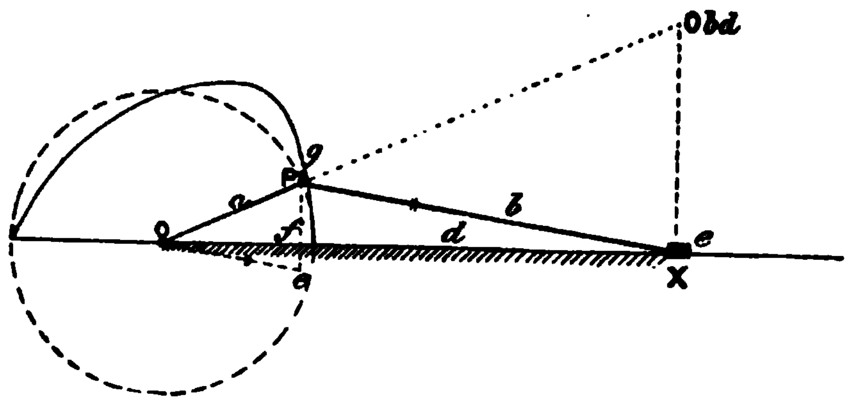


FIG. 155.

$$\frac{\text{velocity P}}{\text{velocity X}} = \frac{\text{O}bd.\text{P}}{\text{O}bd.\text{X}}$$

From O draw a line parallel to the connecting-rod, and from P drop a perpendicular to meet it in *e*. Then the triangle OP*e* is similar to the triangle P.O*bd*.X ; hence—

$$\frac{\text{OP}}{\text{P}e} = \frac{\text{O}bd.\text{P}}{\text{O}bd.\text{X}} = \frac{\text{velocity of pin}}{\text{velocity of cross head}}$$

But the velocity of the crank-pin may be taken to be constant. Let it be represented by the radius of the crank-circle OP ; then to the same scale P*e* represents the velocity of the cross-head. Set up fg = P*e* at several positions of the crank-pin, and draw a curve through them ; then the ordinates of this curve represent the velocity of the crosshead at every point in the stroke, where the radius of the crank-circle represents the velocity of the crank-pin.

When the connecting-rod is of infinite length, or in the case

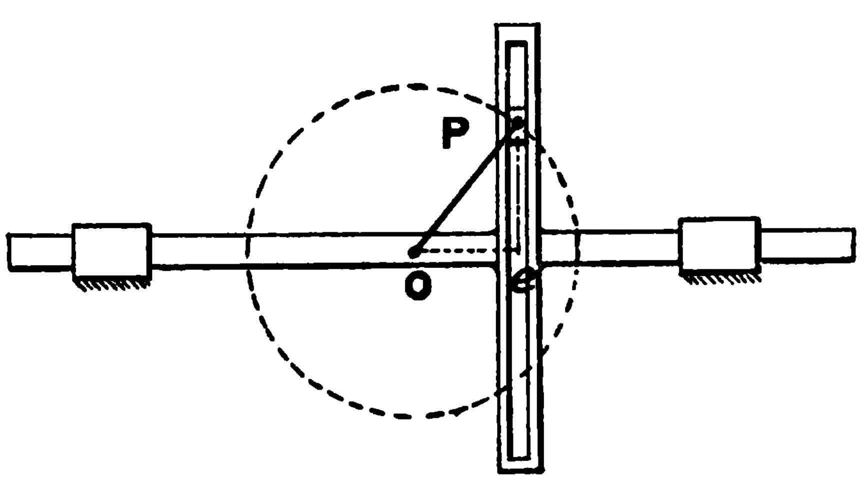


FIG. 156.

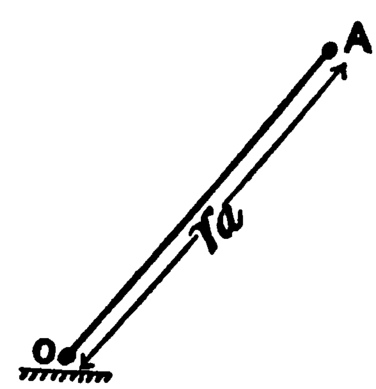


FIG. 157.

of such a mechanism as that shown in Fig. 156, the line O*e* always coincides with the axis, and consequently the crosshead-velocity diagram becomes a semicircle.

Relative Angular Velocities of Bars in Mechanisms.—Every point in a rotating bar has the same angular velocity. Let the bar in Fig. 157 be turning about the point O with an angular velocity ω ; then the linear velocity V_a of a point A situated at a radius r_a is—

$$V_a = \omega r_a, \text{ and } \omega = \frac{V_a}{r_a}$$

In order to find the relative angular velocity of any two links, let the point A (Fig. 158) be first regarded as a point in the bar *a*, and let its radius about O*ad* be r_A. When the

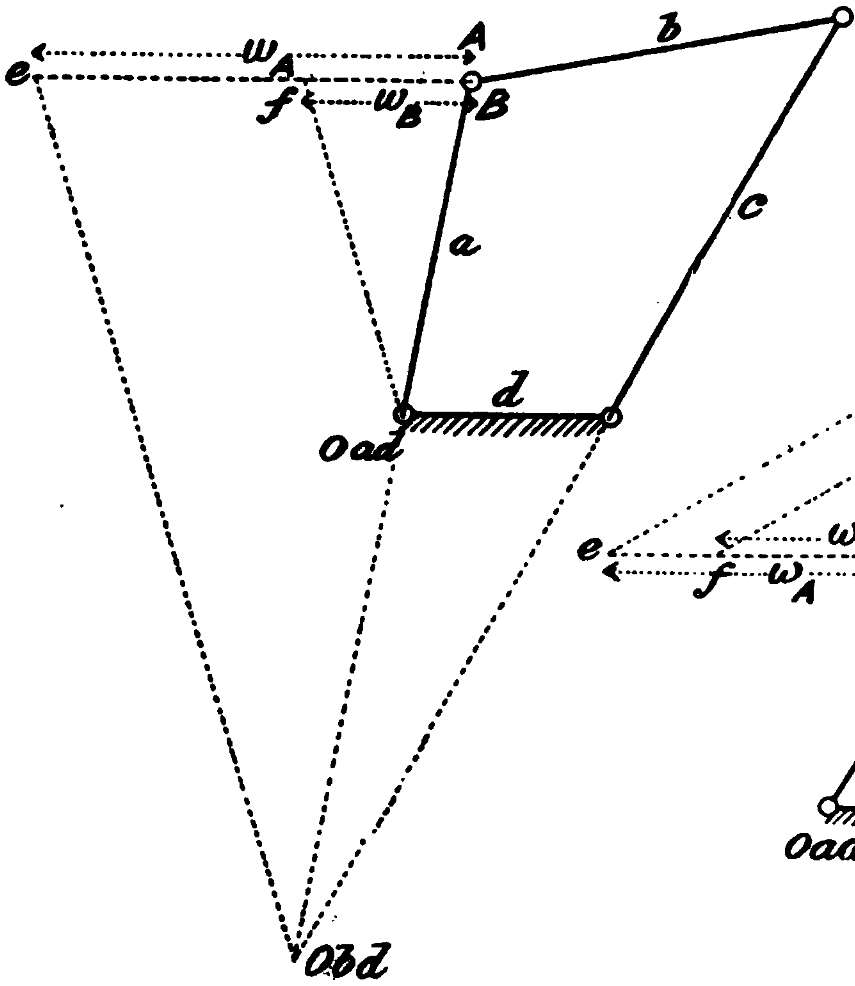


FIG. 158.

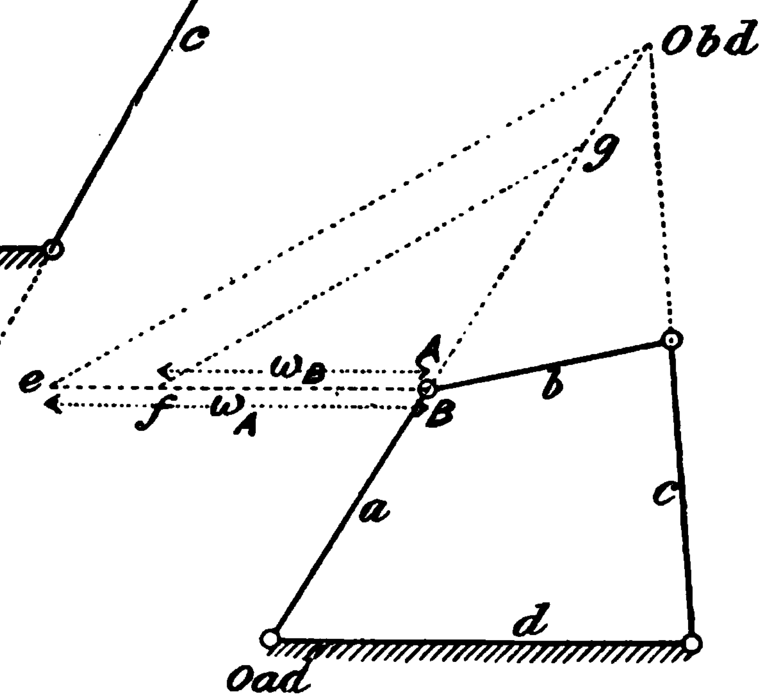


FIG. 159.

point A is regarded as a point in the bar *b*, we shall term it B, and its radius about O*bd*, r_B. Let the linear velocity of A be V_A, and that of B be V_B, and the angular velocity of A be ω_A, and of B be ω_B. Then $V_A = \omega_A r_A$, and $V_B = \omega_B r_B$.

But $V_A = V_B$ as A and B are the same point; hence—

$$\omega_A r_A = \omega_B r_B$$

$$\text{or } \frac{\omega_A}{\omega_B} = \frac{r_B}{r_A} = \frac{(Obd)B}{(Oad)A}$$

This may be very easily obtained graphically thus: Set off a line A*e* in any direction from A, whose length on some given scale is equal to ω_A; join *e*.O*bd*; from O*ad* draw O*ad*.*f* parallel to *e*.O*bd*. Then A*f* = ω_B, because the two triangles A.*f*.O*ad* and A.*e*.O*bd* are similar. Hence—

$$\frac{Ae}{Af} = \frac{(Obd)B}{(Oad)A} = \frac{\omega_A}{\omega_B}$$

In Fig. 159, the distance Ag has been made equal to A.Oad, and gf is drawn parallel to e.Obd. The proof is the same as in the last case. When a is parallel to c, the virtual centre is at infinity, and the angular velocity of b becomes zero.

When finding the relative angular velocity of two non-

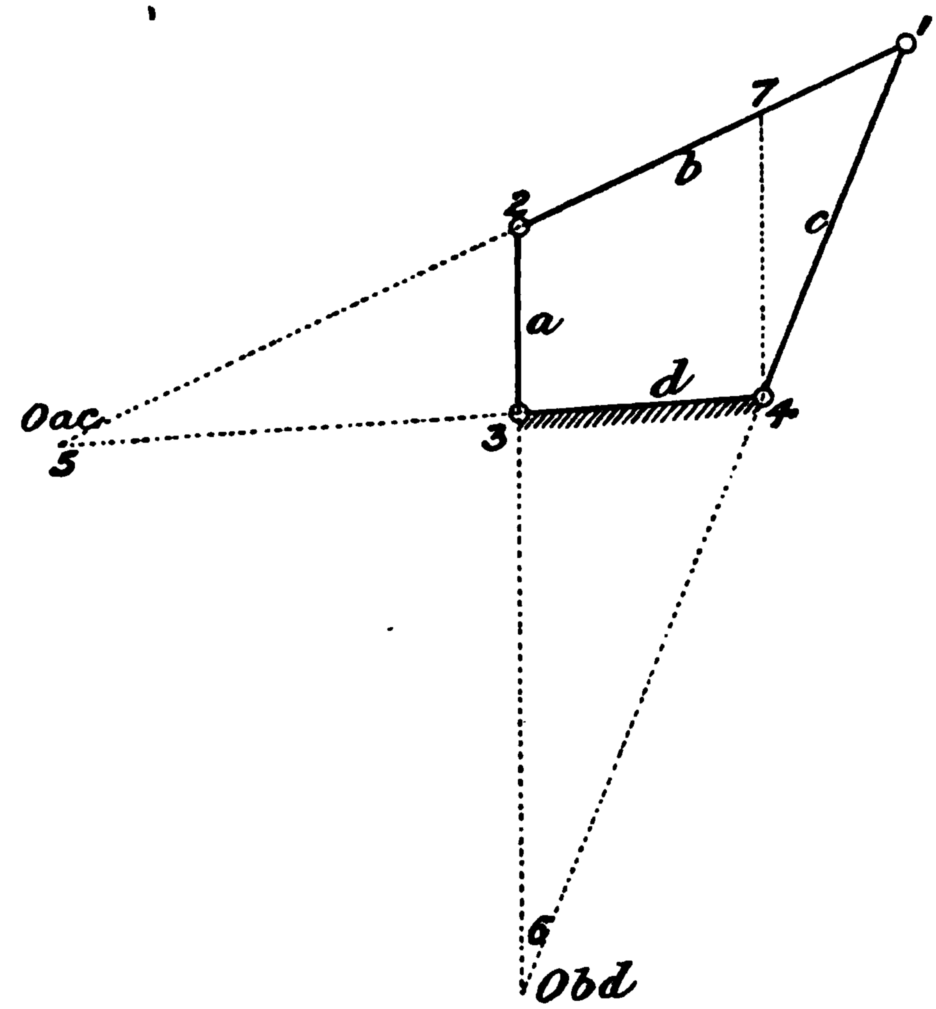


FIG. 160.

adjacent links, such as a and c, we proceed thus: For convenience we have numbered the various points instead of using the more cumbersome virtual centre nomenclature. The radius 1.6 we shall term $r_{1.6}$, and so on.

Then, considering points 1 and 2 as points of the bar b, we have—

$$\frac{V_1}{V_2} = \frac{r_{1.6}}{r_{2.6}}$$

Then, regarding point 1 as a point in bar c, and regarding point 2 as a point in bar a—

$$V_1 = \omega_c \times r_{1.4} \qquad V_2 = \omega_a r_{2.3}$$

Then, substituting the values of V_1 and V_2 in the equation above, we have—

$$\frac{r_{1.6}}{r_{2.6}} = \frac{\omega_c \times r_{1.4}}{\omega_a \times r_{2.3}}, \text{ or } \frac{\omega_c}{\omega_a} = \frac{r_{1.6} \times r_{2.3}}{r_{2.6} \times r_{1.4}}$$

Draw 4.7 parallel to 2.3; then, by the similar triangles 1.2.6 and 1.7.4—

$$\frac{r_{2.6}}{4.7} = \frac{r_{1.6}}{r_{1.4}}$$

$$\text{and } r_{2.6} \times r_{1.4} = r_{1.6} \times 4.7$$

Substituting this value above, we have—

$$\frac{\omega_c}{\omega_a} = \frac{r_{1.6} \times r_{2.3}}{r_{1.6} \times 4.7} = \frac{r_{2.3}}{4.7}, \text{ or} = \frac{5.3}{5.4} \text{ by similar triangles}$$

Thus, if the length r_{23} represents the angular velocity of c, and a line be drawn from 4 to meet the opposite side in 7, 4.7 represents on the same scale the angular velocity of a. Or it may conveniently be done graphically thus: Set off from 3 a

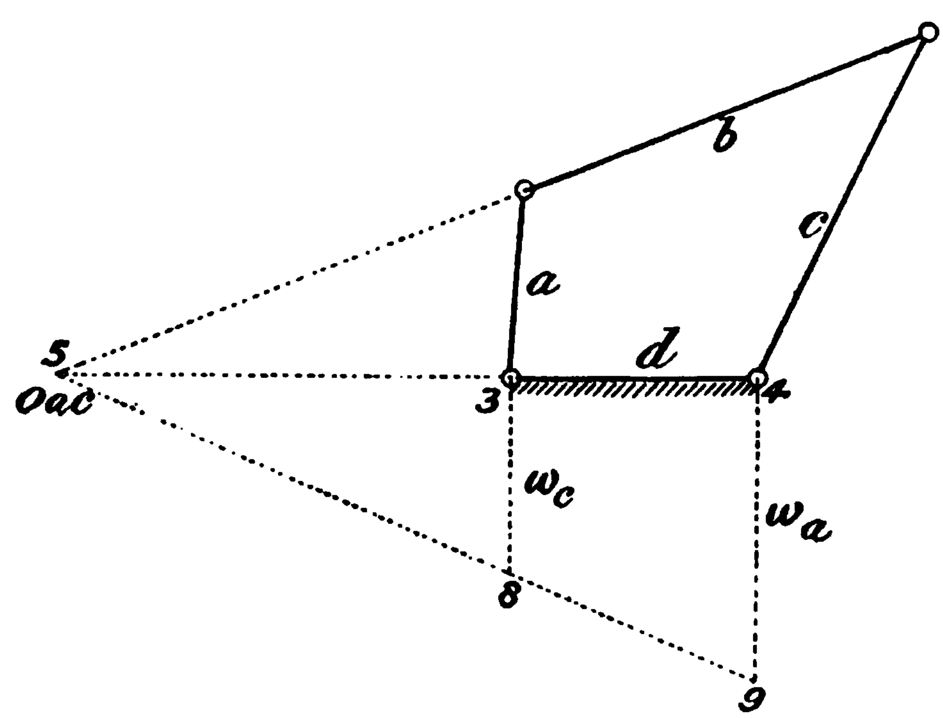


FIG. 161.

line in any direction whose length 3.8 represents the angular velocity of c; from 4 draw a line parallel to 3.8; from 5 draw a line through 8 to meet the line from 4 in 9. Then 4.9 represents the angular velocity of a, the proof of which will be perfectly obvious from what has been shown above.

When b is parallel to d, the virtual centre Oac is at infinity, and the angular velocity of a is then equal to the angular velocity of c.

The Principle of Virtual Velocities applied to Mechanisms.—If a force act on any point of a mechanism and overcome a resistance at any other point, the work done at the two points must be equal if friction be neglected.

In Fig. 162, let the force P act on the link a at the

point 2. Find the magnitude of the force R at the point 4 required to keep the mechanism in equilibrium.

If the bar *a* be given a small shift the path of the point 2

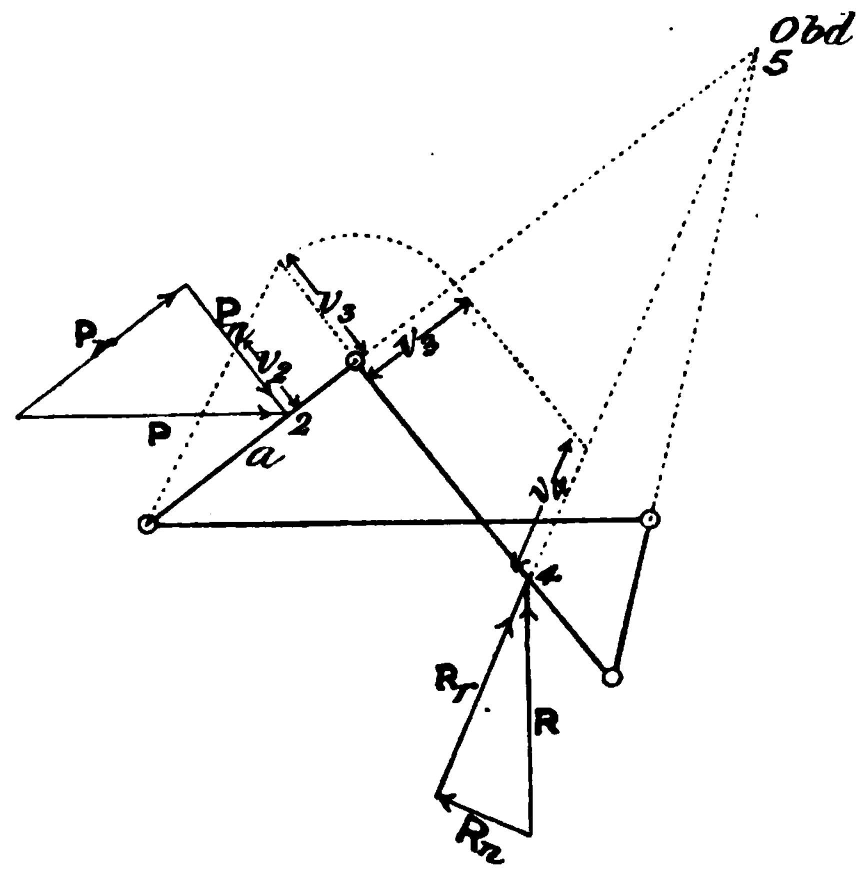


FIG. 162.

will be normal to the link *a*, and the path of the point 4 will be normal to the radius 5.4.

Resolve P along P*r* parallel to *a*, which component, of course, has no turning effort on the bar; also along the normal P*n* in the direction of motion of the point 2.

Likewise resolve R along the radius R*r*, and normal to the radius R*n*.

Now we must find the relative velocity of the points 2 and 4 by methods previously explained, and shown by the construction on the diagram. Then, as no work is wasted in friction, we have—[1]

$$P_n V_2 = R_n V_4$$

$$\text{and } R_n = \frac{P_n V_2}{V_4}$$

Velocity and Acceleration Curves.—In the paragraphs above, we have shown how to find the velocity of any given point of a mechanism relatively to any other. We will now show how to construct a curve to represent such a relation, and how to obtain from it a curve of acceleration.

[1] The small simultaneous displacements are proportional to the velocities.

To illustrate the point, we will choose a simple four-bar mechanism, in which the crank *a* revolves and *c* rocks to and fro.

Let the link *a* revolve with a constant angular velocity. We will construct a curve to show the velocity of the point *f* relatively to the constant velocity of the point *e*.

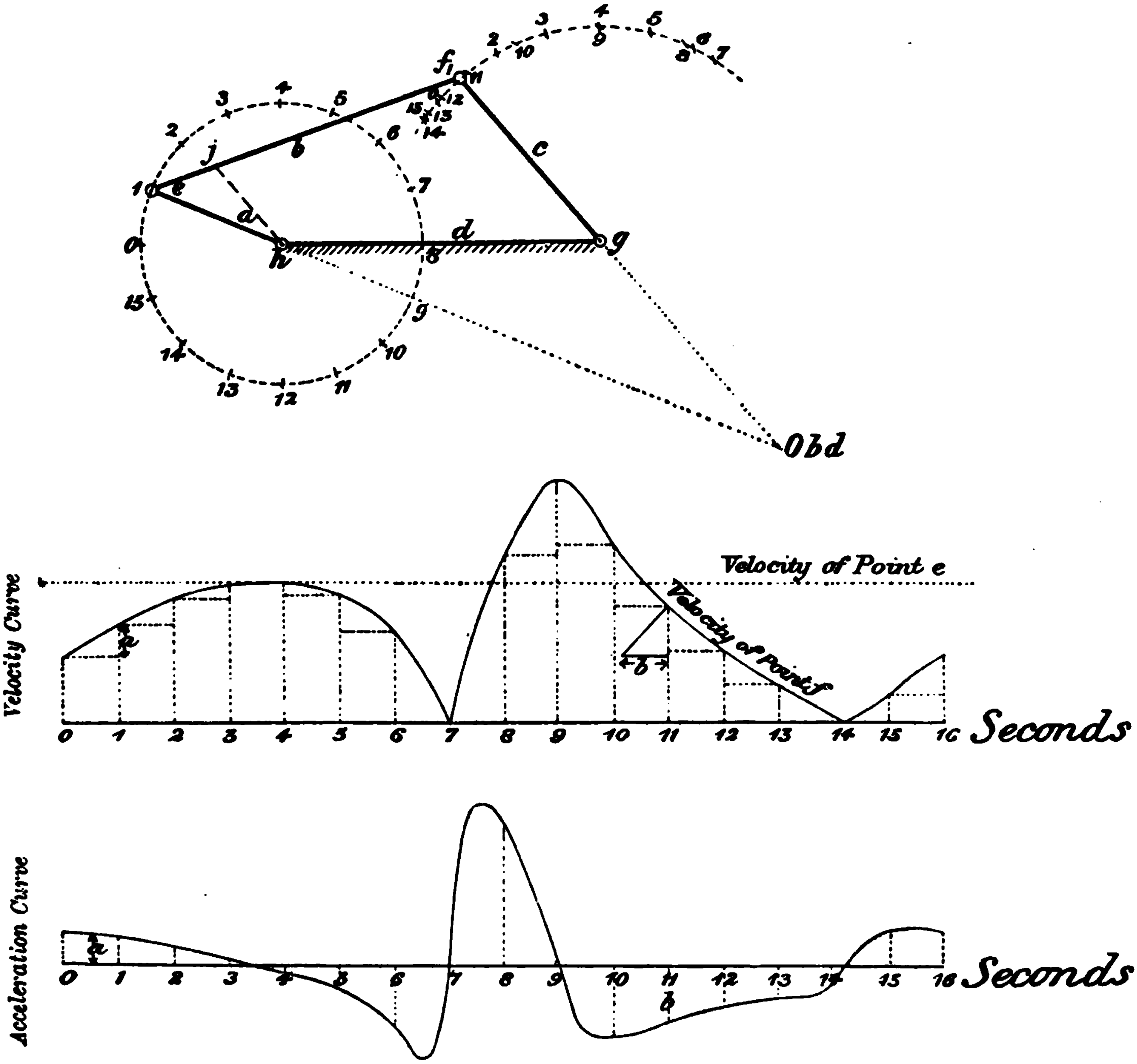


FIG. 163.

Divide the circle that *e* describes into any convenient number of equal parts (in this case 16). The point *f* will move in the arc of a circle struck from the centre *g*; then, by means of a pair of compasses opened an amount equal to *ef*, from each position of *e* set off the corresponding position of *f*

on the arc struck from the centre *g*. By joining up *he*, *ef*, *fg*, we can thus get every position of the mechanism; but only one position is shown in the figure for clearness. Then, in order to find the relative velocity of *e* and *f*, we produce the links *a* and *c* to obtain the virtual centre O*bd*. This will often come off the paper. We can, however, very easily get the relative velocities by drawing a line *hj* parallel to *c*. Then the triangles *hej* and O*bd.e.f* are similar, therefore—

$$\frac{he}{hj} = \frac{Obd.e}{Obd.f} = \frac{\text{velocity of } e}{\text{velocity of } f}$$

the velocity of *e* is constant;[1] therefore take the constant length of the link *a*, viz. *he*, to represent the velocity of *e*; then, from the relation above, *hj* will represent on the same scale the velocity of *f*.

Set off on a straight line the distances on the *e* curve 0.1, 1.2, 2.3, etc., as the base of the velocity curve. At each point set up ordinates equal to *hj* for each position of the mechanism. On drawing a curve through the tops of these ordinates, we get a complete velocity curve for the point *f* when the crank *a* revolves uniformly. The velocity curve for the point *e* is a straight line parallel to the base.

In constructing the acceleration curve, it should be remembered that the acceleration is the rate of change of velocity, *i.e.* the increase or decrease per second in the velocity. Thus, if each of the divisions 0.1, 1.2, 2.3, etc., represent an interval of one second, and horizontals be drawn from the velocity curve as shown, the height α represents the increase in the velocity during the interval 0.1, *i.e.* in this case one second; then on the acceleration diagram the height α is set up in the middle of each space to show the mean acceleration that the point *f* has undergone during the interval 0.1, and so on for each space. A curve drawn through the points so obtained is the acceleration curve for the point *f*. This method has been adopted until point 7 has been reached; then another method has been used. This will perhaps be better explained by a separate diagram, thus—

Let the curve represent the velocity of any point as it moves through space. Let the time-interval between the two dotted lines be *dt*, and the change of velocity of the point

[1] In this case we have assumed each division to be passed over in one second; hence the velocity of *e* is $\frac{2\pi.he}{16} = \frac{he}{2\cdot55}$, *he* is measured in feet.

while passing through that space be dv. Then the acceleration during the interval is $\frac{dv}{dt}$, *i.e.* $\frac{\text{change of velocity}}{\text{interval of time}}$, or the change of velocity in the given interval.

By similar triangles, we have $\frac{dv}{dt} = \frac{zy}{xy}$.

When $dt = 1$ second, $dv =$ acceleration.

Hence, when $xy_0 = 1$ on the velocity scale, z_0y_0, the sub-normal, gives us the acceleration measured on the same scale as the velocity.

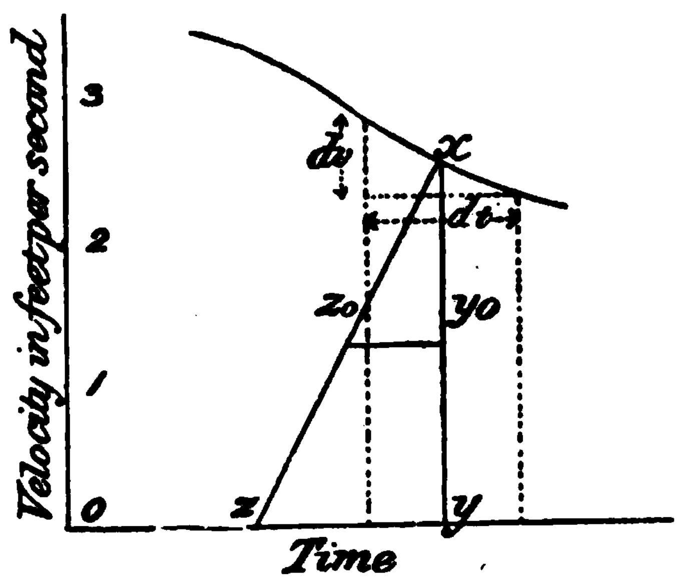


FIG. 164.

The sub-normals to the curve above have been plotted in this way to give the acceleration curve from 7 to 16. The scale of the acceleration curve will be the same as that of the velocity curve.

The reader is recommended to refer to Barker's "Graphical Calculus" for other curves of this character.

Toothed Gearing.—The general principles that have to be considered when deciding upon the form that shall be given to the teeth of wheels, follow directly from the work that we have considered above when dealing with virtual centres and the velocities of different points in mechanisms.

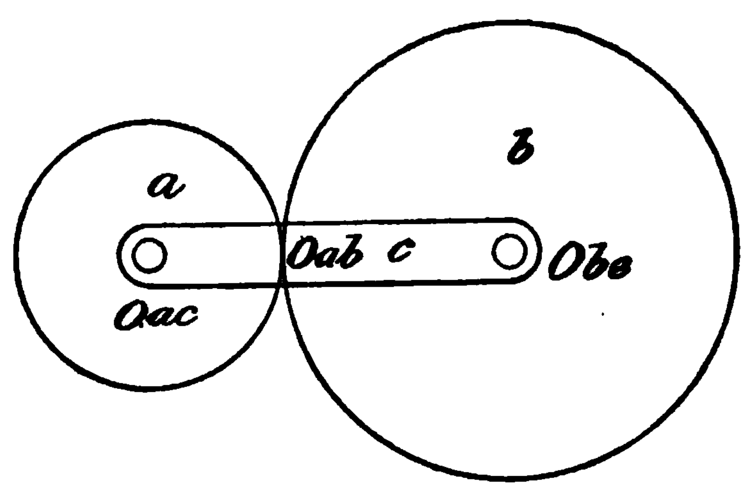


FIG. 165.

In the figure, let two shafts A and B be provided with circular discs, a and b, as shown; let there be sufficient friction at the line of contact to make the one revolve when the other is turned round; then the linear velocity v of the two rims will be the same.

Let the radius of a be r_a, of b be r_b;

the angular velocity of a be ω_a, of b be ω_b;

$N_a =$ the number of revolutions of a in a unit of time;

$N_b =$ the number of revolutions of b in a unit of time.

$$\text{Then } \omega_a = \frac{2\pi r_a N_a}{r_a} = 2\pi N_a, \text{ and } \omega_b = 2\pi N_b$$

$$v = \omega_a r_a = \omega_b r_b$$

$$\text{or } \frac{r_b}{r_a} = \frac{\omega_a}{\omega_b} = \frac{2\pi N_a}{2\pi N_b} = \frac{N_a}{N_b}$$

or the revolutions of each wheel are inversely proportional to their respective radii.

The virtual centre of *a* and *c*, likewise *b* and *c*, is evidently at their permanent centres, and as the three virtual centres must lie in one line (see p. 125), the virtual centre O*ab* must lie on the line joining the centres of *a* and *b*, and must be a point (or axis) common to each. The only point which fulfils these conditions is O*ab*, the point of contact of the two discs. The two circles representing the rims of the discs are known as the *pitch* circles of the gearing. If a large amount of force be transmitted from the one disc to the other, slipping will occur, and the velocity ratios will no longer remain constant; in order to prevent slipping, projections or teeth must be formed on the one wheel to fit into recesses in the other, and they must be of such a form that the velocity ratio at every instant shall be constant, *i.e.* the virtual centre O*ab* must always be in its present position. We have seen above that the direction of motion of any point in a body moving relatively to another body is normal to the virtual radius; hence, if we make a projection or a tooth, say, on *a* (Fig. 166), the direction of motion of any point *d* relatively to *b*, will be a normal to the line drawn from *d* through the virtual centre O*ab*.

Likewise with any point in *b* relatively to *a*. Hence, if a projection on the one wheel is required to fit into a recess in the other, a normal to their surfaces at the point of contact must pass through the virtual centre O*ab*. If such a normal do not pass through O*ab*, the velocity ratio will be altered, and if O*ab* shifts about as the one wheel moves relatively to the other, the motion will be jerky.

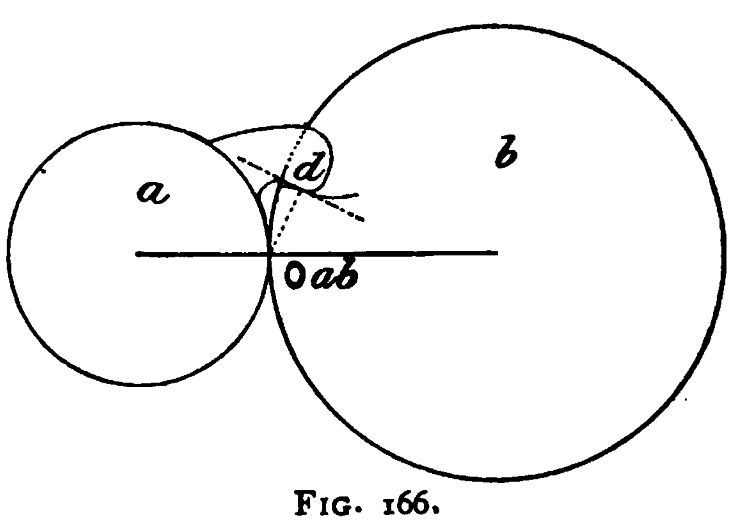


FIG. 166.

Hence, in designing the teeth of wheels, we must so form them that they fulfil the condition that the normal to their surfaces at the point of contact must pass through the virtual

centre of the one wheel relatively to the other, *i.e.* the point where the two pitch circles touch one another, or the point where the pitch circles cut the line joining their centres. An infinite number of forms might be designed to fulfil this condition; but some forms are more easily constructed than others, and for this reason they are chosen.

The forms usually adopted for the teeth of wheels are the

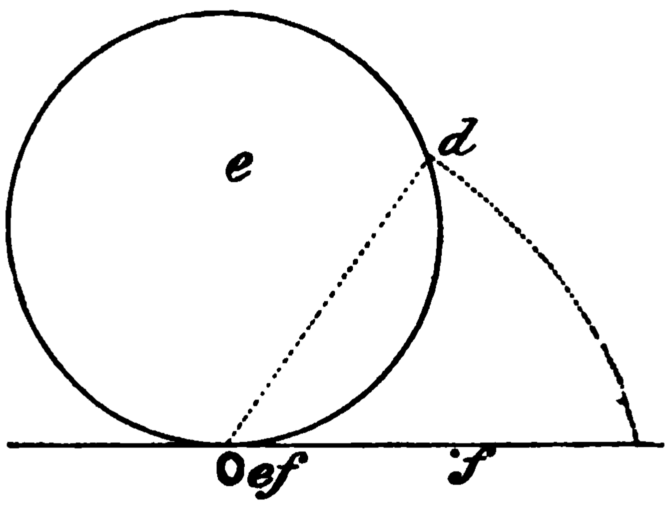


Fig. 167.

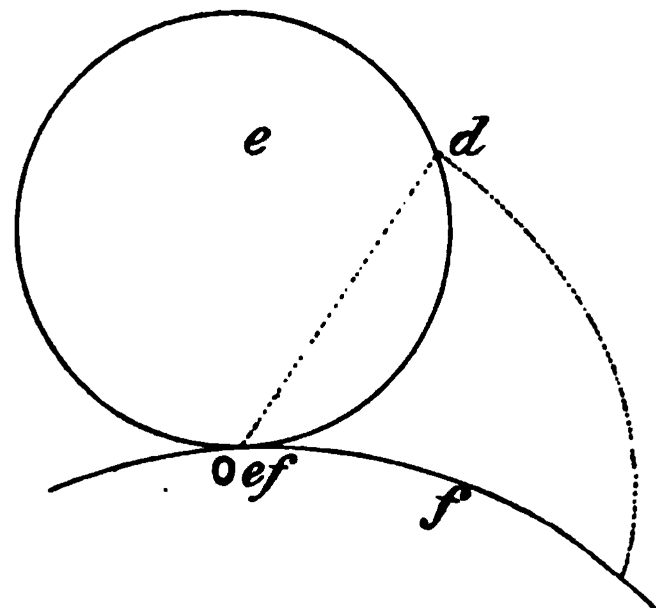


Fig. 168.

cycloid and the involute, both of which are easily constructed and fulfil the necessary conditions.

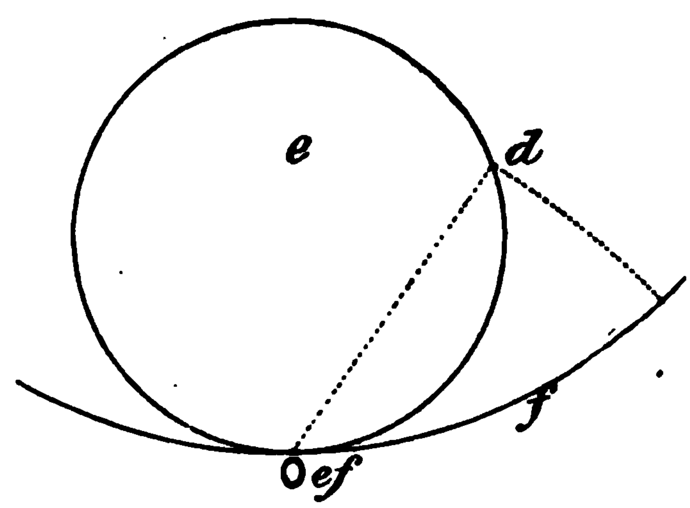


Fig. 169.

If the circle *e* rolls on either the straight line or the arc of a circle *f*, it is evident that the virtual centre is at their point of contact, viz. O*ef*; and the path of any point *d* in the circle moves in a direction normal to the line joining *d* to O*ef*, or normal to the virtual radius. When the circle rolls on a straight line, the curve traced out is termed a cycloid (Fig. 167); when on the outside of a circle, the curve traced out is termed an epicycloid (Fig. 168); when on the inside of a circle, the curve traced out is termed a hypocycloid (Fig. 169).

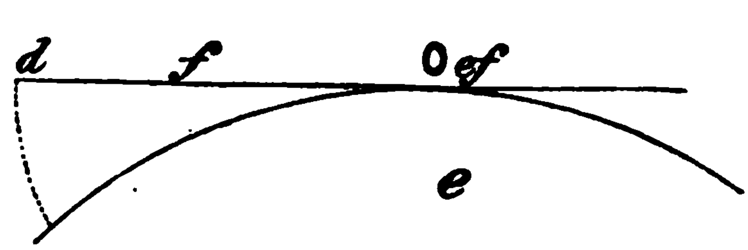
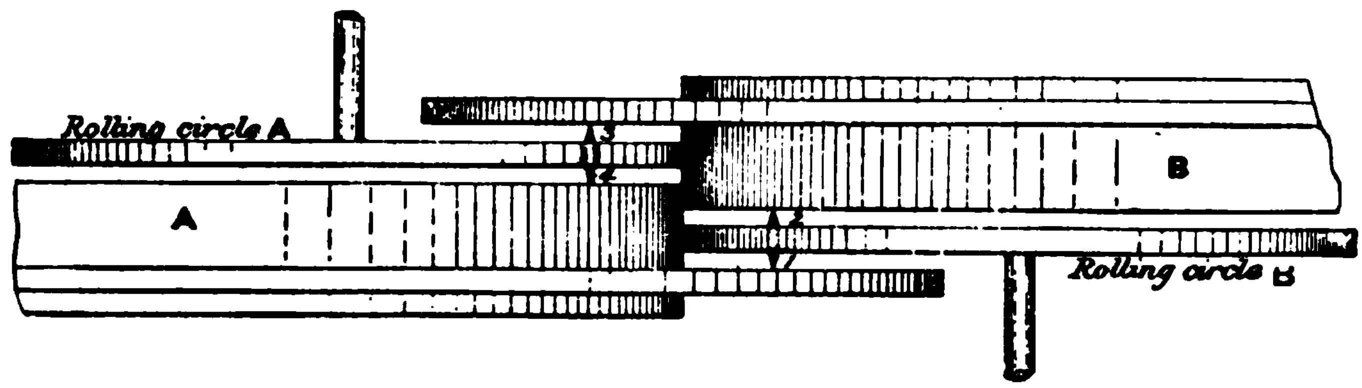


Fig. 170.

If a straight line *f* (Fig. 170) be rolled without slipping on the arc of a circle, it is evident that the virtual centre is at

their point of contact, viz. Of, and the path of any point d in the line moves in a direction normal to the line f, or normal to the virtual radius. The curve traced out by d is an involute.



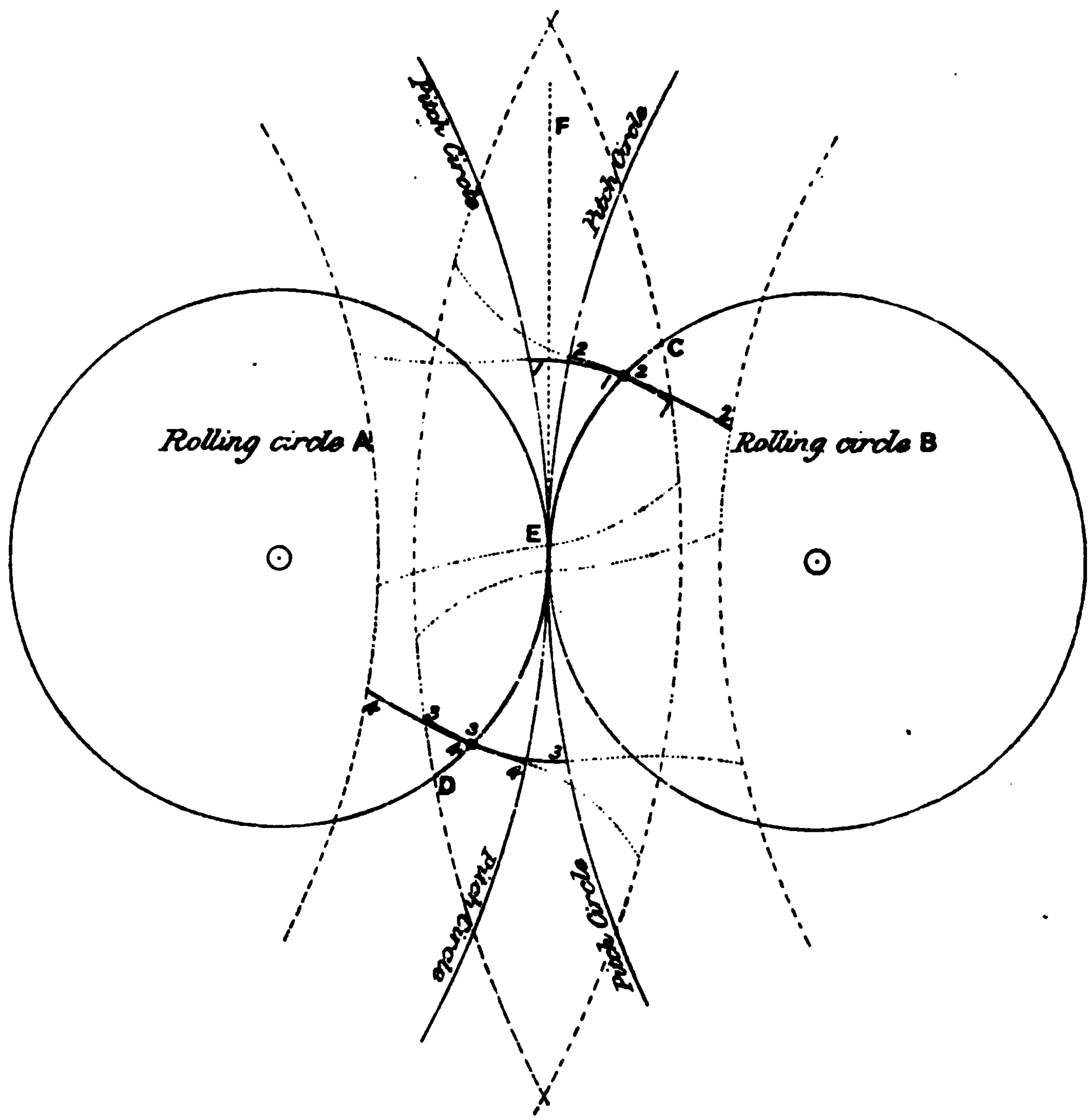


FIG. 171.

It may be described by wrapping a piece of string round a circular disc and attaching a pencil at d; as the string is unwound d moves in an involute.

When setting out cycloidal teeth, only small portions of the cycloids are actually used. The cycloidal portions can be obtained by construction or by rolling a circular disc on the pitch circle. By reference to Fig. 171, which represents a model used to demonstrate the theory of cycloidal teeth, the reason why such teeth gear together smoothly will be evident. A and B are parts of two circular discs of the same diameter as the pitch circles; they are arranged on spindles, so that when the one revolves the other turns by the friction at the line of contact. Two small discs or rolling circles are provided with double-pointed pencils attached to their rims; they are pressed against the large discs, and turn as they turn. Each of the large discs, A and B, is provided with a flange as shown. Then, when these discs and the rolling circles all turn together, the pencil-point 1 traces an epicycloid on the inside of the flange of A, due to the rolling of the rolling circle on A, in exactly the same manner as in Fig. 168; at the same time the pencil-point 2 traces a hypocycloid on the side of the disc B, as in Fig. 169. Then, if these two curves be used for the profiles of teeth on the two wheels, the teeth will work smoothly together, for both curves have been drawn by the same pencil when the wheels have been revolving smoothly. The curve traced on the flange of A by the point 1 is shown in the lower figure, viz. 1.1.1; likewise that traced on the disc B by the point 2 is shown, viz. 2.2.2. In a similar manner, the curves 3.3.3, 4.4.4, have been obtained. The full-lined curves are those actually drawn by the pencils, the remainder of the teeth are dotted in by copying the full-lined curves.

In the model, when the curves have been drawn, the discs are taken apart and the flanges pushed down flush with the inner faces of the discs, then the upper and lower parts of the curves fit together, viz. the curve drawn by 3 joins the part drawn by 2; likewise with 1 and 4.

From this figure it will also be clear that the point of contact of the teeth always lies on the pitch circles, and that contact begins at C and ends at D. The double arc from C to D is termed the "arc of contact" of the teeth. In order that two pairs of teeth may always be in contact at any one time, the arc CD must not be less than the pitch. The direction of pressure between the teeth is evidently in the direction of a tangent to this arc at the point of contact. Hence, the greater the angle the tangent makes to a line EF (drawn normal to the line joining the centres of the wheels), the greater will be the pressure pushing the two wheels apart,

and the greater the friction on the bearings; for this reason the angle is rarely allowed to be more than 30°. In order to keep this angle small, a large rolling circle must be used, but it is rarely more than half the diameter of the smallest wheel in the train. The same rolling circle should be used for all wheels required to gear together.

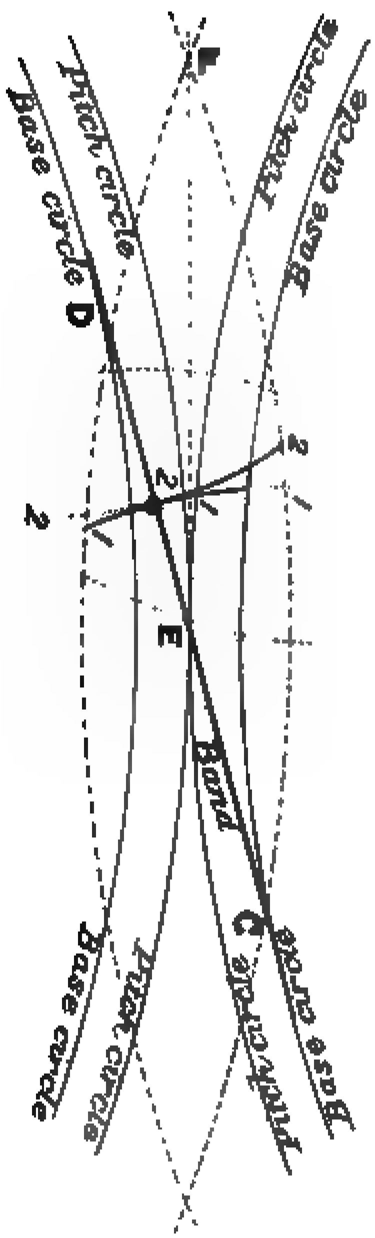


FIG. 172.

The model for illustrating the principle of involute teeth is shown in Fig. 172. Here again A and B are parts of two circular discs connected together with a thin cross-band which rolls off one disc on to the other, and as the one disc turns it makes the other revolve in the opposite direction. The band is provided with a double-pointed pencil, which is pressed against two flanges on the discs; then when the discs turn, the pencil-points describe involutes on the two flanges, in exactly the same manner as that described on p. 139.

Then, from what has been said on cycloidal teeth, it is evident that if such curves be used as profiles for teeth, the two wheels will gear smoothly together, for

they have been drawn by the same pencil as the wheels revolved smoothly together.

The point of contact of the teeth in this case always lies on the band; contact begins at C, and ends at D. The arc of contact here becomes the straight line CD. In order to prevent too great pressure on the axles of the wheels, the angle DEF seldom exceeds $15\frac{1}{2}°$; this gives a base circle $\frac{63}{65}$ of the pitch circle (Unwin).

A simple arrangement for drawing the form of tooth to work with one of any given form has been devised by Professor Hele-Shaw, whose arrangement indicates clearly the actual manner in which one tooth comes into contact with another. This method has been extended by him for drawing cycloidal and involute curves, so as to give any required form of wheel tooth. His paper is printed in full with illustrations, in the Report of the British Association for 1898.

It is beyond the scope of the present work to go beyond the principles of the forms of wheel-teeth. Readers requiring details as to the proportions and strength of teeth, and various other matters regarding the design of toothed gearing, cannot do better than refer to Unwin's "Elements of Machine Design."

CHAPTER VI.

DYNAMICS OF THE STEAM-ENGINE.

Reciprocating Parts.—In Chapter V. we gave the construction for a diagram to show the velocity of the piston at each part of the stroke when the velocity of the crank-pin was assumed to be constant. We there showed that, for an infinitely long connecting-rod or a slotted cross-head (see Fig. 156), such a diagram is a semicircle when the ordinates represent the velocity of the piston, and the abscissæ the distance it has moved through. The radius of the semicircle represents the constant velocity of the crank-pin. We see from such a diagram that the velocity of the reciprocating parts is zero at each end of the stroke, and is a maximum at the middle; hence during the first half of the stroke the velocity is increased, or the reciprocating parts are accelerated, for which purpose energy has to be expended; and during the second half of the stroke the velocity is decreased or the reciprocating parts are retarded, and the energy expended during the first half of the stroke is given back. This alternate expenditure and paying back of energy very materially affects the smoothness of running of high-speed engines, unless some means are adopted for counteracting these disturbing effects.

We will first consider the case of an infinitely long connecting-rod, and see how to calculate the pressure at any part of the stroke required to accelerate and retard the reciprocating parts.

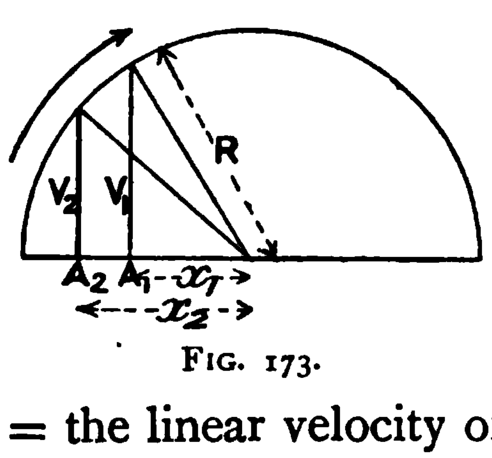


FIG. 173.

The velocity diagram for this case is given in Fig. 173. Let it be drawn to such a scale that the radius R in feet[1] = V feet per second, where V = the linear velocity of the crank-pin, assumed constant; then

[1] This would be an inconveniently large scale to use in practice, but we will retain it here for reasons shortly to be given.

the ordinates V_1, V_2 represent to the same scale the velocity of the piston when it is at the positions A_1, A_2 respectively (see p. 128).

Let the total weight of the reciprocating parts = W. Then—

$$\left.\begin{array}{l}\text{The kinetic energy of the reciprocating} \\ \quad\text{parts at } A_1\end{array}\right\} = \frac{WV_1^2}{2g}$$

$$\left.\begin{array}{l}\text{The kinetic energy of the reciprocating} \\ \quad\text{parts at } A_2\end{array}\right\} = \frac{WV_2^2}{2g}$$

$$\left.\begin{array}{l}\text{The increase of kinetic energy of the recipro-} \\ \quad\text{cating parts during the interval } A_1A_2\end{array}\right\} = \frac{W}{2g}(V_1^2 - V_2^2)$$

This energy must have come from the steam or other motive fluid in the cylinder.

Let P = the pressure on the piston required to accelerate the moving parts.

$$\left.\begin{array}{l}\text{Work done on the piston in accelerating the} \\ \quad\text{moving parts during the interval}\end{array}\right\} = P(x_2 - x_1)$$

But $x_1^2 + V_1^2 = x_2^2 + V_2^2 = R^2$; hence $V_1^2 - V_2^2 = x_2^2 - x_1^2$; and—

$$\left.\begin{array}{l}\text{Increase of kinetic energy of the recipro-} \\ \quad\text{cating parts during the interval}\end{array}\right\} = \frac{W}{2g}(x_2^2 - x_1^2)$$

$$\text{then } P(x_2 - x_1) = \frac{W}{2g}(x_2^2 - x_1^2)$$

$$\text{and } P = \frac{W}{2g}(x_2 + x_1) = \frac{Wx}{g}$$

where x is the mean distance $\frac{x_1 + x_2}{2}$ of the piston from the middle of the stroke.

The diagram is drawn to such a scale that R feet = V feet per second, or 1 foot on the diagram = $\frac{V}{R}$ feet per second; hence x feet on the diagram = $\frac{xV}{R}$ feet per second. Then, expressing the above equation in terms of feet per second, we have—

$$P = \frac{WxV}{gR}$$

and when $x = R = V$ at the beginning and end of the stroke, we have—

$$P = \frac{WV^2}{gR}$$

We shall term P the "acceleration pressure." Thus with an infinitely long connecting-rod the pressure at the end of the stroke required to accelerate or retard the reciprocating parts is equal to the centrifugal force (see p. 19), assuming the parts to be concentrated at the crank-pin, and at any other part of the stroke distant x from the middle the pressure is less in the ratio $\frac{x}{R}$.

Another simple way of arriving at the result given above is as follows: If the connecting-rod be infinitely long, then it always remains parallel to the centre line of the engine; hence the action is the same as if the connecting-rod were rigidly attached to the cross-head and piston, and the whole rotated together as one solid body, then each point in the body would describe the arc of a circle, and would be subjected to the centrifugal force $C = \frac{WV^2}{gR}$, but we are only concerned with the component along the centre line of the piston, marked P in the diagram. It will be seen that P vanishes in the middle of the stroke, and increases directly as the distance from the middle, becoming equal to C at the ends of the stroke.

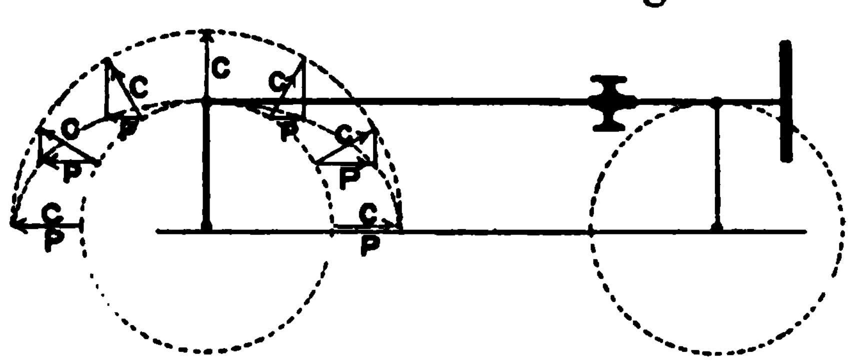


FIG. 174.

When the piston is travelling towards the middle of the stroke the pressure P is positive, and when travelling away from the middle it is negative. Thus, in constructing a diagram to show the pressure exerted at all parts of the stroke, we put the first half above, and the second half below the base-line. We show such a diagram in Fig. 175. The height of any point in the sloping line ab above the base-line represents the pressure

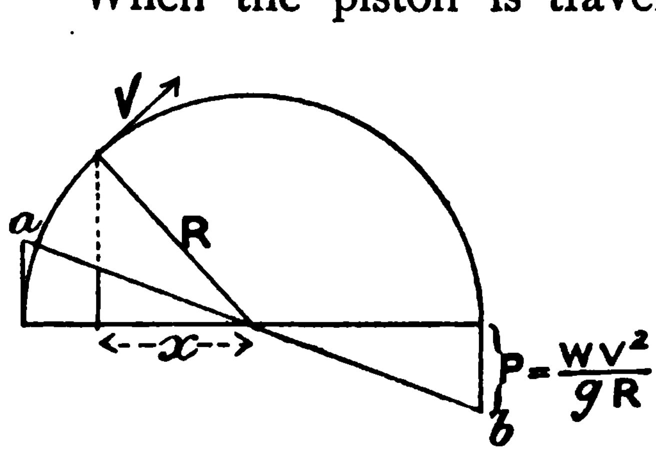


FIG. 175.

at that part of the stroke required to accelerate or retard the moving parts. It is generally more convenient to express the pressure in pounds per square inch, p, rather than the total pressure P; then $\frac{P}{A} = p$, where A = the area of the piston. We will also put $w = \frac{W}{A}$, where w is the weight of the reciprocating parts per square inch of piston. It is more usual to speak of the speed of an engine in revolutions per minute n, rather than the velocity of the crank-pin V in feet per second.

$$V = \frac{2\pi Rn}{60}$$

then, substituting these values of P, W, and V, we have—

$$p = \frac{w 2^2 \pi^2 R^2 n^2}{60^2 g R} = 0{\cdot}00034 w R n^2$$

N.B.—The radius of the crank R is measured in feet.

The quantity w varies between 2 and 6 lbs. per square inch, and occasionally values outside these limits are met with. In the absence of more accurate data, it is usual to take w = 3 lbs. per square inch. Then the above equation becomes—

$$p = 0{\cdot}001 Rn^2$$

$$\text{or } p = \frac{Rn^2}{1000}$$

which is a very convenient form of the expression for committing to memory; even if w be not equal to 3 lbs., the above expression is readily corrected by multiplying by the ratio $\frac{w}{3}$.

When working from the mean velocity of the piston instead of the velocity of the crank-pin V, it should be remembered that V is greater than the piston velocity V_p in the ratio—

$$\frac{\text{semicircumference}}{\text{diameter}} = \frac{\pi}{2} = 1{\cdot}57, \text{ or } V = 1{\cdot}57\ V_p$$

Influence of Short Connecting-rods.—Referring to Fig. 155, we there showed how to construct a velocity diagram

for a short connecting-rod; reproducing a part of the figure, we have—

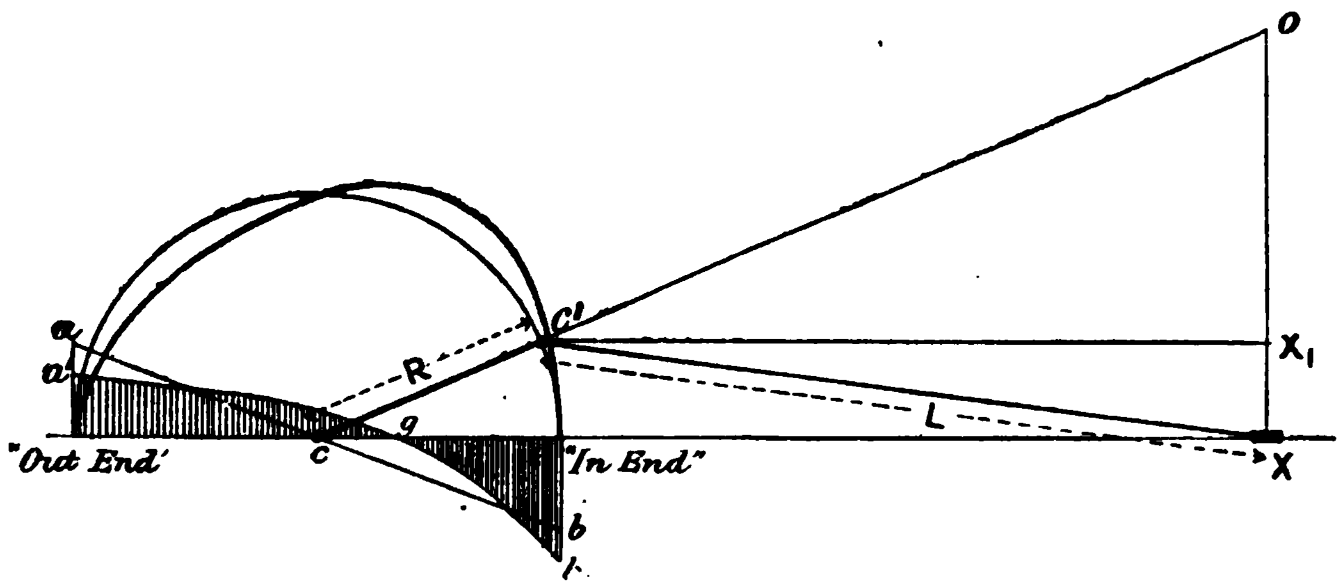


FIG. 176.

Short rod—

$$\frac{\text{cross-head velocity}}{\text{crank-pin velocity}} = \frac{OX}{OC_1}$$

Infinite rod—

$$\frac{\text{cross-head velocity}}{\text{crank-pin velocity}} = \frac{OX_1}{OC_1}$$

$$\frac{\text{velocity of cross-head with short rod}}{\text{velocity of cross-head with long rod}} = \frac{OX}{OX_1}$$

$$\text{But } \frac{OX}{OX_1} = \frac{XC}{X_1C_1} = \frac{L+R}{L}$$

$$\text{at the "in" end of the stroke} = 1 + \frac{R}{L}$$

$$\text{and at the "out" end of the stroke} = 1 - \frac{R}{L}$$

Thus if the connecting-rod is m cranks long, the pressure at the "in" end is $\frac{1}{m}$ greater, and at the "out" end $\frac{1}{m}$ less, than if the rod were infinitely long. The point g, where the curve cuts the base-line, may without sensible error be taken at the point where the connecting-rod is at right angles[1] to the crank. Then, having the three points a', g, b', the curve may be drawn in by eye.

[1] *Engineering*, July 15, 1892, p. 83.

The value of p at each end then becomes (taking $w = 3$)—

$$p = \frac{Rn^2}{1000}\left(1 + \frac{1}{m}\right) \text{ for the "in" end}$$

$$p = \frac{Rn^2}{1000}\left(1 - \frac{1}{m}\right) \text{ for the "out" end}$$

Thus, if the connecting-rod be four cranks long, the pressure p is one-fourth greater or one-fourth less according as it is the inner or outer end of the stroke.

The exact curve $a'b'$ can be obtained by the method given on p. 133, each ordinate of ab being altered in proportion to the acceleration with a short rod to the acceleration with an infinitely long rod.

In arriving at the value of w it is usual to take as reciprocating parts—the piston-head, piston-rod, tail-rod (if any), cross-head, small end of connecting rod and half the plain part of the rod. When air-pumps or other connections are attached to the cross-head, they must be taken into account in calculating the weight of the reciprocating parts; thus—

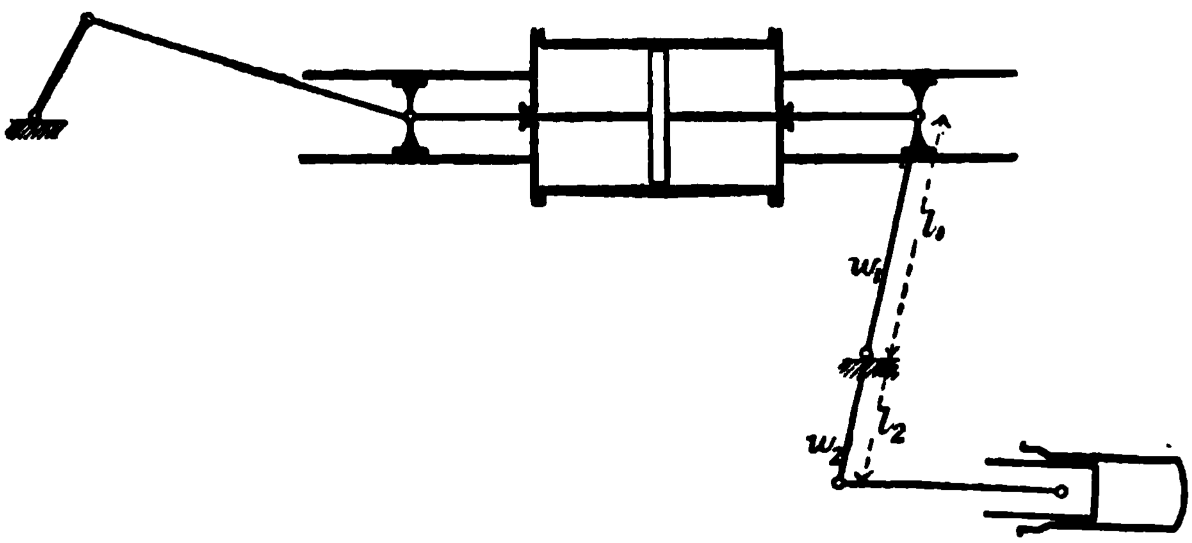


FIG. 177.

$$w = \frac{\text{weight of}\left\{\text{piston} + \text{piston and tail-rods} + \text{both cross-heads} + \text{small end of con. rod} + \dfrac{\text{plain part}}{2} + \text{air-pump plunger}\left(\dfrac{l_2}{l_1}\right)^2 + \dfrac{w_1}{4} + \dfrac{w_2}{2}\left(\dfrac{l_2}{l_1}\right)^2\right\}}{\text{area of piston}}$$

The kinetic energy of the parts varies as the square of the velocity; hence the $\left(\frac{l_2}{l_1}\right)^2$

Correction of Indicator Diagram for Acceleration Pressure.—An indicator diagram only shows the pressure of the working fluid in the cylinder; it does not show the real pressure transmitted to the crank-pin, because some of the energy is absorbed in accelerating the reciprocating parts during the first part of the stroke, and is therefore not available for driving the crank, whereas, during the latter part of the stroke, energy is given back from the reciprocating parts, and there is excess energy over that supplied from the working fluid. But, apart from these effects, a single indicator diagram does not show the impelling pressure on a piston at every portion of the stroke. The impelling pressure is really the difference between the two pressures on both sides of the piston at any one instant, hence the impelling pressure must be measured from the top line of one diagram and the bottom line of the other, as shown in full lines in Fig. 178. This diagram, set out to a straight base-line, is shown in Fig. 179, *aaa*. On the same base-line we have set off the acceleration pressure diagram, *bbb*; then, setting down on the base-line the difference between the two curves, we get the line *ccc*, giving the real pressure transmitted to the crank-pin at each instant. The area of this latter curve is, of course, equal to the area of the indicator diagram. If the engine were of the vertical type the weight of the moving parts *w* must be added or subtracted, according as it is making the up or down stroke; for the upstroke the base line becomes *dd*, and for the downstroke *ee*.

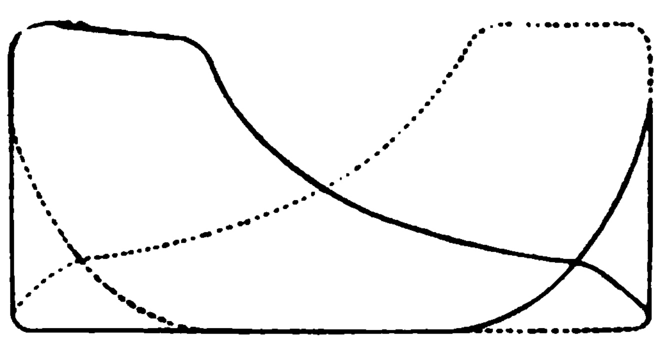


FIG. 178.

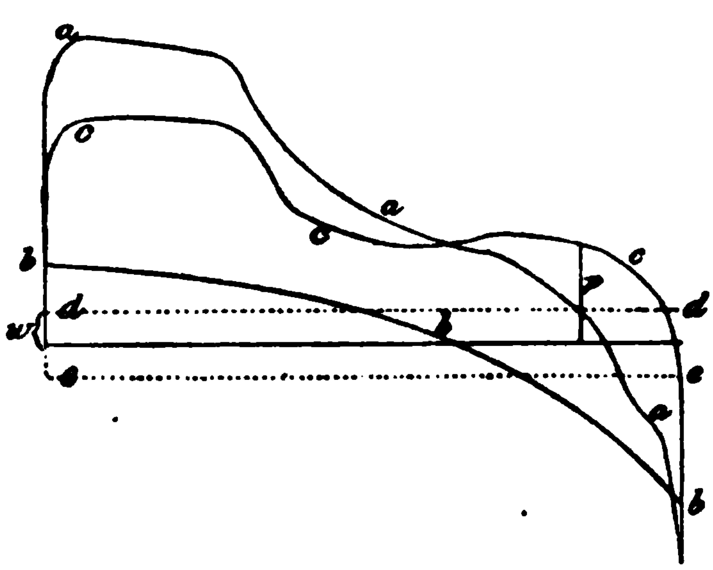


FIG. 179.

When dealing with engines having more than one cylinder, the question of scales must be carefully attended to; that is, the heights of the diagrams must be corrected in such a manner that the mean height of each shall be proportional to the total effort exerted on the piston.

Let the original indicator diagrams be taken with springs of

the following scales, H.P. $\frac{1}{x}$, I.P. $\frac{1}{y}$, L.P. $\frac{1}{z}$. Let the areas of the pistons (allowing for rods) be H.P. X, I.P. Y, L.P. Z. Let all the pistons have the same stroke. Suppose we find that the H.P. diagram is of a convenient size, we then reduce all the others to correspond with it. If, say, the intermediate piston were of the same size as the high-pressure piston, we should simply have to alter the height of the intermediate diagram in the ratio of the springs; thus—

$$\left.\begin{array}{l}\text{Corrected height of I.P. diagram}\\ \quad\text{if pistons were of same size}\end{array}\right\} = \left\{\begin{array}{l}\text{actual height of}\\ \quad\text{intermediate}\\ \quad\text{diagram}\end{array}\right\} \times \frac{\frac{1}{x}}{\frac{1}{y}}$$

$$= \text{actual height} \times \frac{y}{x}$$

But as the cylinders are not of the same size, the height of the diagram must be multiplied by the ratio of the two areas; thus—

$$\left.\begin{array}{l}\text{Height of intermediate diagram}\\ \quad\text{corrected for scales of springs}\\ \quad\text{and for areas of pistons}\end{array}\right\} = \left\{\begin{array}{l}\text{actual height of}\\ \quad\text{intermediate}\\ \quad\text{diagram}\end{array}\right\} \times \frac{y}{x} \times \frac{\text{Y}}{\text{X}}$$

$$= \text{actual height} \times \frac{y\text{Y}}{x\text{X}}$$

Similarly for the L.P. diagram:—

$$\left.\begin{array}{l}\text{Height of L.P. diagram corrected}\\ \quad\text{for scales of springs and for}\\ \quad\text{areas of pistons}\end{array}\right\} = \left\{\begin{array}{l}\text{actual height of}\\ \quad\text{L.P. diagram}\end{array}\right\} \times \frac{z\text{Z}}{x\text{X}}$$

It is probably best to make this correction for scale and area after having reduced the diagrams to the form given in line *ccc* in Fig. 179.

Pressure on the Crank-pin.—The diagram given in Fig. 179 represents the pressure transmitted to the crank-pin at all parts of the stroke. The ideal diagram would be one in which the pressure gradually fell to zero at each end of the stroke, and was constant during the rest of the stroke, such as *a*, Fig. 180.

The curve *b* shows that there is too much compression resulting in a negative pressure $-p$ at the end of the stroke; at the point x the pressure on the pin would be reversed, and, if there were any "slack" in the rod-ends, there would be a

knock at that point, and again at the end of the stroke, when the pressure on the pin is suddenly changed from $-p$ to $+p$.

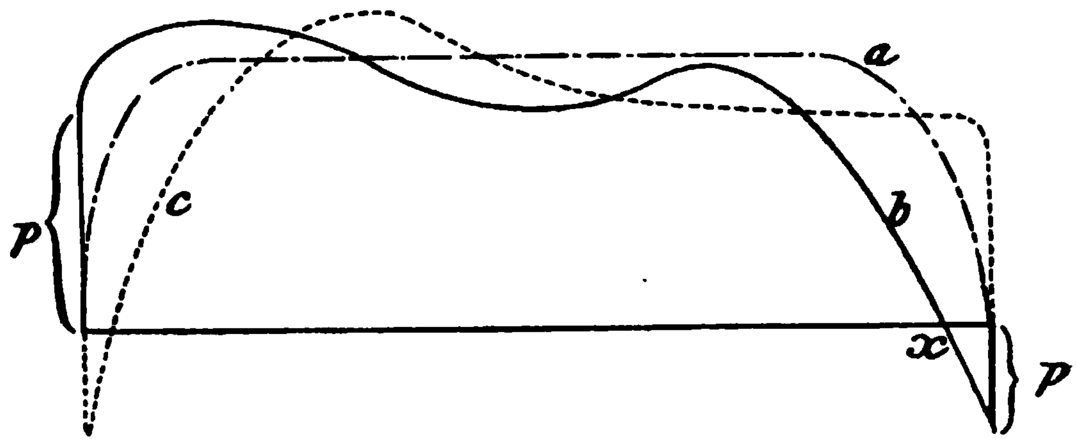


FIG. 180.

These defects could be remedied by reducing the amount of compression and the initial pressure, or by running the engine at a higher speed.

The curve (c) shows that there is a deficiency of pressure at the beginning of the stroke, and an excess at the end. The defects could be remedied by increasing the initial pressure and the compression, or by running the engine at a lower speed.

For many interesting examples of these diagrams, the reader is referred to Rigg's "Practical Treatise on the Steam Engine;" also a paper by the same author, read before the Society of Engineers.

Polar Twisting Moment Diagrams.—From the diagrams of real pressures transmitted to the crank-pin that

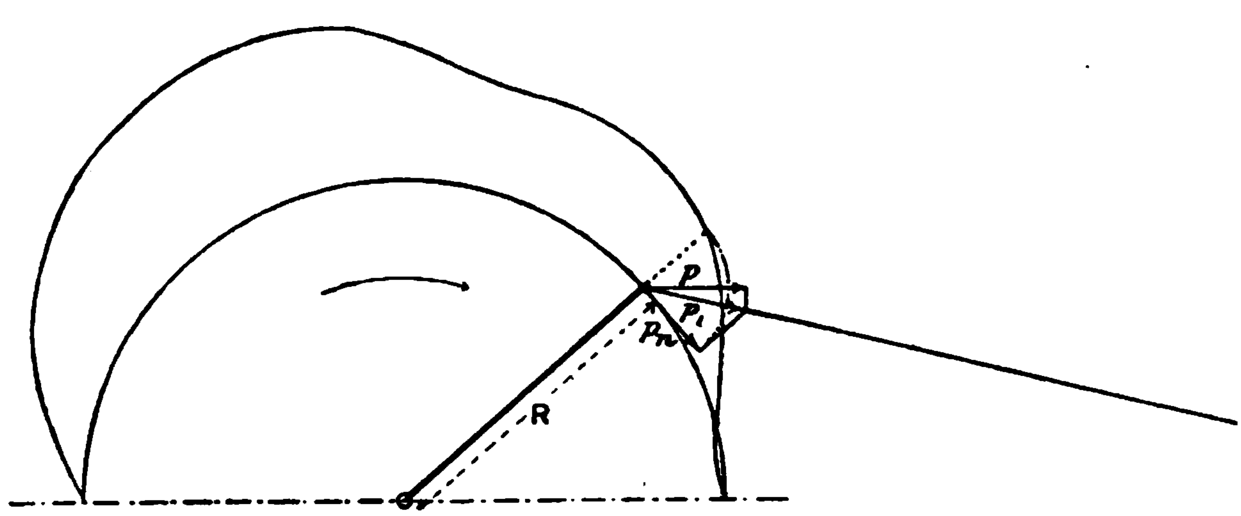


FIG. 181.

we have just constructed, we can readily determine the twisting moment on the crank-shaft at each part of the revolution.

In Fig. 181, let p be the horizontal pressure taken from such a diagram as Fig. 179. Then p_1 is the pressure transmitted

along the rod to the crank-pin. This may be resolved in a direction parallel to the crank and normal to it (p_n); we need not here concern ourselves with the pressure acting along the crank, as that will have no turning effect. The twisting moment on the shaft is then p_nR; R, however, is constant, therefore the twisting moment is proportional to p_n. By setting off values of p_n radially from the crank-circle we get a diagram showing the twisting moment at each part of the revolution. p_n is measured on the same scale, say $\frac{1}{x}$, as the indicator diagram; then, if A be the area of the piston in square inches, the twisting moment in pounds feet $= p_n x$AR, where p_n is measured in inches, and the radius of the crank R is expressed in feet.

When the curve falls inside the circle it simply indicates

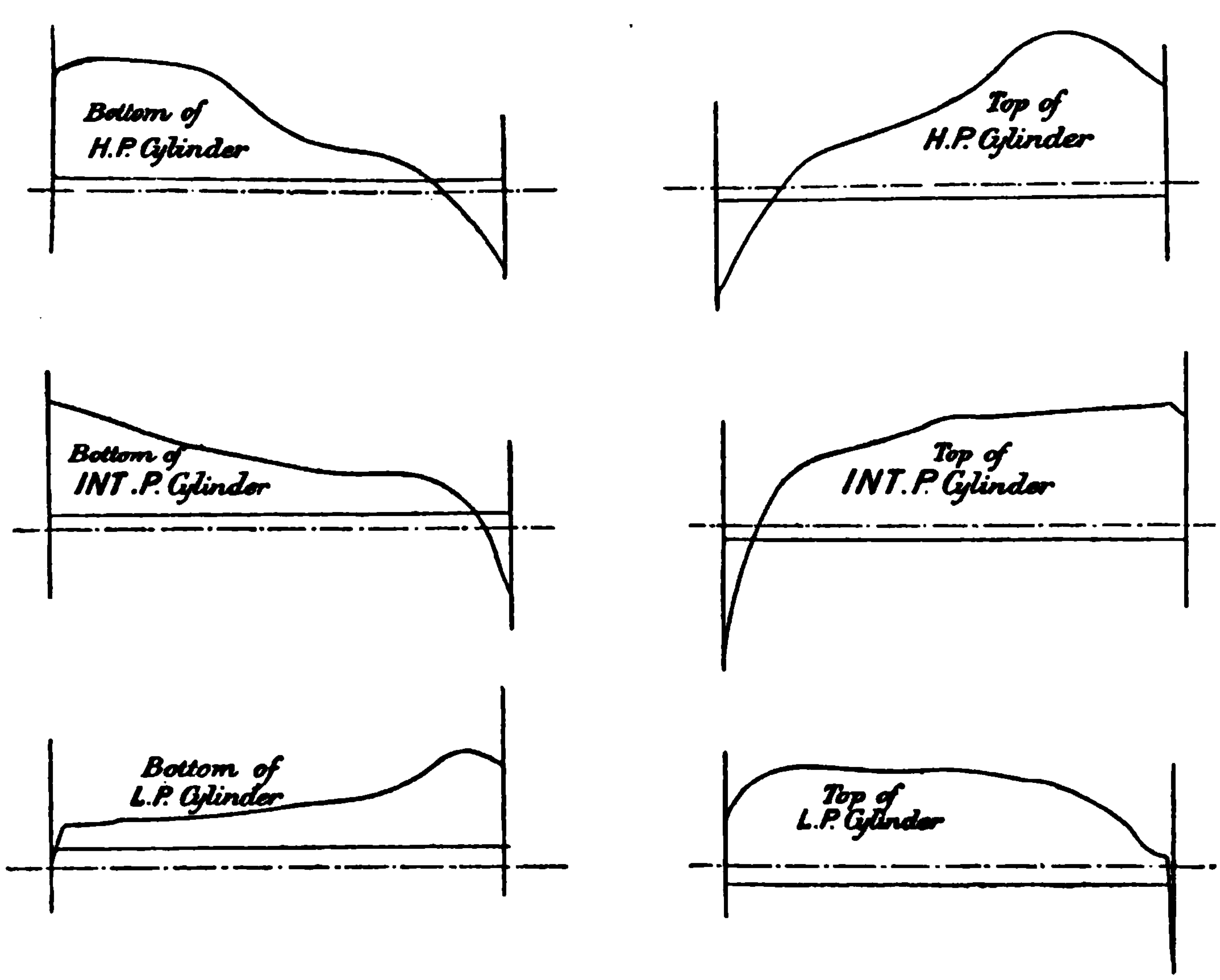


FIG. 182.

that there is a deficiency of driving effort at that place, or, in other words, that the crank-shaft is driving the piston.

In Fig. 182 we have a similar diagram, worked out fully for a vertical triple expansion engine made by Messrs. McLaren of Leeds, and by whose courtesy the author now gives it.

The dimensions of the engine were as follows:—

Diameter of cylinders—

High pressure	.. ...	9·01 inches.
Intermediate	... ...	14·25 ,,
Low pressure	. ..	22·47 ,,
Stroke	... ...	2 feet.

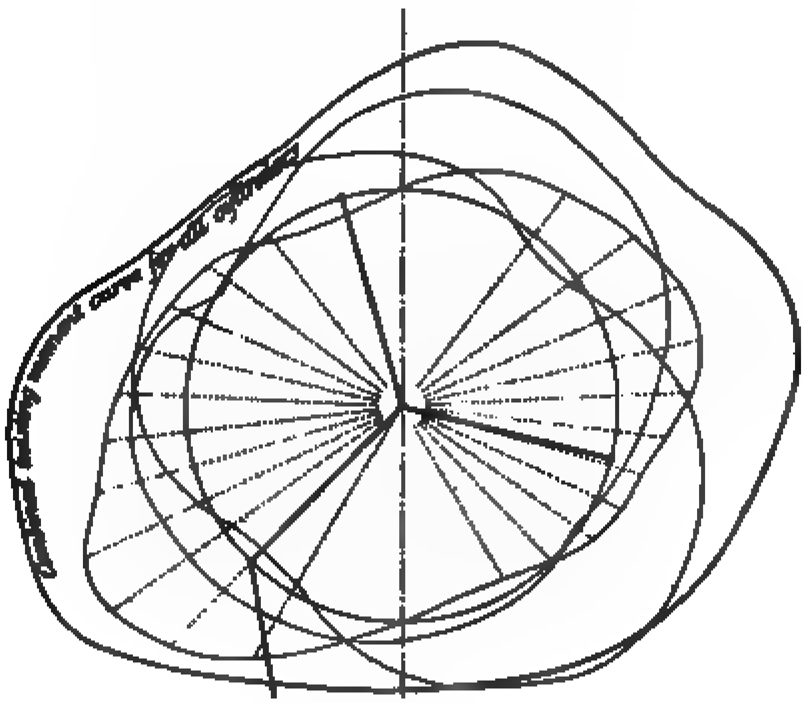


FIG. 182A

The details of reducing the indicator diagrams have been omitted for the sake of clearness; the method of reducing them has been fully described.

Flywheels.—The twisting moment diagram we have just constructed shows very clearly that the turning effort on the crank-shaft is far from being constant; hence, if the moment of resistance be constant, the angular velocity cannot be constant. In fact, the irregularity is so great in a single-cylinder engine, that if it were not for the flywheel the engine would come to a standstill at the dead centre.

A flywheel is put on a crank-shaft with the object of storing energy while the turning effort is greater than the mean, and giving it back when the effort sinks below the mean, thus making the combined effort, due to both the steam and the flywheel, much more constant than it would otherwise be, and thereby making the velocity of rotation more nearly constant. But, however large a flywheel may be, there must always be some variation in the velocity; but it may be reduced to as small an amount as we please by using a suitable flywheel.

In order to find the dimensions of a flywheel necessary for keeping the cyclical velocity within certain limits, we shall make use of the twisting moment diagram, plotted for convenience to a straight instead of a circular base-line, the length of the base being equal to the semicircumference of the crank-pin circle. Such a diagram we give in Fig. 183.[1] Its area is,

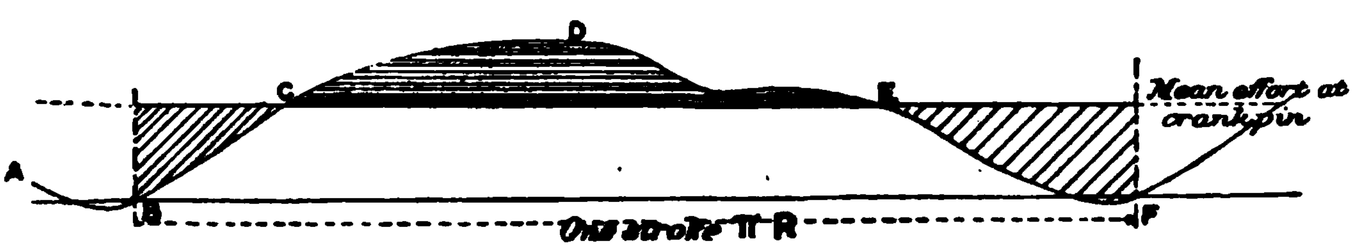


FIG. 183.

of course, equal to the area of the original indicator diagram, from which it was constructed; this check should, indeed, always be applied, to see whether the workmanship is accurate. The length of its base is greater than the length of the original indicator diagram in the ratio of π to 2; the mean height is, consequently, less in the ratio 2 to π. The resistance line, which for the present we shall assume to be straight, is shown dotted; the diagonally shaded portions below the mean line are together equal to the horizontally shaded area above.

During the period AC the effort acting on the crank-pin is

[1] Figures 178, 179, 181, 183 are all constructed from the same indicator diagram.

less than the mean, and the velocity of rotation of the crank-pin is consequently reduced, becoming a minimum at C. During the period CE the effort is greater than the mean, and the velocity of rotation is consequently increased, becoming a maximum at E.

Let V_c = minimum velocity at C of the crank-pin;
V_e = maximum " " E " "
W_f = weight of an equivalent flywheel (in pounds) whose radius is equal to the radius of the crank.

Then the energy stored in the flywheel at C $= \frac{W_f V_c^2}{2g}$

" " " at E $= \frac{W_f V_e^2}{2g}$

" " " during CE $= \frac{W_f}{2g}(V_e^2 - V_c^2)$ (i.).

This energy must have come from the steam; therefore it must be equal to the work represented by the horizontally shaded area CDE $= E_n$.

Let p_e = the mean effective pressure of CDE in pounds per square inch
A = area of piston in square inches
$\overline{CE}$ = the length CE expressed in feet, measured to same scale as πR
V = mean velocity of crank-pin
$= \frac{V_e + V_c}{2}$

hence $2V = V_e + V_c$ . . . . (ii.).

and proportional fluctuation of velocity $= \frac{V_e - V_c}{V}$

Then we have from (i.)—

$$\frac{W_f}{2g}(V_e^2 - V_c^2) = p_e A \overline{CE} = E_n$$

$$\text{and } (V_e - V_c)(V_e + V_c) = \frac{2gE_n}{W_f}$$

by substitution from i. and ii., we have—

$$(V_e - V_c)2V = \frac{2gE_n}{W_f}$$

$$V_e - V_c = \frac{gE_n}{VW_f}$$

$$\left.\begin{array}{l}\text{the proportional fluctuation}\\ \text{of velocity}\end{array}\right\} = \frac{V_e - V_c}{V} = \frac{gE_n}{V^2W_f} = K$$

K is termed the fluctuation coefficient.

K = 0·02 for electric lighting engines;
  = 0·03–0·04 for factory engines;
  = 0·06–0·10 for rough engines.

It should be noticed that 0·02 is a fluctuation of 1 per cent. on either side of the mean; similarly with the other values given.

The equivalent flywheel that we have so far considered, having a radius equal to that of the crank, would be far too heavy and costly. We can, however, enormously reduce the weight without affecting the energy stored, by increasing the radius of the flywheel.

Let R_w = radius of gyration of the flywheel (in feet)
  W_w = weight of the flywheel (in pounds)

$$\text{Then } W_w = W_f\left(\frac{R}{R_w}\right)^2 \quad . \quad . \quad . \quad . \quad . \quad \text{(ii.)}.$$

The ratio $\frac{R}{R_w}$ is usually from $\frac{1}{4}$ to $\frac{1}{5}$.

By substituting in the equation iii., we have—

$$W_w = \frac{gE_nR^2}{KV^2R_w^2} \quad . \quad . \quad . \quad . \quad . \quad \text{(iv.)}.$$

Expressing V in terms of R and N, where N is the number of revolutions per minute, and reducing the constants—

$$W_w = \frac{2937E_n}{KN^2R_w^2} \quad . \quad . \quad . \quad . \quad . \quad \text{(v.)}.$$

It is usual to neglect the energy stored in the revolving shaft, eccentrics, etc., also the arms and boss of the flywheel. R_w is usually taken as the mean radius of the rim.

Taking into account all these small revolving masses very greatly adds to the labour of the calculations, and but very slightly affects the results. The error is all in favour of steady running.

The calculations necessary for arriving at the value of E_n for any proposed flywheel is somewhat long, and the result when obtained has an element of uncertainty about it, because the indicator diagram must be assumed, as the engine so far only exists on paper. The errors involved in the diagram may

not be serious, but the desired result may be arrived at within the same limits of error by the following simpler process. The table of constants given below has been arrived at by constructing such diagrams as that given in Fig. 183 for a large number of cases. They must be taken as fair average values. The length of the connecting-rod, and the amount of pressure required to accelerate and retard the moving parts, affect the result.

Let $E_n = m \times$ work done per stroke.

Referring to Fig. 183, the area CDE is m times the whole area BCDEF.

Then—

$$E_n = \frac{m \times \text{indicated horse power of engine} \times 33000}{2N}$$

Substituting this value in equation v., we have—

$$W_w = \frac{2937 \times 33000 \times \text{I.H.P.} \times m}{2N^3KR_w^2}$$

$$W_w = \frac{48{,}500{,}000 \times \text{I.H.P.} \times m}{N^3KR_w^2} \quad . \quad . \quad \text{(vi.)}.$$

The following table gives approximate values of m. In arriving at these figures it was found that if n = number of cranks, then m varies as $\frac{1}{n^2}$ approximately.

APPROXIMATE VALUES OF m FOR DOUBLE-ACTING STEAM ENGINES.[1]

Cut-off.	Single cylinder.	Two cylinders. Cranks at right angles.	Three cylinders. Cranks at 120°.
0·1	0·35	0·088	0·040
0·2	0·33	0·082	0·037
0·4	0·31	0·078	0·034
0·6	0·29	0·072	0·032
0·8	0·28	0·070	0·031
End of stroke.	0·27	0·068	0·030

[1] The values of m vary much more in the case of two- and three-cylinder engines than in single-cylinder engines. Sometimes the value of m is twice as great as those given, which are fair averages.

m FOR GAS ENGINES.

	Single cylinder.	Two cylinders. Cranks at right angles.
Exploding at every 4th stroke ...	4·5	1·1
,, ,, 8th ,, *i.e.* missing every alternate charge	8·5	2·1

Relation between the Work stored in a Flywheel and the Work done per Stroke.—For many purposes it is convenient to express the work stored in the flywheel in terms of the work done per stroke.

$$\text{The energy stored in the wheel} = \frac{W_w V^2 R_w^2}{2gR^2}$$

Substituting the value of W_w from equation iv., we get—

$$\text{The energy stored in the wheel} = \frac{E_n}{2K}$$

$$\text{and the work done per impulse stroke} = \frac{E_n}{m}$$

$$\left.\begin{array}{l}\text{the number of impulse strokes}\\ \text{stored in flywheel}\end{array}\right\} = \frac{\dfrac{E_n}{2K}}{\dfrac{E_n}{m}} = \frac{m}{2K} \quad \text{(vii.).}$$

In the following table we give the number of strokes that must be stored in the flywheel in order to allow a total fluctuation of speed of 1 per cent., *i.e.* $\frac{1}{2}$ per cent. on either side of the mean. If a greater variation be permissible in any given case, the number of strokes must be divided by the permissible percentage of fluctuation. Thus, if 4 per cent., *i.e.* K = 0·04, be permitted, the numbers given below must be divided by 4.

NUMBER OF STROKES STORED IN A FLYWHEEL FOR DOUBLE-ACTING STEAM ENGINES.[1]

Cut-off.	Single cylinder.	Two cylinders. Cranks at right angles.	Three cylinders. Cranks at 120°.
0·1	18	4·4	2·0
0·2	17	4·1	1·9
0·4	16	3·9	1·8
0·6	15	3·6	1·7
0·8	14	3·5	1·6
End of stroke.	13	3·4	1·5

[1] See note at foot of p. 156.

GAS ENGINES (IMPULSE STROKES).

	Single cylinder.	Two cylinders. Cranks at right angles.
Exploding at every 4th stroke ...	225	56
,, ,, 8th ,, ...	425	112

Shearing, Punching, and Slotting Machines (K **not known**).—It is usual to store energy in the flywheel equal to the work done in two working strokes of the shear, punch, or slotter, amounting to about 15 inch-tons per square inch of metal sheared or punched through.

Gas Engine Flywheels.—The value of m for a gas engine can be roughly arrived at by the following method. The work done in one explosion is spread over four strokes when the mixture explodes at every cycle. Hence the mean effort is only one-fourth of the explosion-stroke effort, and the excess energy is therefore approximately three-fourths of the whole explosion-stroke effort, or three times the mean: hence $m = 3$.

Similarly, when every alternate explosion is missed, $m = 7$.

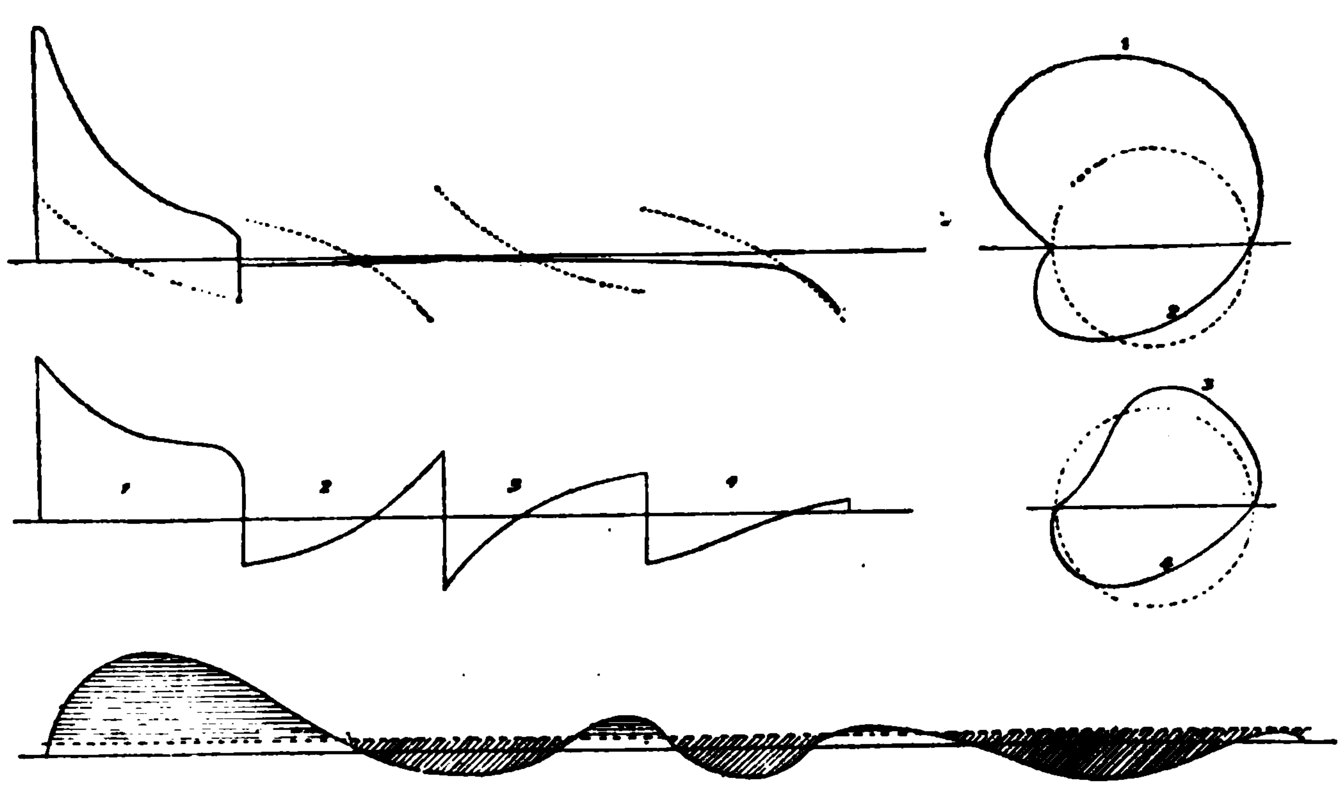


FIG. 184.

By referring to the table, it will be seen that both of these values are too low.

The diagram for a 4-stroke case is given in Fig. 184. It has

been constructed in precisely the same manner as Figs. 179, 181, and 183. When gas engines are used for driving dynamos, a small flywheel is often attached to the dynamo direct, and runs at a very much higher peripheral speed than the engine flywheel. Hence, for a given weight of metal, the small high-speed flywheel stores a much larger amount of energy than the same weight of metal in the engine flywheel.

The peripheral speed of large cast-iron flywheels has to be kept below a mile a minute (see p. 160), on account of their danger of bursting. The small disc flywheels, such as are used on dynamos, are hooped with a steel ring, shrunk on the rim, which allows them to be safely run at much higher speeds than the flywheel on the engine. The flywheel power of such an arrangement is then the sum of the energy stored in the two wheels.

The author's experience leads him to the conclusion that there is no perceptible flicker in the lights when about forty impulse strokes are stored in the flywheels; but there is a distinct flicker with only thirty strokes stored.

Case in which the Resistance varies.—In all the above cases we have assumed that the resistance overcome by the engine is constant. This, however, is not always the case; when the resistance varies, the value of E_m is found thus:

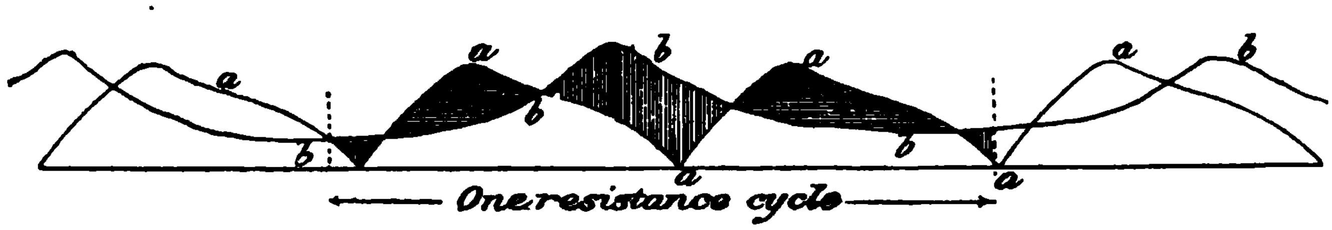


FIG. 185.

The line *aaa* is the engine curve as described above, the line *bbb* the resistance to be overcome, the horizontal shading indicates excess energy, and the vertical deficiency of energy. The excess areas are, of course, equal to the deficiency areas over any complete cycle. The resistance cycle may extend over several engine cycles; an inspection or a measurement will reveal the points of maximum and minimum velocity. The value of m is the ratio of the horizontal shaded areas to the whole area under the line *aaa* described during the complete cycle of operations. For a full treatment of this, the reader is referred to an article by Prof. R. H. Smith in the *Engineer* of January 9, 1885.

Stress in Flywheel Rims.—If we neglect the effects of the arms, the stress in the rim of a flywheel may be treated in

the same manner as the stresses in a boiler-shell or, more strictly, a thick cylinder (see p. 286) in which we have the relation—

$$P_rR_w = ft$$

$$\text{or } P_rR_w = f, \text{ when } t = 1 \text{ inch}$$

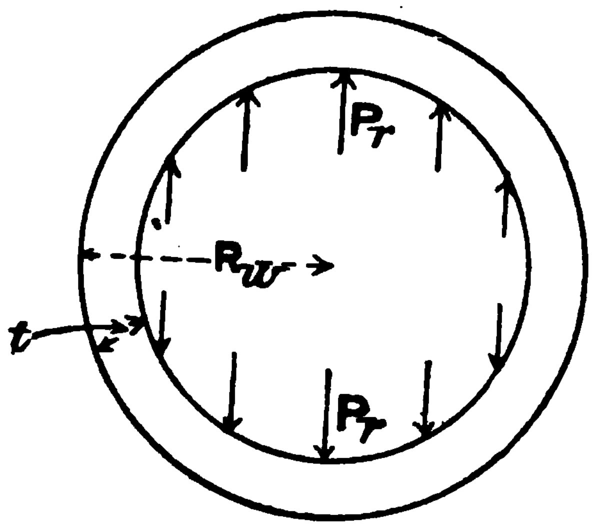


FIG. 186.

The P_r in this instance is the pressure on each unit length of rim due to centrifugal force; we shall find it convenient to take the unit of length as 1 foot, because we take the velocity of the rim in *feet* per second. Then—

$$P_r = \frac{W_rV_w^2}{gR_w}, \text{ and } f = \frac{W_rV_w^2}{g}$$

where W_r = the weight of 1 foot length of rim, 1 square inch in section
= 3·1 lbs. for cast iron

We take 1 *sq. inch* in section, because the stress is expressed in pounds per square *inch*. Then substituting the value of P_r and W_r in the above equations, we have—

$$\frac{3 \cdot 1 V_w^2}{32 \cdot 2} = f$$

$$f = 0 \cdot 096 V_w^2$$

$$\text{or } f = \frac{V_w^2}{10} \text{ (very nearly)}$$

For a fuller treatment, taking into account the effect of the arms, etc., the reader is referred to Unwin's "Elements of Machine Design," Part II.

In English practice V_w is rarely allowed to exceed 100 feet per second, but in American practice much higher speeds are often used, probably due to the fact that American cast iron is much tougher and stronger than the average metal used in England. An old millwright's rule was to limit the speed to a mile a minute, *i.e.* 88 feet per second, corresponding to a stress of about 800 lbs. per square inch.

Bending Stresses in Locomotive Coupling-rods.—Each point in the rod describes a circle (relatively to the engine) as the wheels revolve, hence each particle of the rod is subjected to an upward and downward force equal to the centrifugal force when the rod is in its top and bottom

positions. As we shall express the stress in the rod in pounds per square inch, we must express the bending moment on the

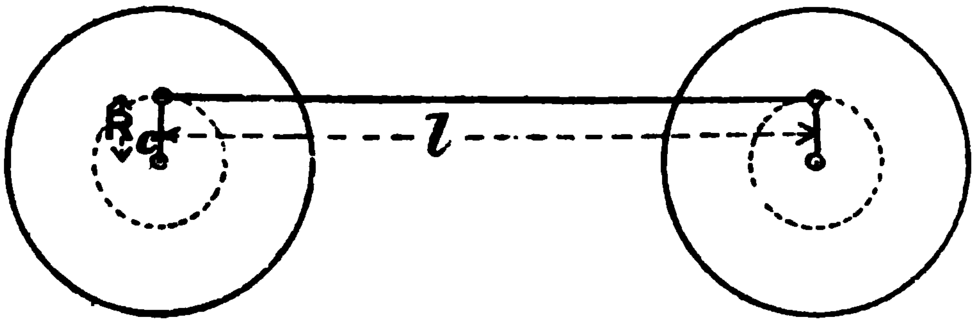


FIG. 187.

rod in pound-inches; hence we take the length of the rod l in inches. In the expression for centrifugal force we have feet units, hence the radius of the coupling crank must be in feet.

$$\left.\begin{array}{r}\text{The centrifugal force acting}\\ \text{on the rod per } \textit{inch} \text{ run}\end{array}\right\} = C = 0{\cdot}00034wR_cN^2$$

where w is the weight of the rod in pounds per inch run, or $w = 0{\cdot}28$A pounds, where A is the sectional area of the rod.

The centrifugal force is an evenly distributed load all along the rod; then, assuming the rod to be parallel, we have—

$$\left.\begin{array}{r}\text{The maximum bending moment}\\ \text{in the middle of the rod, M}\end{array}\right\} \quad \frac{Cl^2}{8} = \frac{fA\kappa^2}{y}$$

(see Chapters IX. and X.)

where κ^2 = the square of the radius of gyration (inch units) about a horizontal axis through the c. of g.;

y = the half-depth of the section (inches).

Then, substituting the value of C, we have—

$$f = \frac{0{\cdot}00034 \times 0{\cdot}28 \times A \times R_c \times N^2 \times l^2 \times y}{8 \times A \times \kappa^2}$$

$$f = \frac{0{\cdot}000012R_cN^2l^2y}{\kappa^2}, \text{ or } = \frac{R_cN^2l^2y}{84{,}000\kappa^2}$$

The value of κ^2 can be obtained from Chapter III. For a rectangular section, $\kappa^2 = \frac{h^2}{12}$; and for an **I** section, $\kappa^2 = \frac{BH^3 - bh^3}{12(BH - bh)}$.

It should be noticed that the stress is independent of the sectional area of the rod, but that it varies inversely as the

square of the radius of gyration of the section; hence the importance of making rods of **I** section, in which the metal is placed as far from the neutral axis as possible. If the stress be calculated for a rectangular rod, and then for the same rod fluted by milling out the sides, it will be found that the fluting very materially strengthens the rod.

In addition to the bending stress in a vertical plane, there is also a direct stress of uniform intensity acting over the section of the rod, sometimes in tension and sometimes in compression; such stresses, if they could be accurately determined, would be added to the skin stress found above.

Bending Stress in Connecting-rods.—When calculating the stresses in a connecting-rod, we may regard it as a rod swinging about the small or gudgeon end. The kinetic energy of the rod in a vertical plane will be due to the velocity of its centre of gyration, assuming the rod to be parallel throughout its length; the centre of gyration will be at a distance $= \frac{l}{\sqrt{3}} = 0{\cdot}577l$ (see p. 78) from the gudgeon end; hence the radius of the centre of gyration as the rod swings will be $0{\cdot}577\ R_c$, and from the expression for the coupling rod we have—

$$f = \frac{0{\cdot}577 R_c N^2 l^2 y}{84{,}000\kappa^2} = \frac{R_c N^2 l^2 y}{146{,}000\kappa^2}$$

When valve gears are driven from connecting-rods, the bending stresses may be considerably increased.

Balancing revolving Axles.

CASE I. "*Standing Balance.*"—If an unbalanced pulley or wheel be mounted on a shaft and the shaft laid across two levelled straight-edges, the shaft will roll until the heavy side of the wheel comes to the bottom.

If the same shaft and wheel were mounted in bearings and rotated rapidly, the centrifugal force acting on the unbalanced portion would cause a pressure on the bearings acting always in the direction of the unbalanced portion; if the bearings were very slack and the shaft light, it would lift bodily at every revolution. In order to prevent this action, a balance weight or weights must be attached to the wheel *in its own plane of rotation*, with the centre of gravity diametrically opposite to the unbalanced portion.

Let W = the weight of the unbalanced portion;
W_1 = " " balance weight,

r = the radius of the c. of g. of the unbalanced portion,
r_1 = the radius of the c. of g. of the balance weight.

Then, in order that the centrifugal force acting on the balance weight may exactly counteract the centrifugal force acting on the unbalanced portion, we must have—

$$0{\cdot}00034\text{WRN}^2 = 0{\cdot}00034\text{W}_1\text{R}_1\text{N}^2$$
$$\text{or WR} = \text{W}_1\text{R}_1$$
$$\text{or WR} - \text{W}_1\text{R}_1 = 0$$

that is to say, the algebraic sum of the moments of the rotating weights about the axis of rotation must be zero, which is equivalent to saying that the centre of gravity of all the rotating weights must coincide with the axis of rotation. When this is the case the shaft will not tend to roll on levelled straight-edges, and therefore the shaft is said to have "standing balance."

When a shaft has standing balance, it will also be perfectly balanced at all speeds, *provided that all the weights rotate in the same plane.*

We must now consider the case in which all the weights do not rotate in the same plane.

CASE II. *Running Balance.*—If we have two or more weights attached to a shaft which fulfil the conditions for standing balance, but yet do not rotate in the same plane, the shaft will no longer tend to lift bodily at each revolution; but it will tend to wobble, that is, it will tend to turn about an axis perpendicular to its own when it rotates rapidly. If the bearings were very slack, it would trace out the surface of a double cone in space as indicated by the dotted lines, and the axis would be constantly shifting its position, *i.e.* it would not be permanent. The reason for this is, that the two centrifugal forces c and c_1 form a couple, tending to turn the shaft about some point A between them. In order to counteract this turning action, an equal and opposite couple must be introduced by placing balance weights diametrically opposite, which fulfil the conditions both for "standing

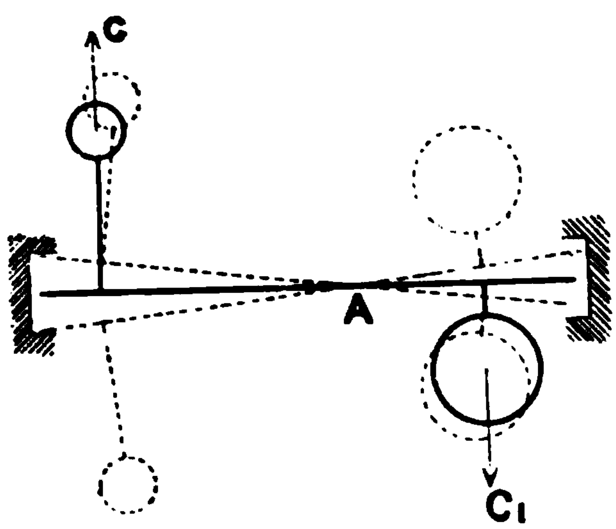


FIG. 188.

balance" and that their centrifugal movements about any point in the axis of rotation are equal and opposite in effect to those

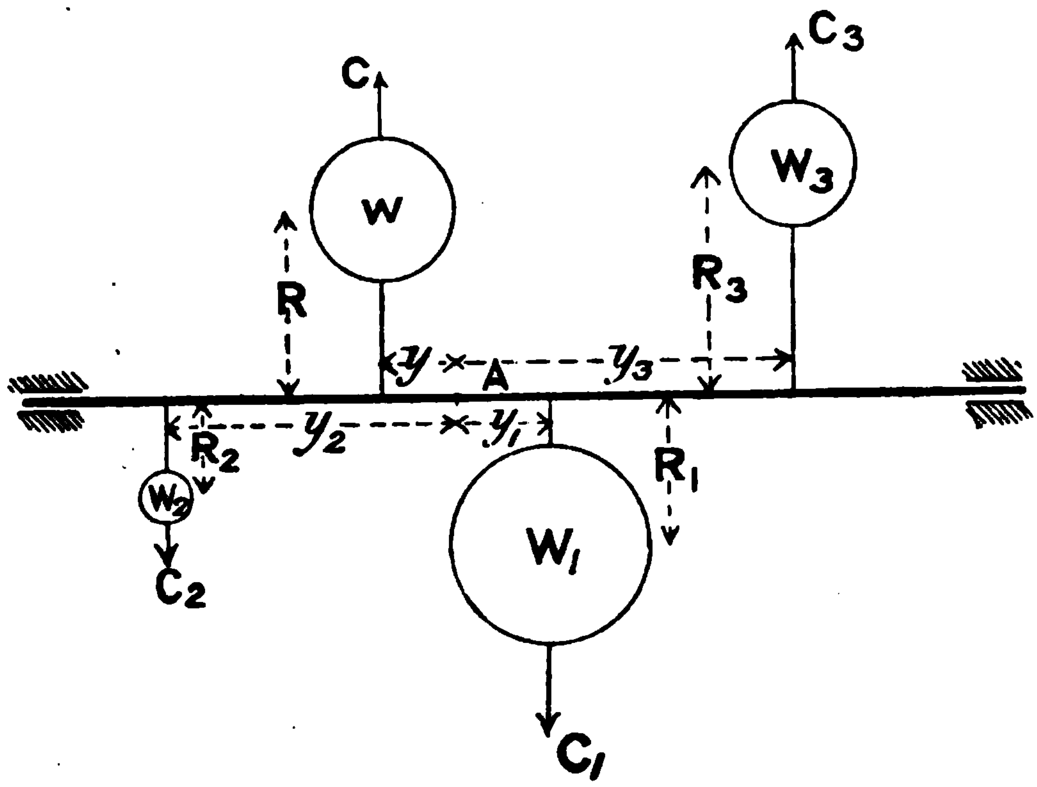


FIG. 189.

of the original weights. Then, of course, the algebraic sum of all the centrifugal moments is zero, and the shaft will have no tendency to wobble, and the axis of rotation will be permanent.

In the figure, let the weights W and W_1 be the original weights, balanced as regards "standing balance," but when rotating they exert a centrifugal couple tending to alter the direction of the axis of rotation. Let the balance weights W_2 and W_3 be attached to the shaft in the same plane as W_1 and W, *i.e.* diametrically opposite to them, also having "standing balance." Then, in order that the axis may be permanent, the following condition must be fulfilled :—

$$cy + c_1y_1 = c_2y_2 + c_3y_3$$
$$0{\cdot}00034\text{N}^2(\text{WR}y+\text{W}_1\text{R}_1y_1) = 0{\cdot}00034\ \text{N}^2(\text{W}_2\text{R}_2y_2+\text{W}_3\text{R}_3y_3)$$
$$\text{or WR}y + \text{W}_1\text{R}_1y_1 - \text{W}_2\text{R}_2y_2 - \text{W}_3\text{R}_3y_3 = 0$$

The point A, about which the moments are taken, may be chosen anywhere along the axis of the shaft without affecting the results in the slightest degree. Great care must be taken with the signs, viz. a + sign for a clockwise moment, and a − sign for a contra-clockwise moment.

The condition for standing balance in this case is—

$$\text{WR} + \text{W}_1\text{R}_1 - \text{W}_2\text{R}_2 - \text{W}_3\text{R}_3 = 0$$

It must be very carefully noted that in each case the balance weight must be placed diametrically opposite to the

weight to be balanced. In some cases this may lead to more than one balance weight in a plane of rotation; the reduction to one equivalent weight is a simple matter, and will be dealt with shortly. Then, remembering this condition, the only other conditions for securing a permanent axis of rotation, or a "running balance," are—

$$\Sigma WR = 0$$
$$\text{and } \Sigma WRy = 0$$

Where ΣWR is the algebraic sum of the moments of all the rotating weights about the axis of rotation, and y is the distance, measured parallel to the shaft, of the plane of rotation of each weight from some given point in the axis of rotation. Thus the c. of g. of all the weights must lie in the axis of rotation.

Balancing Locomotives.—In order that a locomotive may run steadily at high speeds, the rotating and reciprocating parts must be very carefully balanced. If the rotating parts be left unbalanced, there will be a serious blow on the rails every time the unbalanced portion gets to the bottom; this is known as the "hammer blow." If the reciprocating parts be left unbalanced, the engine will oscillate to and fro at every revolution about a vertical axis situated near the middle of the crank-shaft; this is known as the "elbowing action."

By balancing the rotating parts the hammer blow may be overcome, but then the engine will elbow; if, in addition, the reciprocating parts be entirely balanced, the engine will be overbalanced vertically; hence we have to compromise matters by only partially balancing the reciprocating parts. Then, again, the obliquity of the connecting-rod causes the pressure due to the inertia of the reciprocating parts to be greater at one end of the stroke than at the other, a variation which cannot be compensated for by balance weights rotating at a constant radius.

Thus we see that it is absolutely impossible to perfectly balance a locomotive of ordinary design, and the compromise we adopt must be based on experience.

The following symbols will be used in the paragraphs on locomotive balancing:—

W_r, for rotating weights (pounds) to be balanced.

W_ρ, for reciprocating weights (pounds) to be balanced.

W_B, for balance weights; if with a suffix ρ, as $W_{B\rho}$, it will indicate the balance weight for the reciprocating parts, and so on with other suffixes.

R, for radius of crank (feet).
R_c, „ „ coupling-crank.
R_B, „ „ balance weights.

Rotating Parts of Locomotive.—The balancing of the rotating parts is effected in the manner described in the paragraph on standing balance, p. 145, which gives us—

$$W_r R = W_{Br} R_B$$
$$\text{and } W_{Br} = \frac{W_r R}{R_B}$$

The weights included in the W_r vary in different types of engines; we shall consider each as we come to it.

Reciprocating Parts of Locomotive.—We have already shown (p. 145) that the acceleration pressure at the

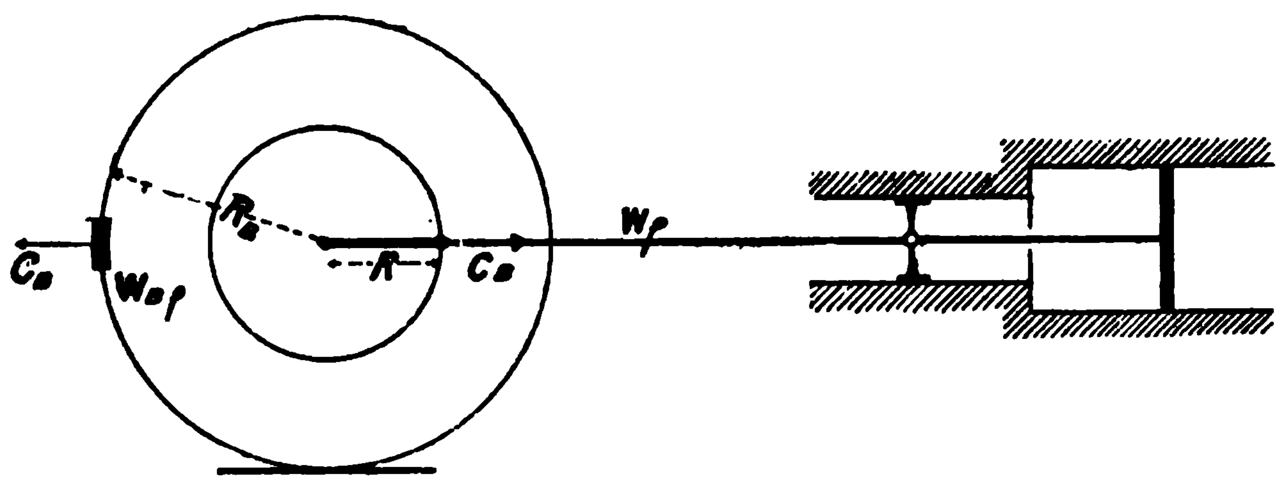


FIG. 190.

end of the stroke due to the reciprocating parts is equal to the centrifugal force, assuming them to be concentrated at the crank-pin, neglecting the obliquity of the connecting-rod.

Then, for the present, assuming the balance weight to rotate in the plane of the crank-pin, in order that the reciprocating parts may be balanced, we must have—

$$C_B = C$$
$$0{\cdot}00034 W_{B\rho} \,.\, R_B \,.\, N^2 = 0{\cdot}00034 W_\rho \,.\, R \,.\, N^2$$
$$W_{B\rho} \,.\, R_B = W_\rho \,.\, R$$
$$\text{and } W_{B\rho} = \frac{W_\rho \,.\, R}{R_B} \quad \ldots \ldots \quad \text{(i.)}$$

On comparing this with the result obtained for rotating parts, we see that reciprocating parts, when the obliquity of the connecting-rod is neglected, may for every purpose be regarded as though their weight were concentrated in a heavy ring round the crank-pin.

Now we come to a much-discussed point. We showed above that with a short connecting-rod of n cranks long, the acceleration pressure was $\frac{1}{n}$ greater at one end and $\frac{1}{n}$ less at the other end of the stroke, than the pressure with an infinitely long rod; hence if we make $W_{B\rho}$ $\frac{1}{n}$ greater to allow for the obliquity of the rod at one end, it will be $\frac{2}{n}$ too great at the other end of the stroke. Thus we really do mischief by attempting to compensate for the obliquity of the rod at either end; we shall therefore proceed as though the rod were of infinite length.

If the reader wishes to follow the effect of the obliquity of the rod at all parts of the stroke, he should consult a paper by Mr. Hill in the *Proceedings of the Institute of Civil Engineers*, vol. civ., or Barker's "Graphic Methods of Engine Design."

There is yet another point upon which there is a great difference of opinion, viz. what proportion of the connecting-rod should be regarded as rotating and what proportion as reciprocating. As a matter of fact, there is no room for difference of opinion here, for an exact solution of the problem is possible, though rather long. Some writers on this subject evidently find much pleasure in indulging in pages of abstruse mathematics on this point, but their labour is in vain, for, do what we may, we cannot perfectly balance an engine as ordinarily built; and as we have to arbitrarily decide upon some proportion of the reciprocating parts that we will balance, viz. *about* two-thirds, it is folly to bother about a matter which may affect the result to 1 or 2 per cent. while we decide to leave unbalanced about 33 per cent. in wholesale fashion.

In this connection, we shall assume that the big end of the connecting-rod and half the plain part rotates, while the small end and the other half of the rod reciprocates.

Inside-cylinder Engine (uncoupled).—In this case we have—

W_ρ = weight of (piston + piston-rod + cross-head + small end of connecting-rod + $\frac{1}{2}$ plain part of rod);

W_r = weight of (crank-pin + crank-webs[1] + big end of connecting-rod + $\frac{1}{2}$ plain part of rod).

[1] See p. 174.

If we could arrange balance weights to rotate in the same plane as the crank-pins, the weight of each would be $W_{Br} + W_{Bp}$, placed at the radius R_B, and if we only counterbalance two-thirds of the reciprocating parts, we should get each balance weight—

$$W_{B0} = \frac{R(\frac{2}{3}W_p + W_r)}{R_B} \quad . \quad . \quad . \quad . \quad . \quad \text{(ii.)}$$

Balance weights cannot, however, be arranged to rotate in the same planes as the crank-pins. They might, of course, be placed opposite the crank-webs, but for many reasons such a position would be inconvenient; they are therefore distributed over the wheels in such a manner that their centrifugal moments about the plane of rotation of the crank-pin is zero. If W be one weight, and W_1 the other, distant y' and y_1' from the plane of the crank, then—

$$WR_By' = W_1R_By_1'$$
$$\text{or } Wy' = W_1y'_1$$

which is equivalent to saying that the centre of gravity of the two weights lies in the plane of rotation of the crank. The object of this particular arrangement is to keep the axis of rotation permanent. Then, considering the vertical crank shown in Fig. 191, by taking moments, we get the equivalent weights at the wheel centres as given in the figure.

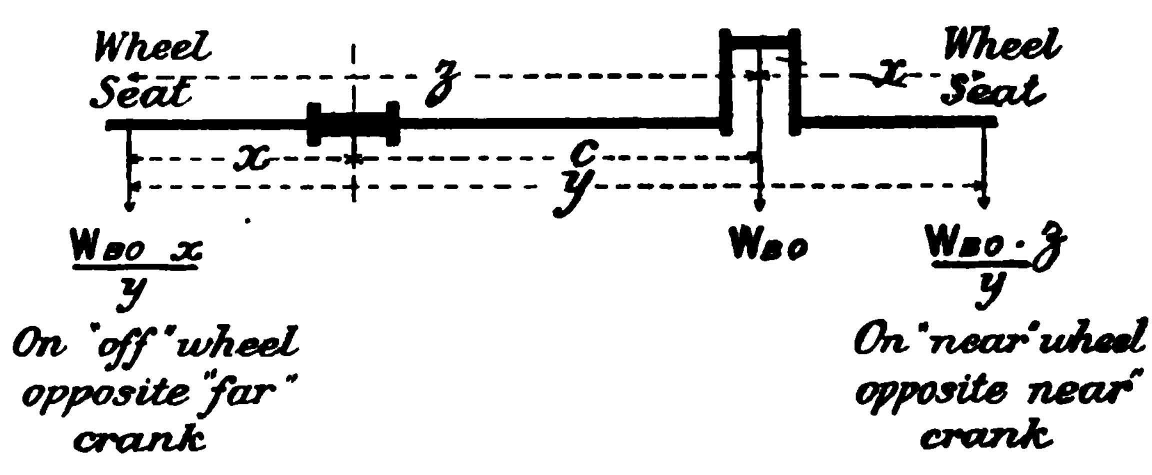


FIG. 191.

We have, from the figure—

$$x = \frac{y}{2} - \frac{c}{2} \qquad x = \frac{y - c}{2}$$

$$z = \frac{y}{2} + \frac{c}{2} \qquad z = \frac{y + c}{2}$$

Substituting these values, we get—

$\frac{W_{B0}}{2y}(y - c) = W_{B1}$, as the proportion of the balance weight on the "off" wheel opposite the far crank

and $\frac{W_{B0}}{2y}(y + c) = W_{B2}$, as the proportion of the balance weight on the "near" wheel opposite near crank

Exactly similar balance weights are required for the other crank. Thus on each wheel we get one large balance weight W_{B2} at N, Fig. 192, opposite the near crank, and one small one W_{B1} at F, opposite the far crank. Such an arrangement would, however, be very clumsy, so we shall combine the two balance weights by the parallelogram of forces as shown, and for them substitute the large weight W_B at M.

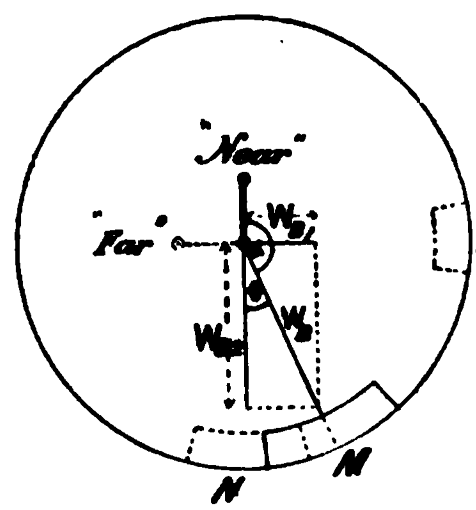


FIG. 192.

Then $W_B = \sqrt{W_{B1}^2 + W_{B2}^2}$

On substituting the values given above for W_{B1} and W_{B2}, we have, when simplified—

$$W_B = \frac{\sqrt{2}W_{B0}}{2y}\sqrt{y^2 + c^2}$$

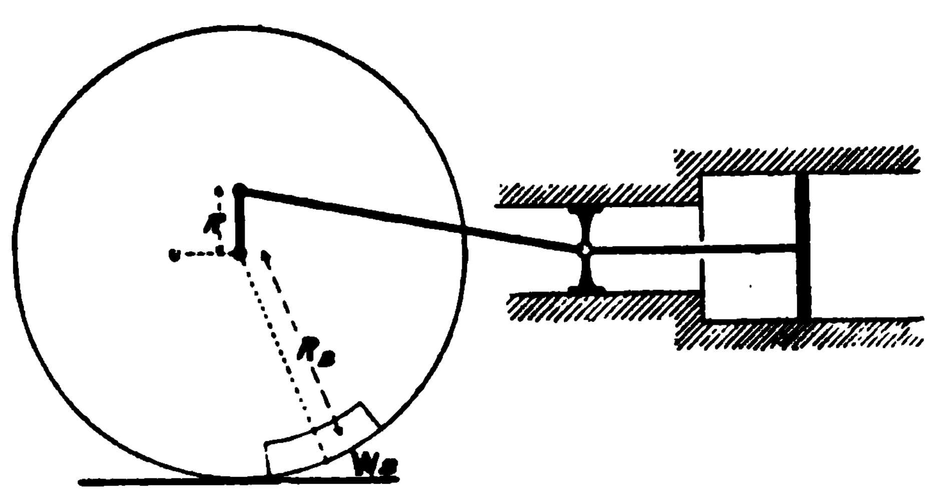


FIG. 193.

In English practice $y = 2{\cdot}5\ c$ (approximately)

On substitution we get—

$$W_B = 0{\cdot}76W_{B0}$$

Substituting from ii., we have—

$$W_B = \frac{0{\cdot}76R(\frac{2}{3}W_p + W_r)}{R_B}$$

Let the angle between the final balance weight and the near crank be a, and the far crank θ.

$$\text{Then } a = 180 - \theta$$
$$\text{and } \tan\theta = \frac{W_{B1}}{W_{B2}} = \frac{y-c}{y+c}$$

Substituting the value of y for English practice, we get—

$$\tan\theta = \frac{1{\cdot}5}{5{\cdot}3} = 0{\cdot}429$$
$$\theta = 23^\circ$$

Now, $\theta = \frac{90}{4}$ very nearly; hence, for English practice, if the quadrant opposite the crank quadrant be divided into four equal parts, the balance weight must be placed on the first of these, counting from the line opposite the near crank.

Outside-cylinder Engine (uncoupled).—W_r and W_p are the same as in the paragraph above. The plane of rotation of the crank-pin in an outside-cylinder engine so nearly coincides with the plane of rotation of the balance weight that they are usually assumed to be one and the same; the error involved is very small.

Then, by the same reasoning as given above, we have—

$$W_B = W_{B0} = \frac{R(\frac{2}{3}W_p + W_r)}{R_B}$$

and the balance weight is placed diametrically opposite the crank.

Inside-cylinder Engine (coupled).—In this case we have W_p the same as in the previous cases.

W_C = the weight of coupling crank-web and pin[1] + coupling rod from a to b, or c to d, or b to c (Fig. 195), as the case may be;

W_{BC} = the weight of the balance weight required to counterbalance the coupling attachments;

R_C = the radius of the coupling crank.

In the case of the driving-wheel of the four-wheel coupled

[1] See p. 174.

engine, we have W_B arrived at in precisely the same manner as in the case of the inside-cylinder uncoupled engine, and $W_{BC} = \frac{R_C W_C}{R_B}$.

The portion of the coupling rod included in the W_C is, in this case, one-half the whole rod. The balance weight W_{BC} is

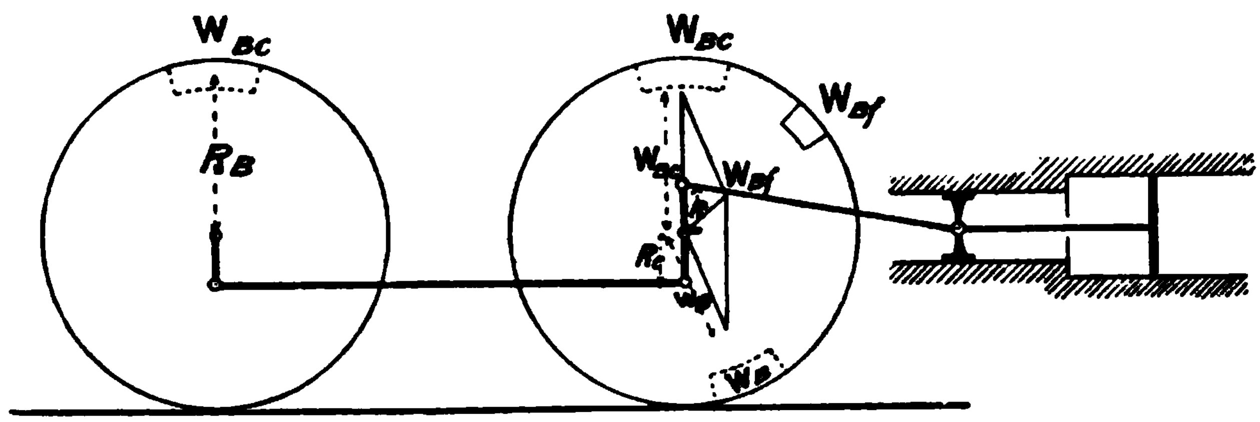


FIG. 194.

placed diametrically opposite the coupling crank-pin. After finding W_B and W_{BC}, they are combined in one weight W_{BF} by the parallelogram of forces, as already described.

With this type of engine the balance weight is always small. Sometimes the weights of the rods are so adjusted that a balance weight may be dispensed with on the driving-wheel.

But in this case it would be better to set back the coupling crank to bring W_{BC} diametrically opposite to W_B, and arrange the rods so that the two weights would be equal.[1]

On the coupled wheel the balance weight W_{BC} is of the same value as that given above, and is placed diametrically opposite the coupling crank-pin.

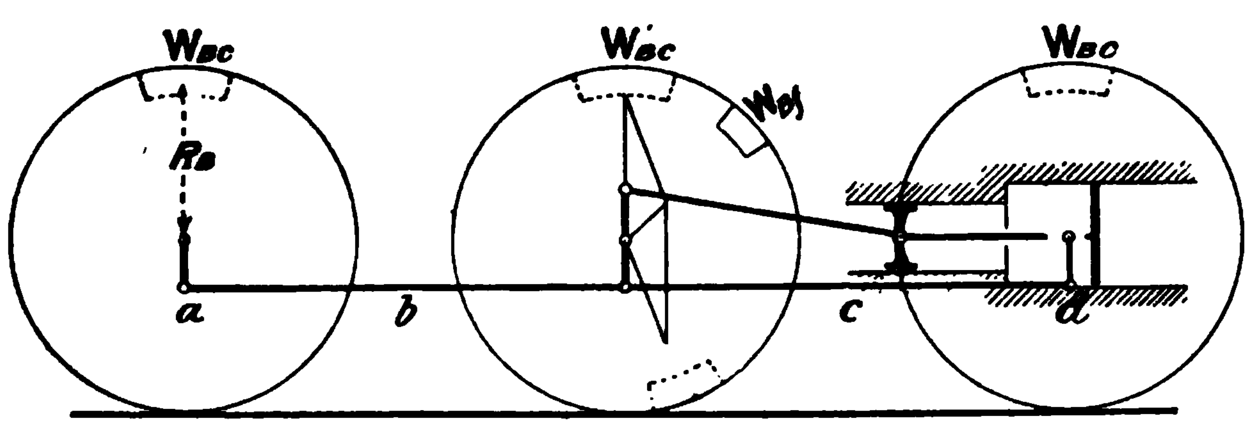


FIG. 195.

In the six-wheel coupled engine the method of treatment is precisely the same, but one or two points require notice.

[1] The author is not aware that this has ever been done.

$$W'_{BC} = \frac{R_C W_C}{R_B}$$

The portion of the coupling rod included in the W_C is from *b* to *c*; whereas in the W_{BC} the portion is from *a* to *b* or *c* to *d*.

Coupling cranks[1] have been placed *with* the crank-pins; the balance weights then become very much greater. They are treated in precisely the same way.

Some locomotive-builders evenly distribute the balance weights on coupled engines over all the wheels: most authorities strongly condemn this practice. Space will not allow of this point being discussed here.

Outside-cylinder Engine, coupled.

W_p is the same as before;

W_r is the weight of crank-web[2] and pin + coupling rod from *a* to *b* + big end of connecting-rod + half plain part of rod;

W_C is the same as in the last paragraph;

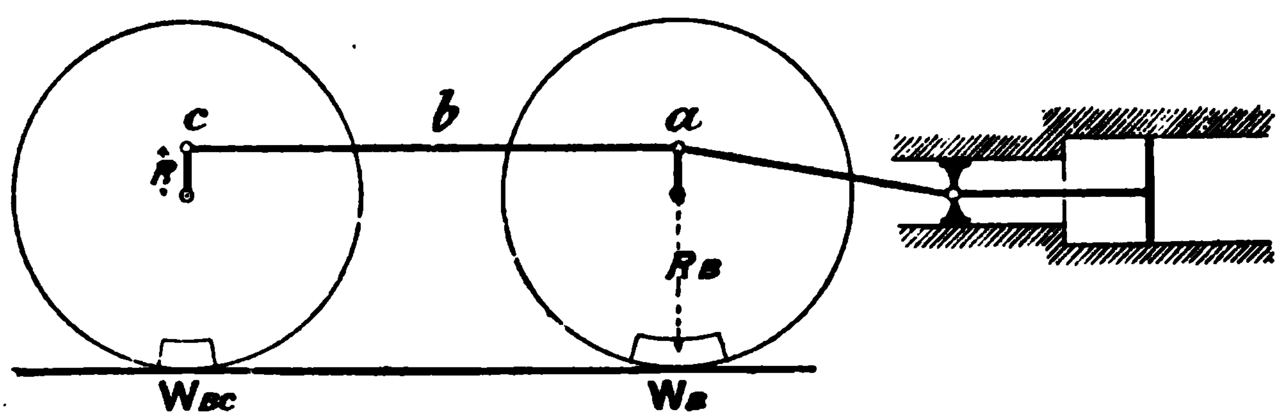


FIG. 196.

$$R_C = R;$$

$$W_B = W_{B0} = \frac{R(\frac{2}{3}W_p + W_r)}{R_B}$$

$$W_{BC} = \frac{RW_C}{R_B}$$

The six-wheel coupled engine is treated in a similar way; the remarks in the last paragraph also apply here.

Centre of Gravity of Balance Weights and Crank-webs.—The usual methods adopted for finding the position and weight of balance weights are long and tedious; the following method will be found more convenient. The effective balance weight is the whole weight, *minus* the weight of the spokes embedded.

[1] See *Proc. Inst. C.E.*, vol. lxxxi., p. 122.   [2] See p. 174.

Let Figs. 197, 198, 199 represent sections through a part of the balance weight and a spoke; then, instead of dealing

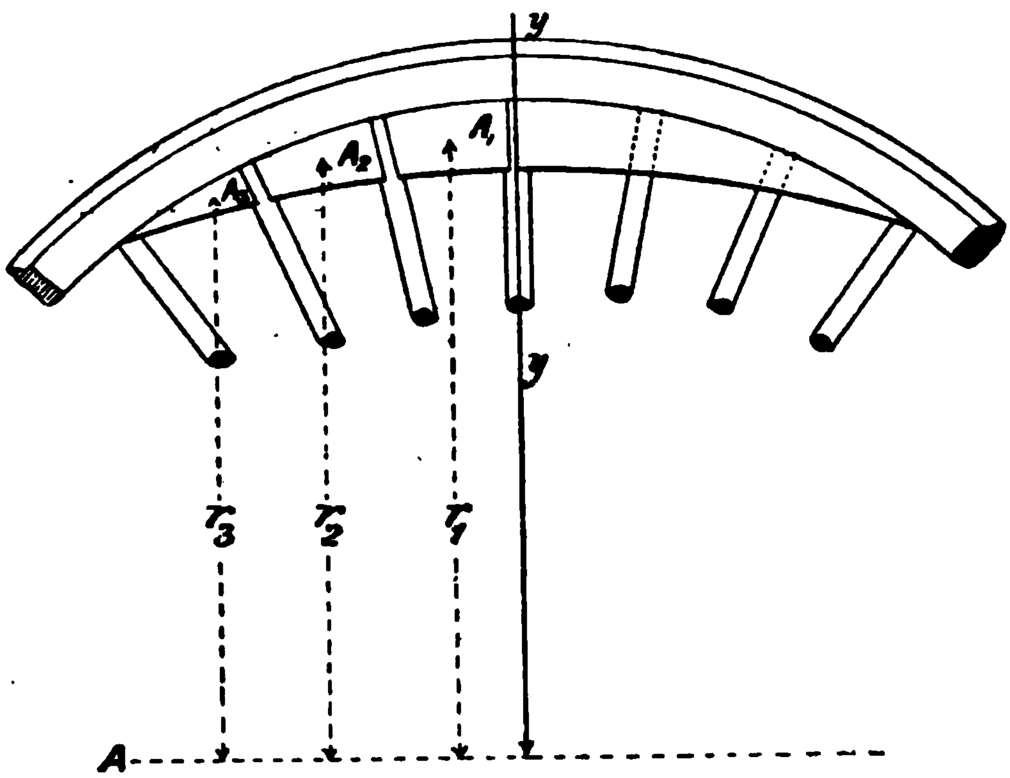
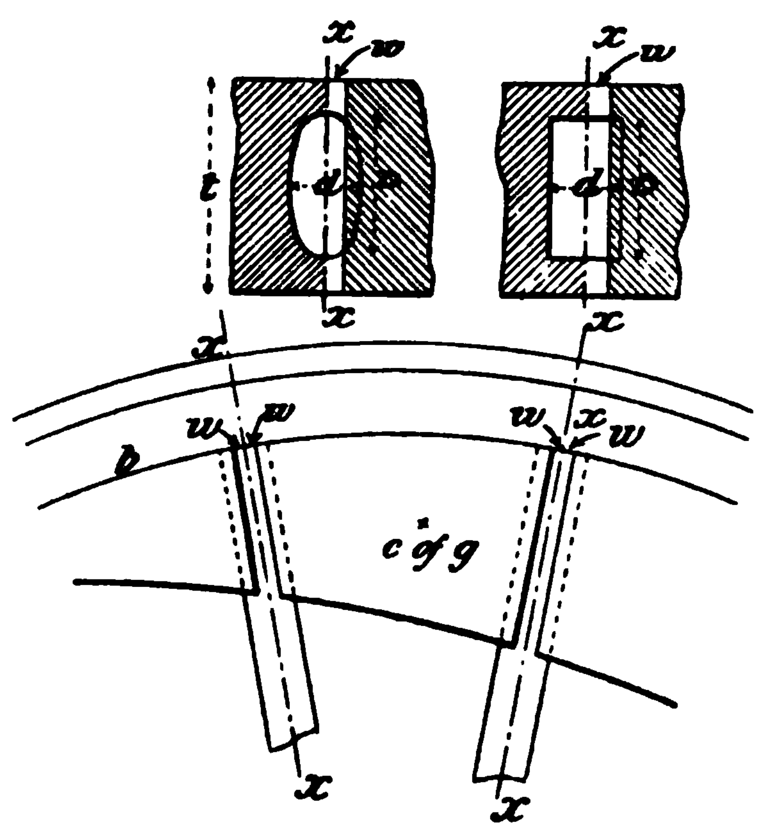


FIG. 197.

first with the balance weight as a whole, and then deducting the spokes, we shall deduct the spokes first. Draw the centre lines of the spokes *x*, *x*, and from them set off a width *w* on each side as shown, where wt = half the area of the spoke; in the case—



FIGS. 198, 199.

$$\left.\begin{array}{r}\text{of the ellipti-}\\ \text{cal spoke, } wt\end{array}\right\} = \frac{0{\cdot}785\,\mathrm{D}d}{2}$$

$$w = \frac{0{\cdot}392\,\mathrm{D}d}{t}$$

$$\left.\begin{array}{r}\text{of the rectan-}\\ \text{gular spoke, } wt\end{array}\right\} = \frac{\mathrm{D}d}{2}$$

$$w = \frac{0{\cdot}5\,\mathrm{D}d}{t}$$

By doing this we have not altered either the weight or the position of the centre of gravity of the section of the balance weight, but we have reduced it to a much simpler form to deal with. If a centre line *yy* be drawn through the

balance weight, the segments on either side of it and the portion on only one side of this line need be dealt with.

Measure the area of each segment when thus treated. Let them be A_1, A_2, A_3; then the weight of the whole balance weight is the sum of these segments—

$$W_B = 2tw_m(A_1 + A_2 + A_3)$$

where w_m = the weight per cubic inch of the metal.

For a cast-iron weight—

$$W_B = 0{\cdot}52t(A_1 + A_2 + A_3)$$

For a wrought-iron or cast-steel weight—

$$W_B = 0{\cdot}56t(A_1 + A_2 + A_3)$$

all dimensions being in inches.

The centre of gravity of each section can be calculated, but it is far less trouble to cut out pieces of cardboard to the shape of each segment, and then find the position of the centre of gravity by balancing, as described on p. 75. Measure the distance of each centre of gravity from the line AB drawn through the centre of the wheel.

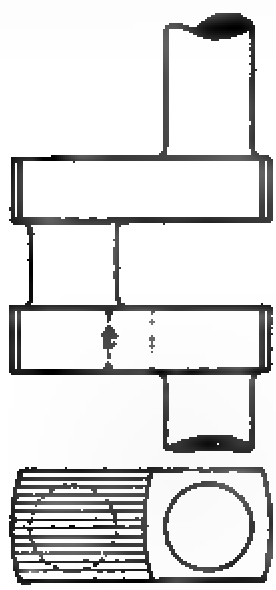


FIG. 200.

FIG. 201.

Let them be r_1, r_2, r_3 respectively; then the radius of the centre of gravity of the whole weight (see Fig. 197)—

$$R_B = \frac{A_1r_1 + A_2r_2 + A_3r_3}{A_1 + A_2 + A_3} \text{ (see p. 58)}$$

$$\text{and } W_BR_B = \left\{\begin{matrix} 0{\cdot}52 \\ \text{or} \\ 0{\cdot}56 \end{matrix}\right\} t(A_1r_1 + A_2r_2 + A_3r_3)$$

If there were more segments than those shown, we should get further similar terms in the brackets.

When dealing with cranks, precisely the same method may be adopted for finding their weight and the position of the centre of gravity.

In the figures, the weight of the crank $= 2tw_m \times$ shaded areas. The position of the centre of gravity is found as before, but no material error will be introduced by assuming it to be at the crank-pin.

Governors.—The function of a flywheel is to keep the speed of an engine approximately constant during one revolution or one cycle of its operations, but the function of a governor is to regulate the number of revolutions or cycles that the engine makes per minute. In order to regulate the speed, the supply of energy must be varied proportionately to the resistance overcome; this is usually achieved automatically by a governor consisting essentially of a rotating weight suspended in such a manner that its position relatively to the axis of rotation varies as the centrifugal force acting upon it, and therefore as the speed. As the position of the weight varies, it either directly or indirectly opens and closes the valve through which the energy is supplied, closing it when the speed rises, opening it when it falls.

The governor weight shifts its position on account of a change in speed, hence some variation of speed must always take place when the resistance is varied, but the change in speed can be reduced to a very small amount by suitably arranging the governor.

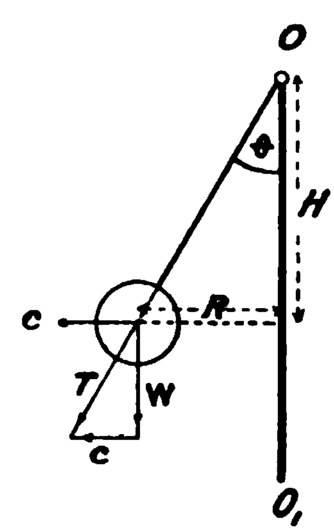


FIG. 202.

Simple Watt Governor.—Let the ball shown in the figure be suspended by an arm pivoted at O, and let it rotate round the axis OO_1 at a constant velocity. The ball is kept in equilibrium by the three forces—W, the weight of the ball acting vertically downwards (we shall for the present neglect the weight of the arm and its attachments, also friction on the joints); C, the centrifugal force acting horizontally; T, the tension in the supporting arm. The relative value of these forces is easily found by constructing the triangle of forces as shown.

Let H = height of the governor in feet;

h = " " " inches;

R = radius of the ball in feet;
N = number of revolutions made by the governor per second;
n = number of revolutions made by the governor per minute;
V = velocity (linear) in feet per second of the balls.

Then, from similar triangles, we have—

$$\frac{H}{R} = \frac{W}{C} = \frac{W}{\frac{WV^2}{gR}} = \frac{gR}{V^2}$$

$$H = \frac{gR^2}{V^2}$$

$$\text{but } V = 2\pi RN$$

$$\text{hence } H = \frac{gR^2}{4\pi^2R^2N^2} = \frac{0 \cdot 816}{N^2}$$

Expressing the height in inches, and the speed in revolutions per minute, we get—

$$h = \frac{35230}{n^2}$$

Thus we see that the height at which a simple Watt governor will run is entirely dependent upon the number of revolutions per minute at which it runs. The size of the balls and length of arms make no difference whatever as regards the height (when the balls are "floating").

The following table gives the height of a simple Watt governor for various speeds:—

Revolutions per minute (n).	Height of governor in inches (h).	Change of height corresponding to a change of speed of 10 revolutions per minute.
		Inches.
50	14·09	—
(54·2)	(12·00)	—
60	9·79	4·30
70	7·19	2·60
80	5·51	1·68
90	4·35	1·16
100	3·52	0·83
110	2·91	0·61
120	2·45	0·46

These figures show very clearly that the change of height corresponding to a given change of speed falls off very rapidly as the height of the governor decreases or as the apex angle θ increases; but as the governing is done entirely by a change in the height of the governor in opening or closing a throttle or other valve, it will be seen that the regulating of the motor is much more rapid when the height of the governor is great than when it is small, hence, if we desire to keep the speed within narrow limits, we must keep the height of the governor as great as possible or the apex angle θ as small as possible within reasonable limits.

Suppose, for instance, that a change of height of 2 inches were required to fully open or close the throttle or other valve; then, if the governor were running at 60 revolutions per minute, the 2-inch movement would correspond to about 7 per cent. change of speed; at 80, 15 per cent.; at 100, 24 per cent.; at 120, 36 per cent.

The greater the change of height corresponding to a given change of speed, the greater is said to be the *sensitiveness* of the governor.

A simple Watt governor can be made as sensitive as we please by running it with a very small apex angle, but it then becomes very cumbersome, and, moreover, it then possesses very little "power" to overcome external resistances.

Loaded Governor.—In order to illustrate the principle of the loaded governor, suppose a simple Watt governor to be loaded as shown. The broken lines show the position of the governor when unloaded.

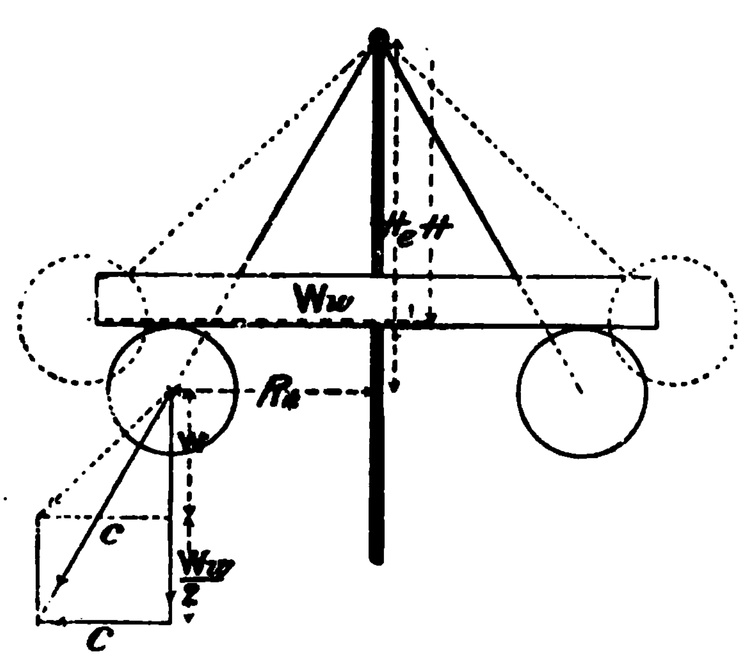


FIG. 203.

When the load W_w is placed on the balls, the "equivalent height of the simple Watt governor" is increased from H to H_e. Then, constructing the triangle of forces as before, we have—

$$\frac{H_e}{R_e} = \frac{W + \frac{W_w}{2}}{C}$$

Then, by precisely the same reasoning as in the case given above, we have—

$$H_e = \frac{0{\cdot}816}{N^2}\left(\frac{W + \frac{W_w}{2}}{W}\right)$$

$$\text{or } h_e = \frac{35200}{n^2}\left(\frac{W + \frac{W_w}{2}}{W}\right)$$

If W_w be m times the weight of one ball, we have—

$$h_e = \frac{35230}{n^2}\left(1 + \frac{m}{2}\right)$$

m varies from 10 to 50.

This expression must, however, be used with caution. Consider the case of a simple Watt governor both when unloaded and when loaded as shown in Figs. 202 and 203. If the *same* governor be taken in both instances, it is evident that its maximum height, *i.e.* when it just begins to lift, also its minimum apex angle, will be the same whether loaded or unloaded, and cannot in any case be greater than the length of the suspension arm measured to the centre of the ball. The speed of the loaded governor corresponding to any given height will, however, be greater than in the case of the unloaded governor in the ratio $\sqrt{1 + \frac{m}{2}}$ to 1, and if the *engine* runs at the same speed in both cases, the governor must be geared up in this ratio, but the alteration in height for any given alteration in the speed of the engine will be the same in both cases, or, in other words, the sensitiveness will be the same whether loaded or unloaded. We shall later on show, however, that the loaded governor is better on account of its greater power.

In the author's opinion most writers on this subject are in error; they compare the sensitiveness of a loaded governor at heights which are physically impossible (because greater even than the length of the suspension arms), with the much smaller, but possible, heights of an unloaded governor. If the reader wishes to appeal to experiment he can easily do so, and will find that the sensitiveness actually is the same in both cases.

The following table may help to make this point clear. m has been chosen as 16; then $\sqrt{1 + \frac{m}{2}} = 3$.

On comparing the last column of this table with that for the unloaded governor, it will be seen that they are identical, or the sensitiveness is the same in the two cases.

Revolutions per minute of governor.	Height of loaded governor in inches.	Change of height corresponding to a change of speed of 30 revolutions per minute of governor, or 10 revolutions per minute of engine.
		Inches.
150	14·09	
180	9·79	4·30
210	7·19	2·60
240	5·51	1·68
270	4·35	1·16
300	3·52	0·83
330	2·91	0·61
360	2·45	0·46

If, by any system of leverage, the weight W_w moves up and down x times as fast as the balls, the above expression becomes—

$$h_e = \frac{35230}{n^2}\left(1 + \frac{xm}{2}\right)$$

The method of loading shown in Fig. 203 is not convenient. The usual method of loading a governor is that shown in the Porter governor, Fig. 204.

Porter and other Loaded Governors.—If a governor be arranged with a spring, as in Fig. 206, the W_w in the above equations is the load on the spring. We shall shortly see that such an arrangement is very faulty if a sensitive governor be required.

In the type of governor shown in Fig. 207, which is frequently met with, springs are often used instead of a dead weight. The value of x is usually a small fraction, consequently a huge weight would be required to give the same results as a Porter or similar type of governor. But it has other inherent defects which will shortly be apparent.

Isochronous Governors.—A perfectly isochronous governor will go through its whole range with the slightest variation in speed, but such a governor is practically useless

for governing an engine, for reasons shortly to be discussed. But when designing a governor which is required to be very

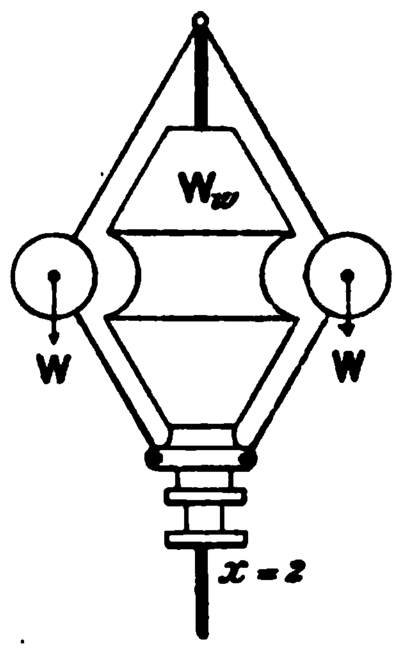


FIG. 204.

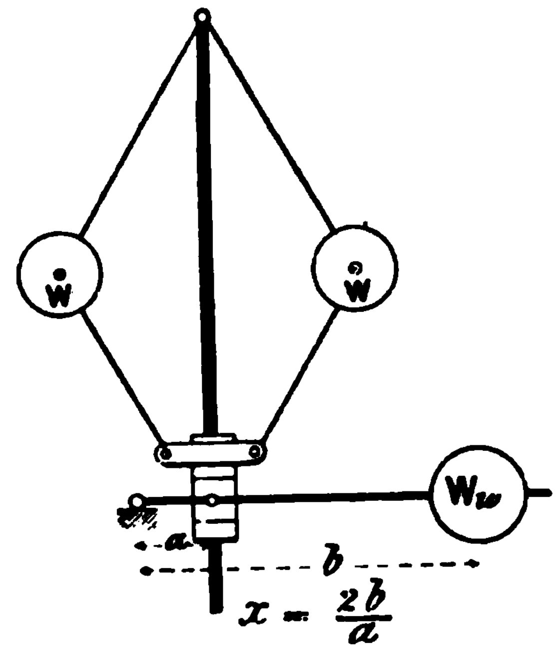


FIG. 205.

sensitive, we sail as near the wind as we dare, and make it very nearly isochronous. In the governors we have considered,

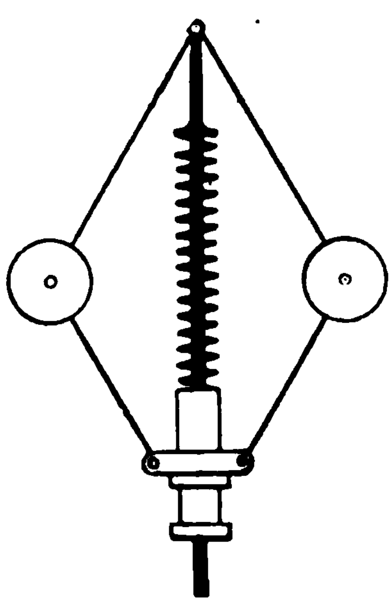


FIG. 206.

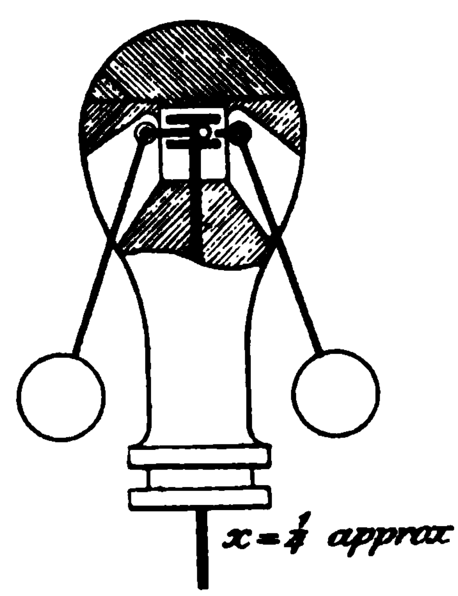


FIG. 207.

the height of the governor has to be altered in order to alter the throttle or other valve opening. If this could be accomplished without altering the height of the governor, it could also be accomplished without altering the speed, and we should have an isochronous governor. Such a governor can be constructed by causing the balls to move in the arc of a parabola, the axis being the axis of rotation. Then, from the properties of the parabola, we know that the height of the governor, *i.e.* the subnormal to the path of the balls, is constant for all positions of the balls, and therefore of the sleeve which actuates the governing valve. Examples of such governors are

to be found in many books on the steam-engine. We shall, however, only consider an approximate form which is very commonly used, viz. the crossed-arm governor.

The curve *abc* is a parabolic arc; the axis of the parabola is O*b*; then, if normals be drawn to the curve at the highest and lowest positions of the ball, they intersect at some point *d* on the other side of the axis. Then, if the balls be suspended from this point, they will move in an approximately parabolic arc, and the governor will therefore be approximately isochronous. If it be desired to make the governor more stable, the points *d*, *d* are brought in nearer the axis. The virtual centre of the arms is at their intersection; hence the height of the governor is H, which is approximately constant. The equivalent height can be raised by adding a central weight as in a Porter governor. It, of course, does not affect the sensitiveness, but it increases the power of the governor to overcome resistances. The speed at which a crossed-arm governor lifts depends upon the height in precisely the same manner as in the simple Watt governor.

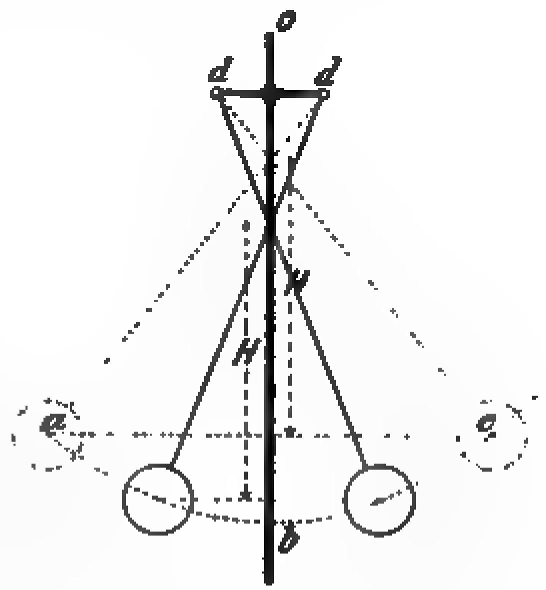


Fig. 208.

Another well-known and highly successful isochronous governor is the "Wilson-Hartnell" governor.

Fig. 209.

In the diagram, *c* is the centrifugal force acting on the ball, and *p* the pressure due to the spring, *i.e.* one-half the total pressure. As the balls fly out the spring is compressed, and as the pressure increases directly as the compression, the pressure *p* increases directly (or very nearly so) as the radius *r* of the balls; hence we may write $p = Kr$, where K is a constant depending on the stiffness of the spring.

Let $r_0 = nr$, frequently $n = 1$

$$\text{Then } cr_0 = pr$$
$$\text{and } 0{\cdot}00034\text{W}r^2n\text{N}^2 = \text{K}r^2$$
$$\text{and N}^2 = \frac{\text{K}}{0{\cdot}00034\text{W}n}$$

For any given governor the weight W of the ball is constant; hence the denominator of the fraction is constant, whence N^2, and therefore N, is constant; *i.e.* there is only one speed at which the governor will float, and any increase or decrease in the speed will cause the balls to fly right out or in, or, in other words, will close or fully open the governing valve, therefore the governor is isochronous.

There are one or two small points that slightly affect the isochronous character of the governor. For example, the weight of the ball, except when its arm is vertical, has a moment about the pivot. Then, except when the spring arm is horizontal, the centrifugal force acting on the spring arm tends to make the ball fly in or out according as the arm is above or below the horizontal.

We shall shortly show how the sensitiveness can be varied by altering the compression on the spring.

Crank-shaft Governors.—The governing of steam-engines is often effected by varying the point at which the steam is cut off in the cylinder. Any of the forms of governor that we have considered can be adapted to this method, but the one which lends itself most readily to it is the crank-shaft governor, which alters the cut-off by altering the throw of the eccentric. We will consider one typical instance only, the McLaren governor, chosen because it contains so many good points, and, moreover, has a great reputation for governing within extremely fine limits. Fig. 210.

The eccentric E is attached to a plate pivoted at A, and suspended by spherical-ended rods at B and C. A curved cam DD, attached to this plate, fits in a groove in the governor weight W in such a manner that, as the weight flies outwards due to centrifugal force, it causes the eccentric plate to tilt, and so bring the centre of the eccentric nearer to the centre of the shaft, or, in other words, to reduce its eccentricity, and consequently the travel of the valve, thus causing the steam to be cut off earlier in the stroke. The cam DD is so arranged that when the weight W is right in, the cut-off is as late as the slide-valve will allow it to be. Then, when the weight is right

out, the travel of the valve is so reduced that no steam is admitted to the cylinder. A spring SS is attached to the weight arm to supply the necessary centripetal force. The speed of the engine is regulated by the tension on this spring. In order to alter the speed while the engine is running, the lower end of the spring is attached to a screwed hook, F. The nut G is in the form of a worm wheel; the worm spindle is provided with a small milled wheel, H. If it be desired to alter the speed when running, a leather-covered lever is pushed into gear, so that the rim of the wheel H comes in contact with it at each revolution, and is thereby turned through a small amount, thus tightening or loosening the spring as the case may be. If the lever bears on the one edge of the wheel H, the spring is tightened and the speed of the engine increased, and if on the other edge the reverse. The spring S is attached to the weight arm as near its centre of gravity as possible, in order to eliminate friction on the pin J when the engine is running.

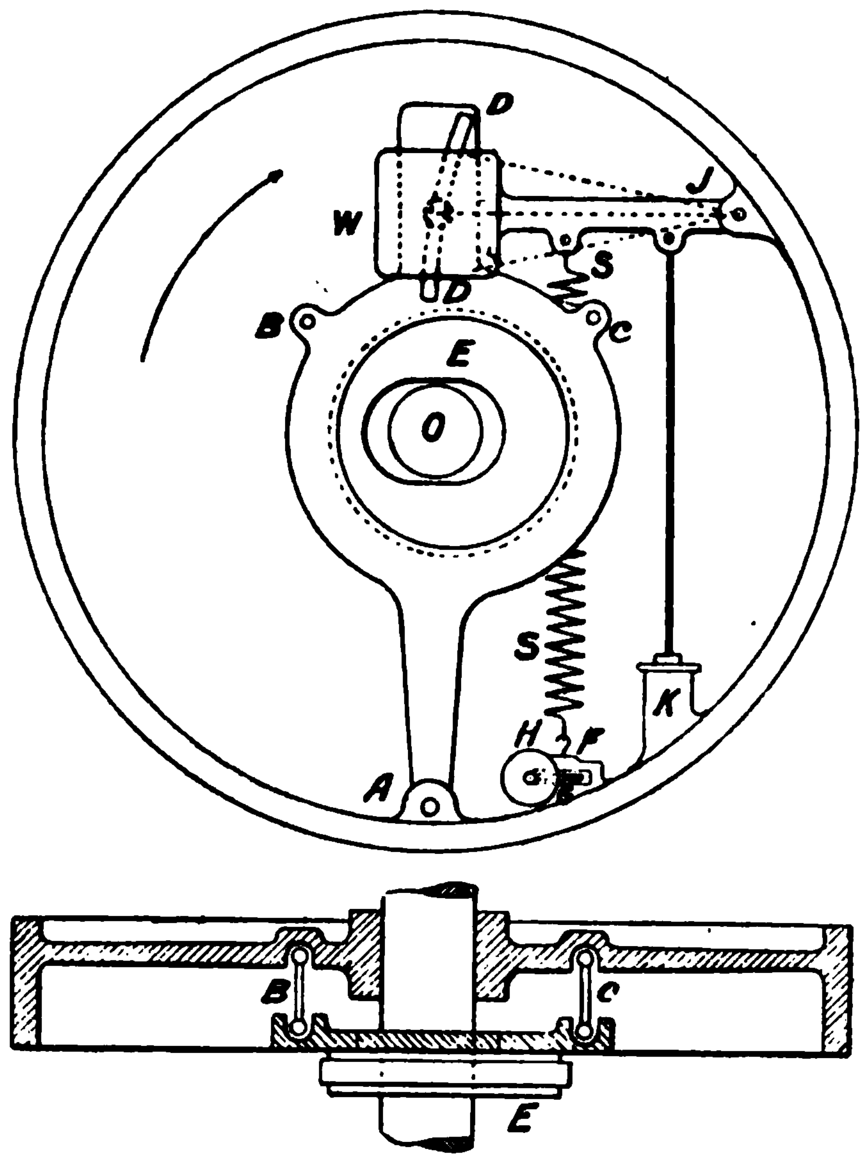


FIG. 210.

The governor is designed to be extremely sensitive, and, in order to prevent hunting, a dashpot K is attached to the weight arm.

In the actual governor two weights are used, coupled together by rods running across the wheel. The figure must be regarded as purely diagrammatic.

It will be seen that this governor is practically isochronous, for the load on the spring increases as the radius of the weight, and therefore, as explained in the Hartnell governor, as the centrifugal force.

The sensitiveness can be varied by altering the position of suspension, J. In order to be isochronous, the path of the weight must as nearly as possible coincide with a radial line drawn from O, and the direction of S must be parallel to this radial line.

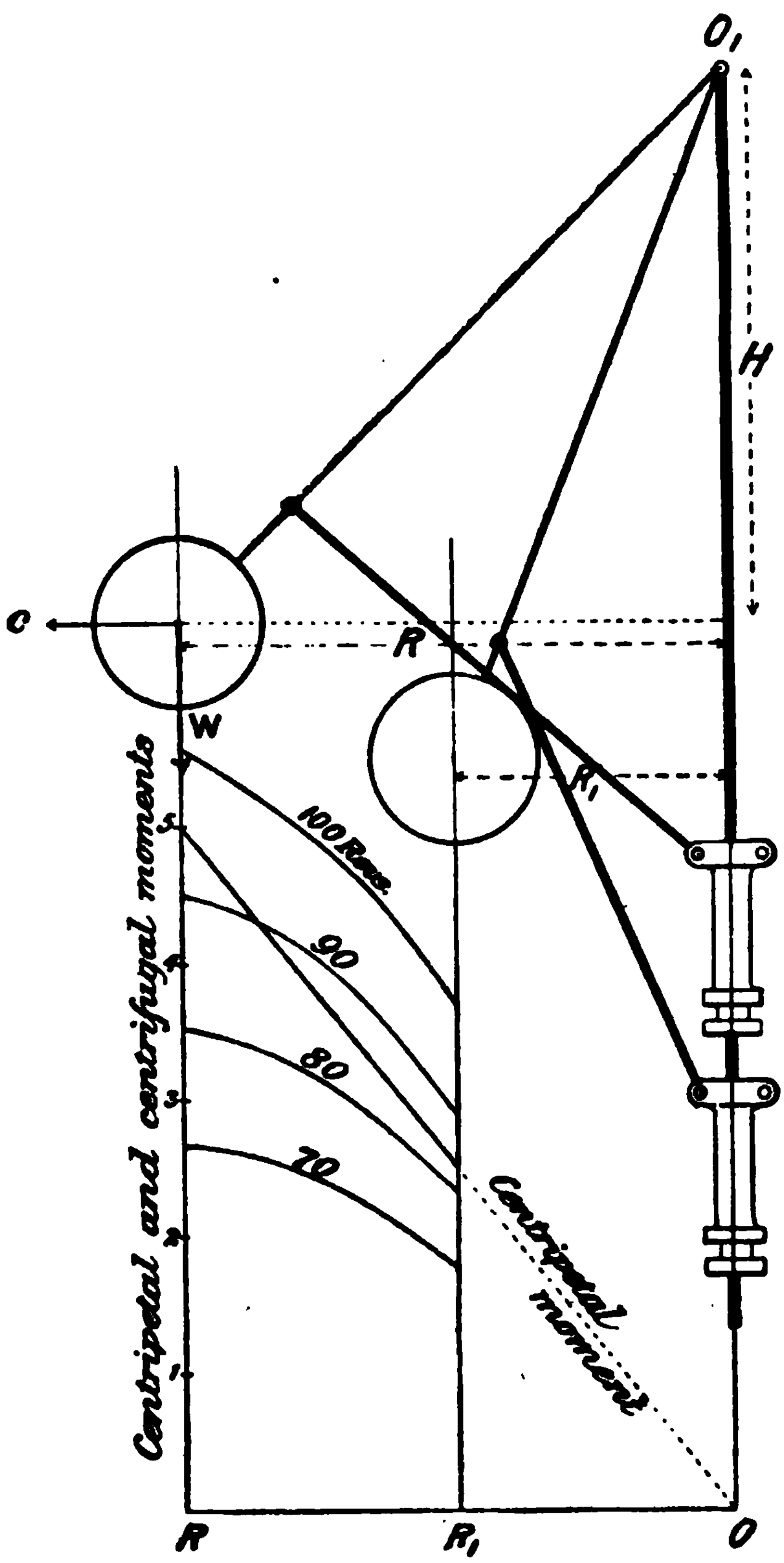


FIG. 211.

Sensitiveness of Governors.—The sensitiveness and behaviour of a governor when running can be very conveniently studied by means of a diagram showing the rate of increase of the centrifugal and centripetal moments as the governor balls fly outwards. These diagrams are the invention of Mr. Wilson Hartnell, who first described them in a paper read before the Institute of Mechanical Engineers in 1882.

In Fig. 211 we give such a diagram for a simple Watt governor, neglecting the weight of the sleeve, etc. The axis OO′ is the axis of rotation. The ball is shown in its two extreme positions. The ball is under the action of two moments —the centrifugal moment CH and the centripetal moment WR, which of course must be equal for all positions of the ball, unless the ball is being accelerated or retarded. The centrifugal moment is tending to carry the ball outwards, the centripetal to bring it back. The four numbered curves show the relation between the moment tending to make the balls fly out (ordinates) and the position of the balls. The centripetal moment line shows the relation between the moment tending to bring the balls back and the position of the balls, which is independent of the speed.

$$\begin{aligned}\text{We have CH} &= 0{\cdot}00034\ \text{WRN}^2\text{H}\\ &= 0{\cdot}00034\text{WN}^2(\text{RH})\\ &= \text{KRH}\end{aligned}$$

The quantity 0·00034 WN^2 is constant for any given ball running at any given speed. Values of KRH have been calculated for various positions and speeds, and the curves plotted.

The value of WR varies, of course, directly as the radius; hence the centripetal line is straight, and passes through the origin O. From this we see that the governor begins to lift at a speed of about 82 revolutions per minute, but gets to a speed of about 94 before the governor lifts to its extreme position. Hence, if it were intended to run at a mean speed of 88 revolutions per minute, it would, if free from friction, vary about 9 per cent. on either side of the mean, and when retarded by friction it will be far worse.

If the centrifugal and centripetal curves coincided, the governor would be isochronous. If the slope of the centrifugal curve be less than the centripetal, the governor is too stable, *i.e.* not sufficiently sensitive; but if, on the other hand, the slope of the centrifugal curve be greater than the centripetal, the governor is too sensitive, for as soon as the governor begins to lift, the centrifugal moment, tending to make the balls fly out, increases more rapidly than the centripetal moment, tending to keep the balls in—consequently the balls are accelerated, and fly out to their extreme position, completely closing the governing valve, which immediately causes the engine to slow down. But as soon as this occurs, the balls close right in and fully open the governing valve, thus causing the engine to race and the balls to fly out again, and so on. This alternate racing and slowing down is known as *hunting*, and is the most common defect of governors intended to be sensitive.

It will be seen that this action cannot possibly occur with a simple Watt governor unless there is some disturbing action.

Friction of Governors.—So far, we have neglected the effect of friction on the sensitiveness, but it is in reality one of the most important factors to be considered in connection with sensitive governors. Many a governor is practically perfect on paper—friction neglected—but is to all intents and purposes useless in the material form on an engine, on account of retardation due to friction. The friction is not merely due to

the pins, etc., of the governor itself, but to the moving of the governing valve or its equivalent and its connections.

In Fig. 212 we show how friction affects the sensitiveness of a governor. The vertical height of the shaded portion represents the friction moment that the governor has to overcome. Instead of the governor lifting at 80 revolutions per minute, the speed at which it should lift if there were no friction, it does not lift till the speed gets to about 92 revolutions per minute; likewise on falling, the speed falls to 64 revolutions per minute. Thus with friction the speed varies about 22 per cent. above and below the mean. Unfortunately, very little experimental data exists on the friction of governors and their attachments,[1] but a designer cannot err by doing his utmost to reduce it even to the extent of fitting all joints, etc., with ball bearings.

Fig. 212.

In the well-known Pickering governor, the friction of the governor itself is reduced to a minimum by mounting the balls on a number of thin band springs instead of arms moving on pins. The attachment of the spring at the c. of g. of the weight and arm, as in the McLaren governor, is a point also worthy of attention. We will now examine in detail several types of governor by the method just described.

Porter Governor.—In this case the centripetal force is greatly increased, while the centrifugal is unaffected. The central weight W_w rises twice as fast as the balls (Fig. 204), hence we get the weight W_w acting on each ball in the manner shown in Fig. 213. Resolve W_w in the directions of the two arms as shown: it is evident that *ab*, acting along the upper

[1] See Paper by Ransome, *Proc. Inst. C.E.*, vol. cxiii.; also the question has recently been investigated by one of the author's students, Mr. Eurich, who finds that when oiled a Watt governor lags behind to the extent of 7·5 per cent., and when unoiled 17·5 per cent.

arm, has no moment about O, but $bd = W_0$ has a centripetal moment W_0R_0; then we have—

$$CH = WR + W_0R_0$$

Values of each have been calculated and plotted as before in Fig. 211. In the central spring governor W_w varies as the balls lift; in other respects the construction is the same.

It should be noticed that the centripetal and centrifugal moment curves much more closely coincide as the height of the governor increases; thus the sensitiveness increases with the height, a conclusion we have already come to by another process of reasoning.

Crossed-arm Governor.—In this governor H is constant, and as C varies directly as the radius for any given speed, it is evident that the centripetal and centrifugal lines are both straight and coincident, hence the governor is isochronous.

Wilson Hartnell Governor. — In constructing this diagram (Fig. 214) we have neglected the moment of the ball weight on either side of the suspension pin, and the other slight irregularities, but in a big governor they are of importance, and should be taken into account.

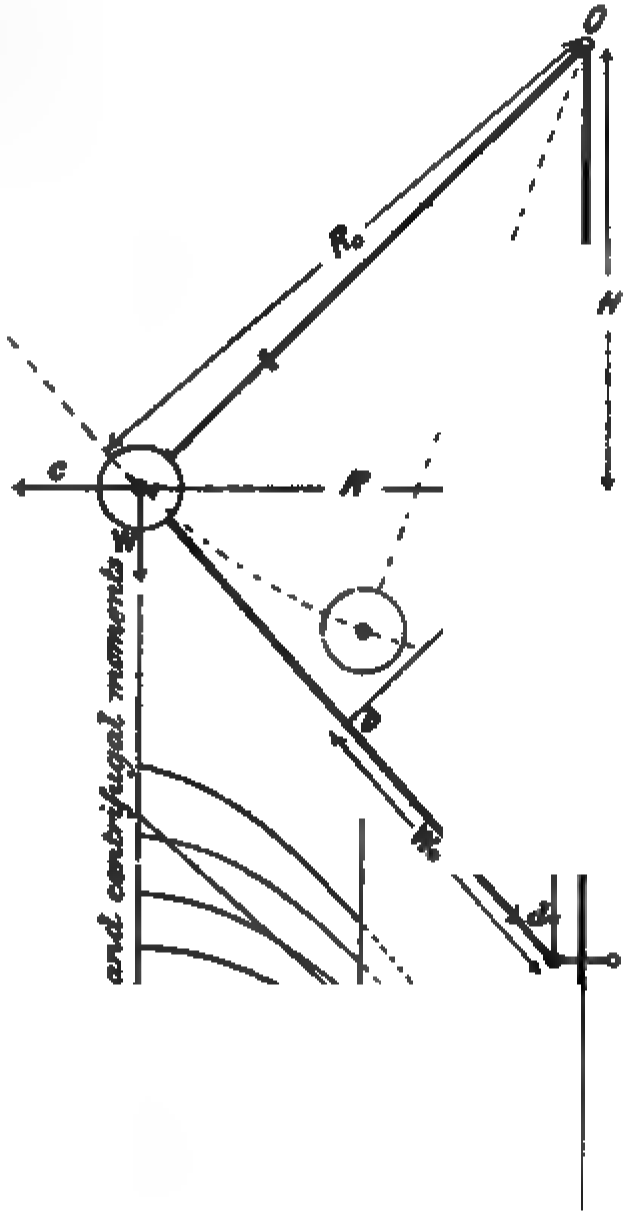


FIG. 213.

We have shown that $cr_0 = pr$, and that c varies as R, likewise p varies as R, hence both the centrifugal and the centripetal moment curves are straight, and when the governor is isochronous,

both pass through the origin. When a spring is loaded in either tension or compression, the strain is proportional to the load applied; hence, if an initial load be put on the governor spring, the spring pressure curves will always be parallel to one another as shown in dotted lines. If the initial load be too small, the governor will be too stable, and if too great, too sensitive, *i.e.* it will cause the engine to hunt. The position for the governor to be isochronous is when the centripetal and centrifugal curves are coincident, *i.e.* when both pass through the origin O. It may happen, however, that when the spring is adjusted to make the governor isochronous, the speed is not that which is desired, and in order to obtain it, either a weaker or stiffer spring will be required. Instead, however, of getting a new spring, the stiffness can be readily varied by altering the effective length of the spring. This is most conveniently done by casting a gun-metal nut round the coils of the spring, like a cork round a cork-screw. If the spring be previously warmed and dipped in loam and water so as to coat the spring, the nut will not bind when cold; then by screwing this nut up or down the effective length of the spring can be varied at will, and the exact stiffness obtained.

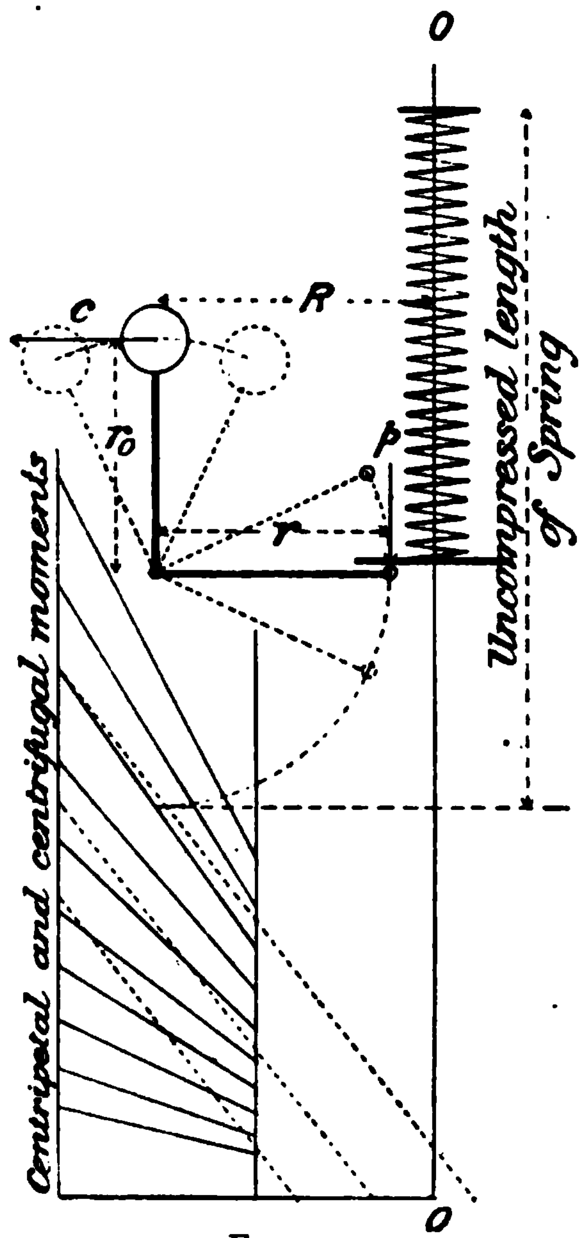


FIG. 214.

McLaren's Crankshaft Governor.—In this governor we have $CR_0 = SR_s$; but C varies as R, hence if there be no tension on the spring when R is zero, it will be evident that S will vary directly as R; but C also varies in the same manner, hence the centrifugal and centripetal moment lines are nearly straight and coincident. The centrifugal lines are not absolutely straight, because the weight does not move exactly on a radial line from the centre of the crank-shaft.

Governor Dashpots.—A dashpot consists essentially of a cylinder with a leaky piston, around which oil, air, or other fluid has to leak. An extremely small force will move the piston

slowly, but very great resistance is offered by the fluid if a rapid movement be attempted.

Very sensitive governors are therefote generally fitted with

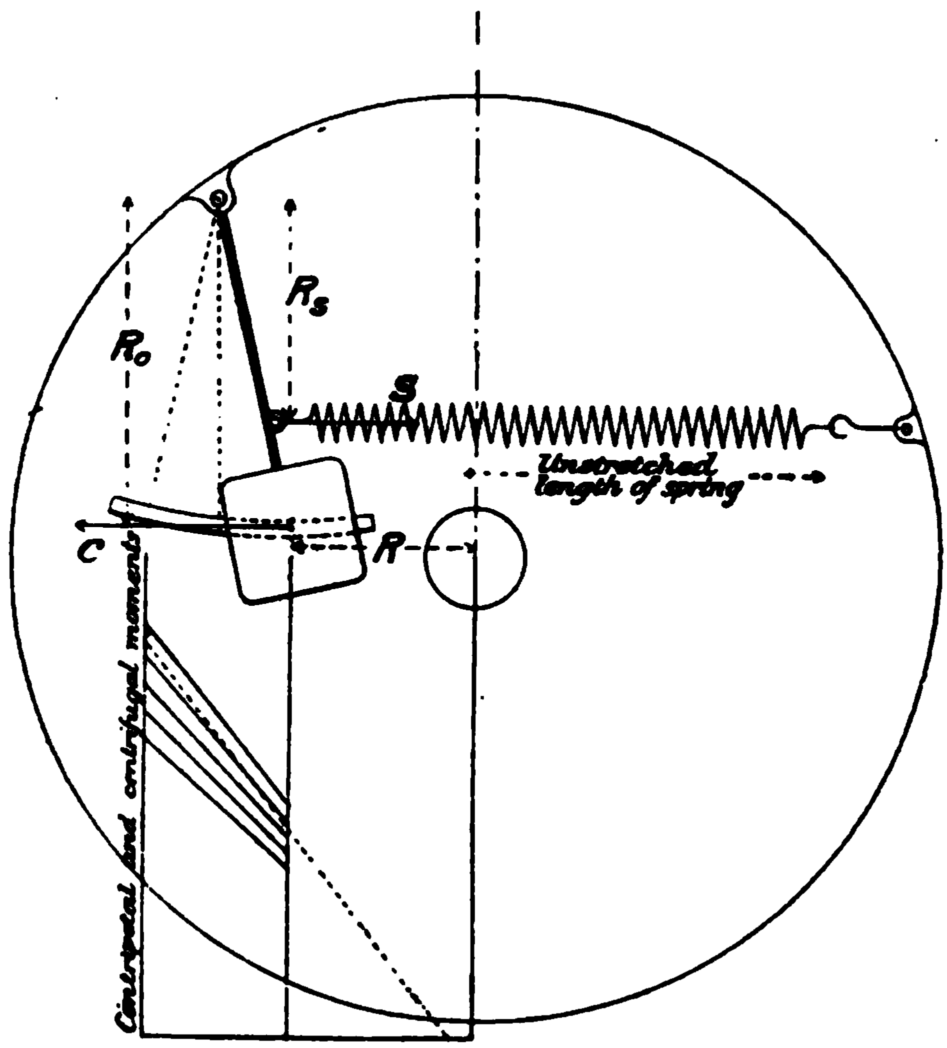


Fig. 215.

dashpots, to prevent them from suddenly flying in or out, and thus causing the engine to hunt.

If a governor be required to work over a very wide range of power, such as all the load suddenly thrown off, a sensitive, almost isochronous governor with dashpot gives the best result; but if very fine governing be required over small variations of load, a slightly less sensitive governor without a dashpot will be the best.

However good a governor may be, it cannot possibly govern well unless the engine be provided with sufficient fly-wheel power. If an engine have, say, a 2-per-cent. cyclical variation and a very sensitive governor, the balls will be constantly fluctuating in and out during every stroke.

Power of Governors.—The power of a governor is its power of overcoming external resistances. The greater the power, the greater the external resistance it will overcome with a given alteration in speed.

Nearly all governor failures are due to their lack of power.

The useful energy stored in a governor is readily found thus, approximately :—

Simple Watt governor, crossed arm and others of a similar type—

Energy = weight of both balls × vertical rise of balls

Porter and other loaded governors—

Energy = weight of both balls × vertical rise of balls + weight of central weight × its vertical rise

Spring governors—

$$\text{Energy} = \text{weight of both balls} \times \text{vertical rise (if any) of balls} + n\left(\frac{\text{max. load on spring} + \text{min. load on spring}}{2}\right) \times \frac{\text{the stretch or compression of spring}}{2}$$

where n = the number of springs employed; express weights in pounds, and distances in feet.

The following may be taken as a rough guide as to the energy that should be stored in a governor to get good results; it is always better to store too much rather than too little energy in a governor.

Type of governor.	Foot-pounds of energy stored per I.H.P. of engine.
For trip gears and where small resistances have to be overcome	0·5–0·75
For fairly well balanced throttle-valves ...	0·75–1
For automatic expansion gears in which the driving mechanism is not reversible (see p. 227)	5–8
For automatic expansion gears in which the driving mechanism is reversible	12–14 (large engines), up to 40 for small engines.

Generally speaking, it is better to so design the governor that the valve-gear cannot react upon it, then no amount of pressure on the valve-gear will alter the height of the governor; that is to say, the reversed efficiency of the mechanism which alters the cut-off must be negative, or the efficiency of the mechanism must be less than 50 per cent. On referring to the McLaren governor, it will be seen that no amount of pressure on the eccentric will cause the main weight W to move in or out.

CHAPTER VII.

FRICTION.

When one body, whether solid, liquid, or gaseous, is caused to slide over the surface of another, a resistance to sliding is experienced, which is termed the "friction" between the two bodies.

Many theories have been advanced to account for the friction between sliding bodies, but none are quite satisfactory. To attribute it merely to the roughness between the surfaces is but a very crude and incomplete statement; the theory that the surfaces somewhat resemble a short-bristled brush or velvet pile much more nearly accounts for known phenomena, but still is far from being satisfactory.

However, our province is not to *account* for what happens, but simply to observe and classify, and, if possible, to sum up our whole experience in a brief statement or formula.

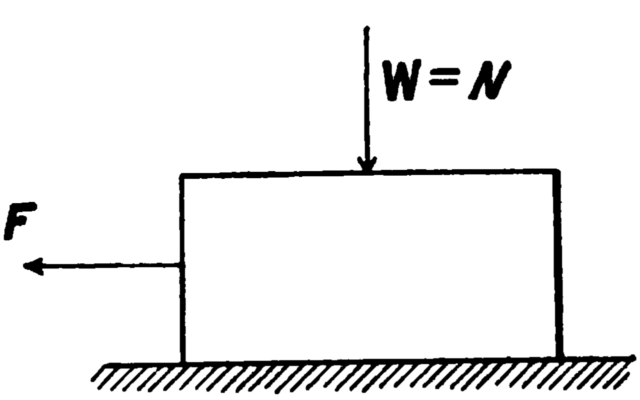


Fig. 216.

Frictional Resistance (F).—If a block of weight W be placed on a horizontal plane, as shown, and the force F applied parallel to the surface be required to make it slide, the force F is termed the frictional resistance of the block. The normal pressure between the surfaces is N.

Coefficient of Friction (μ).—Referring to the figure above, the ratio $\frac{F}{W}$ or $\frac{F}{N} = \mu$, and is termed the coefficient of friction. It is, in more popular terms, the ratio the friction bears to the normal pressure between the surfaces. It may be found by dragging a block along a plane surface and measuring F and N, or it may be found by tilting the surface as in Fig. 217. The plane is tilted till the block just begins to slide. The vertical

weight W may be resolved normal N and parallel to the plane F. The friction is due to the normal pressure N, and the force

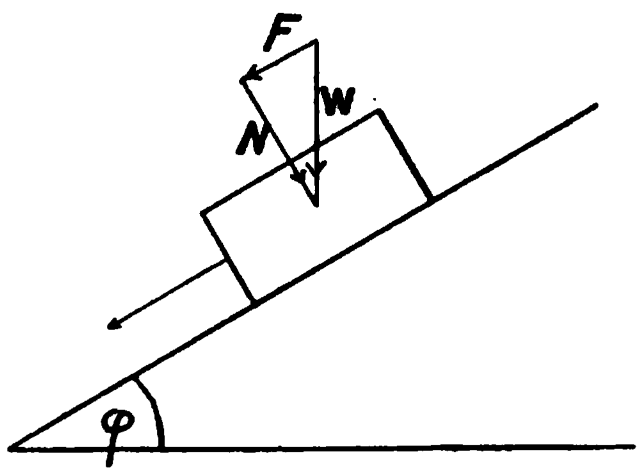


FIG. 217.

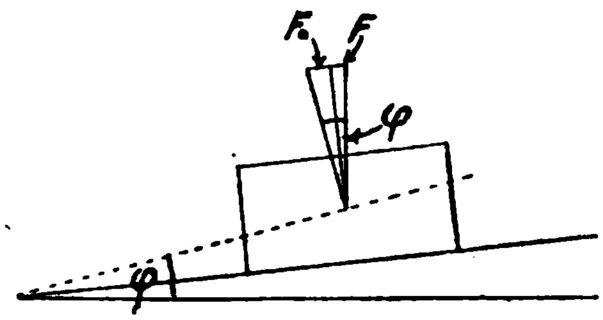


FIG. 218.

required to make the body slide is F; then the coefficient of friction $\mu = \frac{F}{N}$ as before. But $\frac{F}{N} = \tan\,\phi$, where ϕ is the angle the plane makes to the horizontal when sliding just commences.

The angle ϕ is termed the "friction angle," or "angle of repose." The body will not slide if the plane be tilted at an angle less than the friction angle, and a force F_0, Fig. 218,

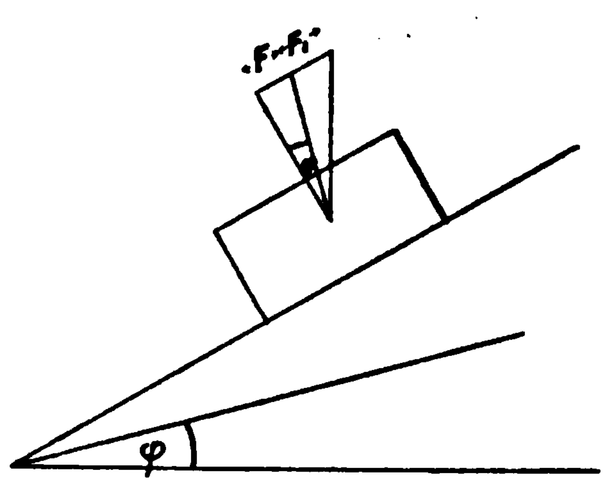


FIG. 219.

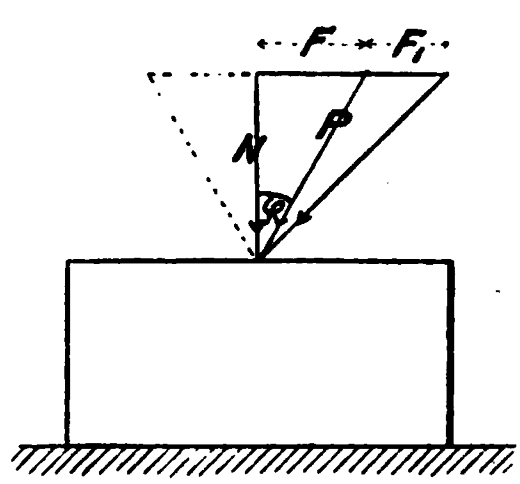


FIG. 220.

will have to be applied parallel to the plane in order to make it slide. Whereas, if the angle be greater than ϕ, the body will be accelerated due to the force F_1, Fig. 219.

There is yet another way of looking at this problem. Let the body rest on a horizontal plane, and let a force P be applied at an angle to the normal; the body will not begin to slide until the angle becomes equal to the angle ϕ, the angle of friction. If the line representing P be revolved round the normal, it will describe the surface of a cone in space, the apex angle being 2ϕ; this cone is known as the "friction cone."

If the angle with the normal be less than ϕ, the block will not slide, and if greater the block will be accelerated, due to the force F_1.

If P be very great compared with the area of the surfaces in contact, the surfaces will seize or cling to one another, and if continued the surfaces will be torn or abraded.

Friction of Dry Surfaces.—The experiments usually quoted on the friction of dry surfaces are those made by Morin and Coulomb; they were made under very small pressures and speeds, hence the laws deduced from them only hold very imperfectly for the pressures and speeds usually met with in practice. They are as follows:—

1. The friction is directly proportional to the normal pressure between the two surfaces.
2. The friction is independent of the area of the surfaces in contact for any given normal pressure, *i.e.* it is independent of the *intensity* of the normal pressure.
3. The friction is independent of the velocity of rubbing.
4. The friction between two surfaces at rest is greater than when they are in motion, or the friction of rest[1] is greater than the friction of motion.
5. The friction depends upon the nature of the surfaces in contact.

We will now see how the above laws agree with experiments made on a larger scale.

The first two laws are based on the assumption that the coefficient of friction is constant for all pressures; this, however, is not the case.

The curves in Fig. 221 show approximately the difference between Coulomb's law and actual experiments carried to high pressures. At the low pressure at which the early workers worked, the two curves practically agree, but at higher pressures the friction falls off, and then rises until seizing takes place.

Instead of the frictional resistance being

$$F = \mu N$$

it is more nearly given by—

$$F = \mu N^{0.97}, \text{ or } F = \mu \sqrt[1.03]{N}$$

where μ has the following values:—

[1] The friction of rest has been very aptly termed the "sticktion."

Wood on wood	0·25 to 0·50
Metal ,, ...	0·20 to 0·60
Metal on metal	0·15 to 0·30
Leather on wood	0·25 to 0·50
,, metal	0·30 to 0·60
Stone on stone	0·40 to 0·65

These coefficients must always be taken with caution; they vary very greatly indeed with the state of the surfaces in contact.

The third law given above is far from representing facts; in

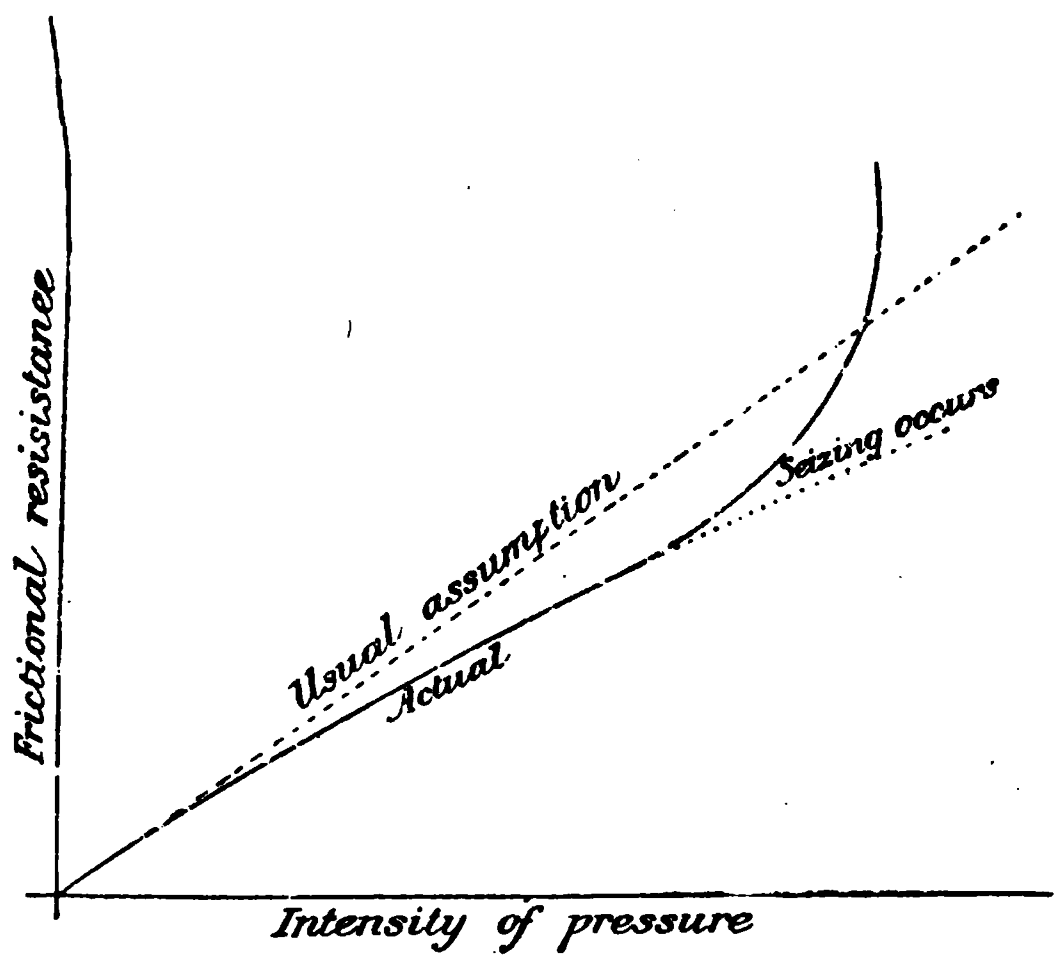


FIG. 221.

the limit the fourth law becomes a special case of the third. If the surfaces were perfectly clean, and there were no film of air between, this law would probably be strictly accurate, but all experiments show that the friction decreases with velocity of rubbing.

The following empirical formula fairly well agrees with experiments :—

Let μ = coefficient of friction;
K = a constant to be determined by experiment;
V = the velocity of sliding.

$$\text{Then } \mu = \frac{K}{2 \cdot 4\sqrt{V}}$$

The following experiments by Westinghouse and Galton, on steel tyres on steel rails will serve to illustrate this point:—

Speed miles per hour ...	10	15	25	38	45	50
Coefficient of friction ...	0·110	0·087	0·080	0·051	0·047	0·040

For other instances, see *Proc. Inst. M.E.*, 1883, p. 660.

The fourth law has been observed by nearly every experimenter on friction; the following figures by Morin and others will suffice to make this clear—

Materials.	Coefficient of friction.	
	Rest.	Velocity 3 to 5 ft. sec.
Wood on wood	0·54	0·46
" "	0·69	0·43
Metal on metal	0·34	0·26
Stone on stone	0·74	0·63
Leather on iron	0·59	0·52

The figures already quoted quite clearly demonstrate the truth of the fifth law given above.

Special Cases of Sliding Bodies.—In the cases we are about to consider, we shall, for sake of simplicity, assume that Coulomb's laws hold good.

Oblique Force required to make a Body Slide on a horizontal plane.—If an oblique force P act upon a block of weight W, making an angle θ with the direction of sliding, we can find the magnitude of P required to make the block slide; the total normal pressure on the plane is the normal component of P, viz. n, together with W. From a draw a line making an angle ϕ (the friction angle) with W, cutting P in

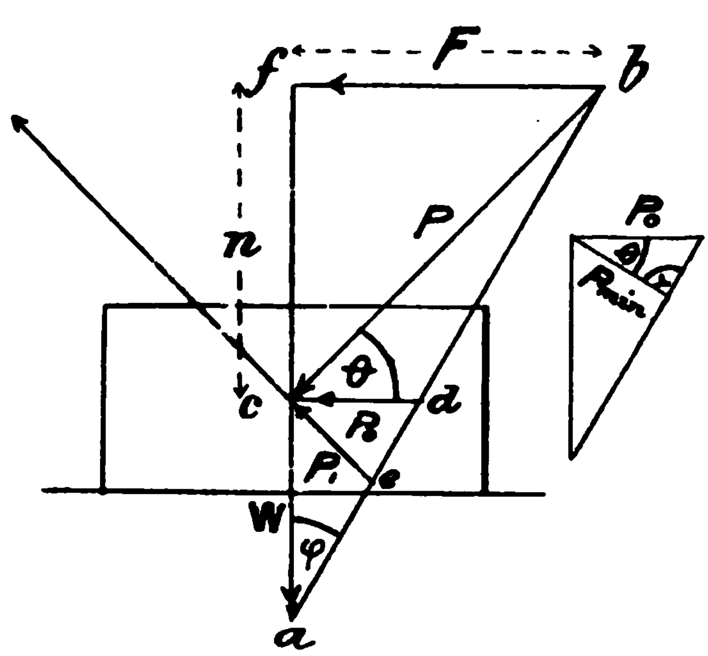


FIG. 222.

the point b; then bc, measured to the same scale as W, is the magnitude of the force P required to make the body slide. The frictional resistance is F, and the total normal pressure $n + W$, hence $F = \mu(n + W)$. When $\theta = 0$, $P_0 = cd = \mu W$.

When θ is negative, it simply indicates that P_1 is pulling away from the plane: the magnitude is given by ce. From the figure it is clear that the least value of P is when its direction is normal to ab, *i.e.* when $-\theta = \phi$; then—

$$\begin{aligned} P_{min} &= P_0 \cos \phi = \mu W \cos \phi \\ &= \tan \phi W \cos \phi \\ &= W \sin \phi \end{aligned}$$

It will be seen from the figure that P is infinitely great when ab is parallel to bc—that is, when P is just on the edge of the friction cone, or when $90 - \theta = \phi$. When $\theta = -90°$, P acts vertically upwards and is equal to W.

A general expression for P is found thus:

$$\begin{aligned} n &= P \sin \theta \\ F &= P \cos \theta = \mu(W \mp n) \\ F &= \mu(W \mp P \sin \theta) \end{aligned}$$

and $P(\cos \theta \mp \mu \sin \theta) = \mu W$

$$\text{hence } P = \frac{\mu W}{\cos \theta \mp \mu \sin \theta} = \frac{\tan \phi W}{\cos \theta \mp \mu \sin \theta}$$

$$P = \frac{\sin \phi W}{\cos \phi \cos \theta \mp \sin \phi \sin \theta}$$

$$P = \frac{W \sin \phi}{\cos (\phi \mp \theta)}$$

When P acts upwards away from the plane, the + sign is to be used in the denominator; and for the minimum value of P, $\phi = -\theta$; then the denominator is unity, and $P = W \sin \phi$, the result given above, but arrived at by a different process.

Thus, in order to drag a load, whether sliding or on wheels, along a plane, the line of pull should be upwards, making an angle with the plane equal to the friction angle.

Force required to make a Body slide on an Inclined Plane.—A special case of the above is that in which the plane is inclined to the horizontal at an angle α. Let the block of weight W rest on the inclined plane as shown. In order to make it slide up the plane, work must be done in lifting the block as well as overcoming the friction. The pull required to raise the block is readily obtained thus: Set down a line bc to represent the weight W, and from c draw a line cd, making

an angle α with it; then, if from b a line be drawn parallel to the direction of pull P_1, the line bd_1 represents to the same scale as W the required pull if there were no friction. An examination of the diagram will at once show that bd_1c is simply the triangle of forces acting on the block; the line cd_1 is, of course, normal to the plane.

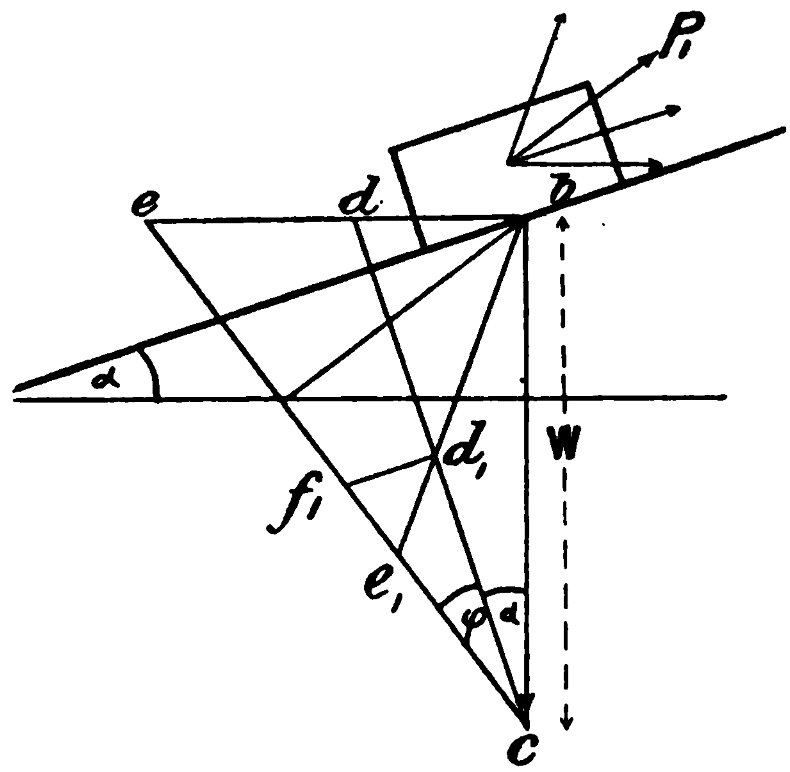


FIG. 223.

When friction is taken into account, draw the line ce, making an angle ϕ (the friction angle) with cd; then be_1 gives the pull P_1 required to drag the block up the plane including friction. For it will be seen that the normal pressure on the plane is cd_1 and that the friction parallel to the direction of sliding, viz. normal to cd, is—

$$\mu \,.\, \overline{cd_1} = \tan\phi \,.\, \overline{cd_1} = \frac{d_1f_1}{d_1c} \times \overline{cd_1}$$
$$= d_1f_1$$

Then resolving d_1f_1 in the direction of the pull, we get the pull required to overcome the friction d_1e_1; hence the total pull required to both raise the block and to overcome the friction is be_1.

Least Pull.—The least pull required to pull the block up the plane is when be has its least value, *i.e.* when be is normal to ce; the direction of pull then makes an angle ϕ with the plane, or $\theta = \phi$, for cd is normal to the plane, and ce makes an angle ϕ with cd.

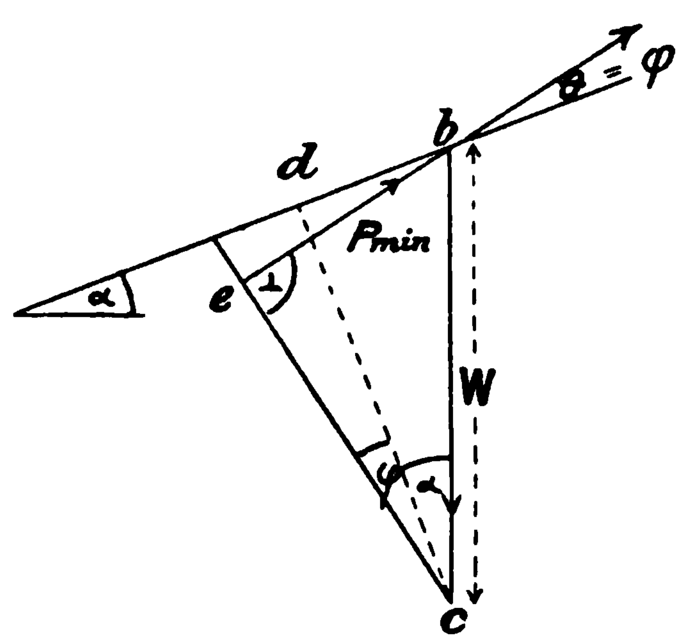


FIG. 224.

$$\text{Then } P_{min} = W \sin(\phi + \alpha) \quad . \quad . \quad . \quad . \quad \text{(i.)}$$

Horizontal Pull.—When the body is raised by a horizontal pull, we have (Fig. 225)—

$$P_b = W \tan (\phi + \alpha) \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(ii.)}$$

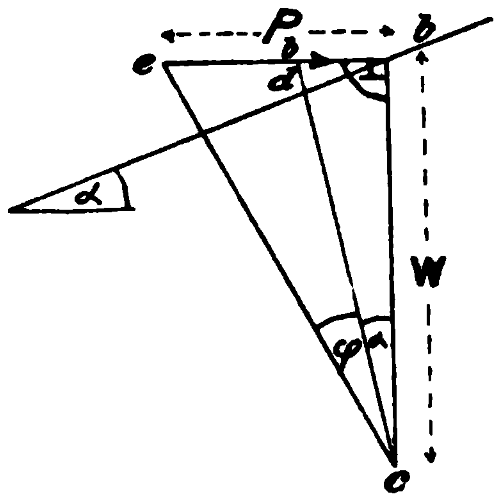


FIG. 225.

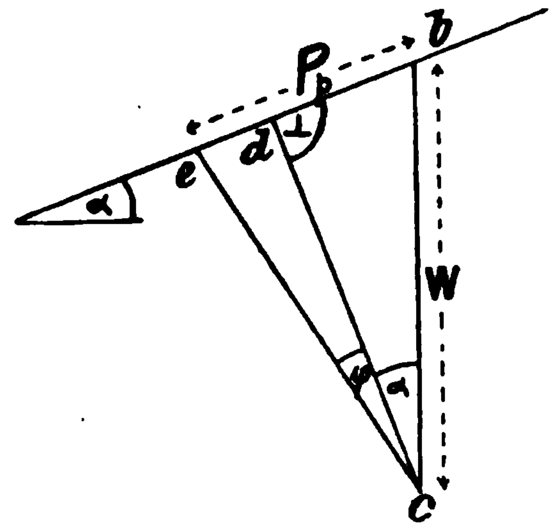


FIG. 226.

Thus, when the pull is horizontal, the effect of friction is equivalent to making the slope steeper by an amount equal to the friction angle.

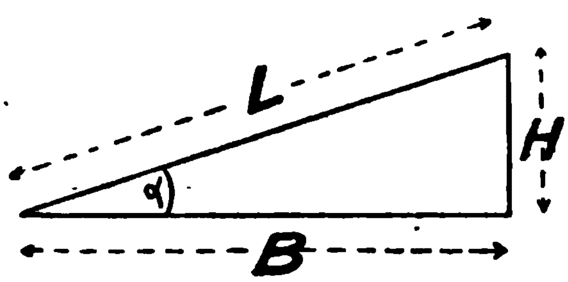


FIG. 227.

Parallel Pull. — When the body is raised by a pull parallel to the plane, we have (Fig. 226)—

$$P_p = ed + db$$

$$\text{But } \overline{ed} = \overline{dc} \tan \phi = \mu \overline{dc}$$

$$\text{and } \overline{dc} = W \,.\, \cos \alpha$$

$$\text{therefore } \overline{ed} = \mu \,.\, W \,.\, \cos \alpha$$

$$\text{and } \overline{db} = W \sin \alpha$$

$$\text{hence } P_p = W(\mu \,.\, \cos \alpha + \sin \alpha) \quad . \quad . \quad . \quad . \quad \text{(iii.)}$$

This may be expressed thus (see Fig. 227):

$$P_p = W\left(\mu\frac{B}{L} + \frac{H}{L}\right)$$

$$\text{or } P_pL = W\mu B + WH$$

$$\left.\begin{array}{l}\text{or, Work done in}\\ \text{dragging a body}\\ \text{of weight W up a}\\ \text{plane, by a force}\\ \text{acting parallel}\\ \text{to the plane}\end{array}\right\} = \left\{\begin{array}{l}\text{work done in dragging}\\ \text{the body through the}\\ \text{same horizontal dis-}\\ \text{tance against friction}\end{array}\right\} + \left\{\begin{array}{l}\text{work done}\\ \text{in lifting}\\ \text{the body}\end{array}\right.$$

General Case.—When the body is raised by a pull making an angle θ with the plane—

$$P = \frac{P_{min}}{\cos(\theta - \phi)}$$

Substituting the value of P_{min} from equation (i.)

$$P = \frac{W \sin(\phi + a)}{\cos(\theta - \phi)} \quad . \quad \text{(iv.)}$$

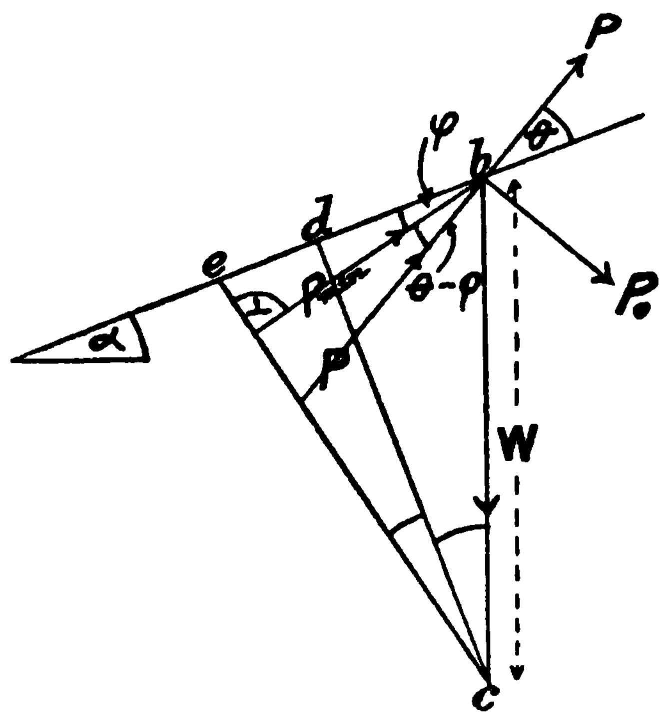


FIG. 228.

All the above expressions may be obtained from this.

When the direction of pull is parallel to ec, viz. P_0, it will only meet ec at infinity—that is, an infinitely great force would be required to make it slide; but this is impossible, hence the direction of pull must make an angle to the plane $\theta < (90 - \phi)$ in order that sliding may take place.

We must now consider the case in which a body is dragged down a plane, or simply allowed to slide down. If the angle a be less than ϕ, the body must be dragged down, and if a be greater, a force must be applied to prevent it from rushing down and being accelerated.

Least pull when body is lowered, $\phi < a$ (Fig. 229).

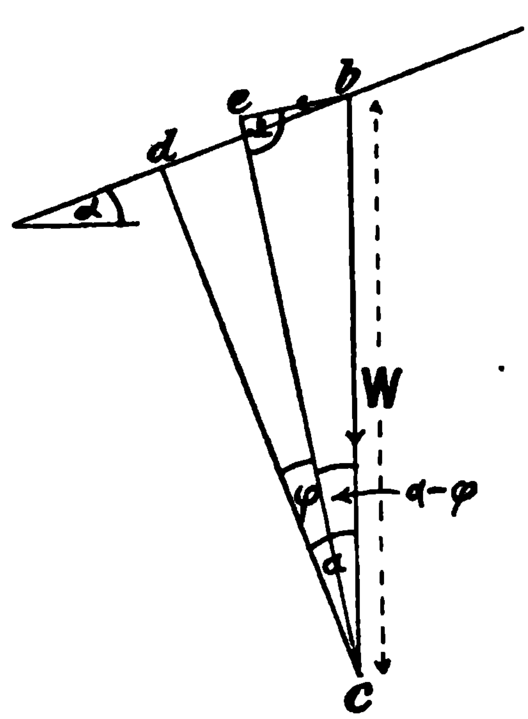


FIG. 229.

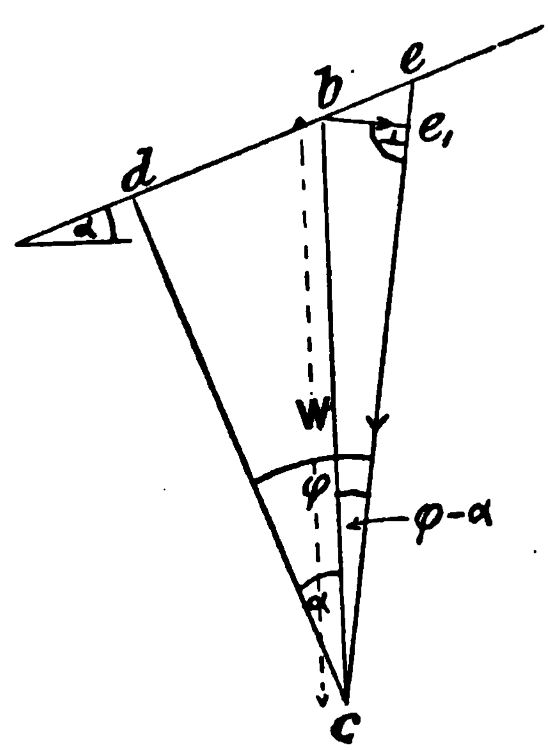


FIG. 230.

NOTE.—The friction now assists the lowering, hence ce is set off to the right of cd.

$$P_{min} = \overline{be} = W \sin (\alpha - \phi) \text{ and } \theta = \phi \quad . \quad . \quad . \quad \text{(v.)}$$

$\overline{be_1}$ is the least force acting up the plane required to prevent the body from sliding down (Fig. 230).

$$P_{min} = \overline{be_1} = W \sin (\phi - \alpha) \quad . \quad . \quad . \quad . \quad \text{(vi.)}$$

when $\phi = \alpha$, P_{min} is of course zero.

The remaining cases are arrived at in a similar manner; we will therefore simply state them.

	$\phi < \alpha$	$\phi > \alpha$
Least pull	$W \sin (\alpha - \phi)$	$W \sin (\phi - \alpha)$
Parallel pull ...	$W(\sin \alpha - \mu \cos \alpha)$	$W(\mu \cos \alpha - \sin \alpha)$
Horizontal pull ...	$W \tan (\alpha - \phi)$	$W \tan (\phi - \alpha)$
General case ...	$\dfrac{W \sin (\alpha - \phi)}{\cos (\theta + \phi)}$	$\dfrac{W \sin (\phi - \alpha)}{\cos (\theta - \phi)}$

NOTE.—If the line of pull comes below the plane, the angle θ takes the $-$ sign.

In the case of the parallel pull, it is worth noting that when $\phi < \alpha$, we have—

Total work done = work done in lowering the body − work done in dragging the body through the horizontal distance against friction

and when $\phi > \alpha$ we have the same relation, but the work done is negative, that is, the body has to be retarded.

Efficiency of Inclined Planes.—If an inclined plane be used as a machine for raising or lowering weights, we have—

$$\text{Efficiency} = \frac{\text{useful work done (}i.e.\text{ without friction)}}{\text{actual work done (with friction)}}$$

Inclined Plane when raising a Load.—The maximum efficiency occurs when the pull is least, *i.e.* when $\theta = \phi$. The useful work done without friction is when $\phi = 0$; then—

$$\text{The work done without friction} = \frac{\text{LW} \sin \alpha}{\cos \theta} \text{ from (iv.)}$$

$$\text{,, \quad ,, \quad with \quad ,,} \quad = \text{LW} \sin (\phi + \alpha) \text{ from (i.)}$$

where L = the distance through which the body is dragged;
α = the inclination of the plane to the horizon;
θ = the inclination of the force to the plane;
ϕ = the friction angle.

$$\text{Then maximum efficiency} = \frac{\frac{\mathrm{LW}\sin\alpha}{\cos\theta}}{\mathrm{LW}\sin(\phi+\alpha)} = \frac{\sin\alpha}{\cos\phi\sin(\phi+\alpha)} \quad \text{(vii.)}$$

When the pull is horizontal, $\theta = \alpha$, and—

$$\text{efficiency} = \frac{\sin\alpha}{\cos\alpha \, . \tan(\phi+\alpha)} = \frac{\tan\alpha}{\tan(\phi+\alpha)} \quad \text{(viii.)}$$

when the pull is parallel, $\theta = 0$, and $\cos\theta = 1$;

$$\text{efficiency} = \frac{\sin\alpha \, . \cos\phi}{\sin(\alpha+\phi)} \quad . \quad . \quad . \quad . \quad \text{(ix.)}$$

General case, when the line of pull makes an angle θ with the direction of sliding—

$$\text{efficiency} = \frac{\sin\alpha \, . \cos(\theta-\phi)}{\cos\theta \, . \sin(\phi+\alpha)} \quad . \quad . \quad . \quad \text{(x.)}$$

Friction of Wedge.—This is simply a special case of the inclined plane in which the pull is horizontal, or when it acts normal to W. We then have from equation (ii.) P = W tan $(\phi + \alpha)$ for a single inclined plane; but here we have two inclined planes, each at an angle α, hence W moves twice as far for any given movement of P, as in the single inclined plane; hence—

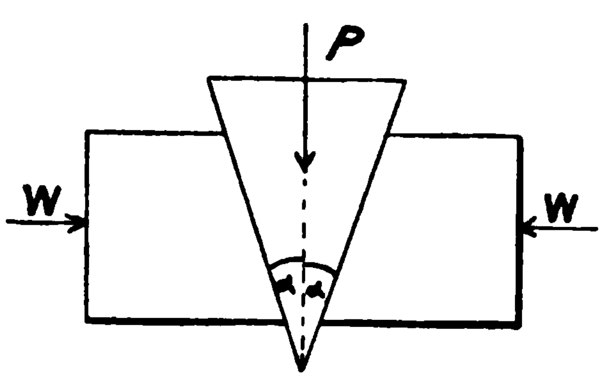


FIG. 231.

$$\mathrm{P} = 2\mathrm{W}\tan(\phi+\alpha) \text{ for a wedge}$$

The wedge will not hold itself in position, but will spring back, if the angle α be greater than the friction angle ϕ.

From the table on p. 200 we have the pull required to withdraw the wedge—

$$-\mathrm{P} = 2\mathrm{W}\tan(\alpha-\phi)$$

The efficiency of the wedge is the same as that of the inclined plane, viz.—

$$\text{efficiency} = \frac{\tan \alpha}{\tan (\alpha + \phi)} \text{ when overcoming a resistance} \quad \text{(xi.)}$$

$$\left.\begin{array}{r}\text{reversed}\\ \text{efficiency}\end{array}\right\} = \frac{\tan (\alpha - \phi)}{\tan \alpha} \text{ when withdrawing from a resistance}$$

(see page 226)

Efficiency of Screws and Worms. Square Thread.—A screw thread is in effect a narrow inclined plane wound round a cylinder, hence the efficiency is the same as that of an inclined plane. We shall, however, work it out by another method.

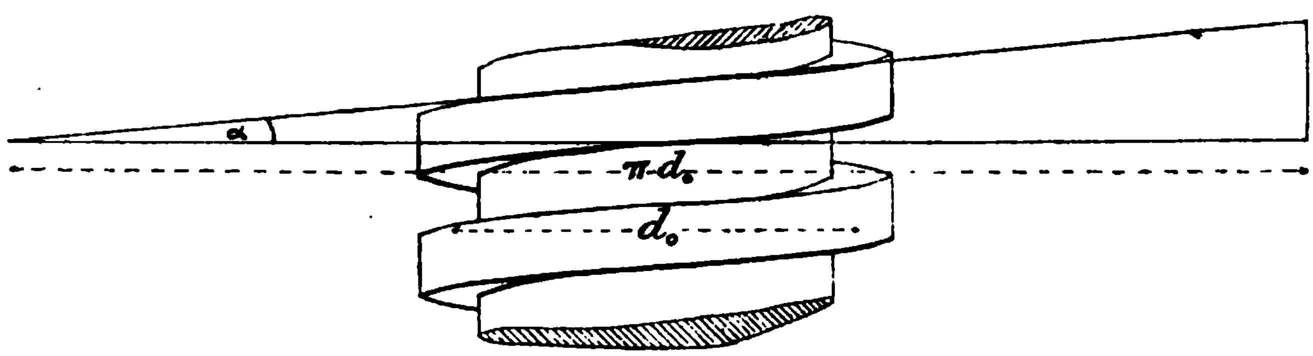


FIG. 232.

Let p = the pitch of the screw;
d_0 = the mean diameter of the threads;
W = the weight lifted.

$$\left.\begin{array}{l}\text{The useful work done per revolution}\\ \quad\text{on the nut without friction}\end{array}\right\} = \mathrm{W}p = \mathrm{W}\pi d_0 \tan \alpha$$

$$\left.\begin{array}{l}\text{The force applied at the mean radius of}\\ \quad\text{the thread required to raise nut}\end{array}\right\} = \mathrm{P} = \mathrm{W} \tan (\alpha + \phi)$$

(see equation ii.)

$$\left.\begin{array}{l}\text{The work done in turning the nut}\\ \quad\text{through one complete revolution}\end{array}\right\} = \mathrm{P}\pi d_0$$

$$= \mathrm{W}\pi d_0 \tan (\alpha + \phi)$$

$$\text{efficiency when raising the weight} = \frac{\mathrm{W}\pi d_0 \tan \alpha}{\mathrm{W}\pi d_0 \tan (\alpha + \phi)}$$

$$= \frac{\tan \alpha}{\tan (\alpha + \phi)}$$

This result will be found to be the same as that obtained for the inclined plane (equation xi.). The expression $\dfrac{\tan \alpha}{\tan (\alpha+\phi)}$ has its maximum value when $\alpha = 45^\circ - \dfrac{\phi}{2}$, and its value is then—

$$\text{maximum efficiency} = \frac{1 - \tan\frac{\phi}{2}}{1 + \tan\frac{\phi}{2}}$$

The proof of this is somewhat long; perhaps the easiest way of arriving at it is to calculate several values, plot a curve, and from it find a maximum.

In addition to the friction on the threads, the friction on the thrust collar of the screw must be taken into account. The thrust collar may be assumed to be of the same diameter as the thread; then the

$$\left.\begin{array}{l}\text{efficiency of screw thread}\\ \text{and thrust collar}\end{array}\right\} = \frac{\tan \alpha}{\tan(\alpha + 2\phi)} \text{ (approx.)}$$

In the case of a nut the radius at which the friction acts will be about $1\frac{1}{2}$ times that of the threads; we may then say—

$$\left.\begin{array}{l}\text{efficiency of screw thread and nut}\\ \text{bedding on a flat surface}\end{array}\right\} = \frac{\tan \alpha}{\tan(\alpha + 2{\cdot}5\phi)}$$

If the angle of the thread be very steep the screw will be reversible, that is, the nut will drive the screw. By similar reasoning to that given above, we have—

$$\text{reversed efficiency} = \frac{\tan(\alpha - \phi)}{\tan \alpha} \text{ (see p. 226)}$$

Such an instance is found in the Archimedian drill brace, and another in the twisted hydraulic crane-post used largely on board ship. By raising and lowering the twisted crane-post, the crane, which is in reality a part of a huge nut, is slewed round as desired.

Triangular Thread.—In the triangular thread the normal pressure on the nut is greater than in the square-threaded screw, in the ratio of $\frac{W_0}{W} = \frac{1}{\cos\frac{\theta}{2}}$, and $W_0 = \frac{W}{\cos\frac{\theta}{2}}$, where θ is the angle of the thread. In the Whitworth thread the angle θ is 55°, hence $W_0 = 1{\cdot}13$ W. In the Sellers thread $\theta = 60^\circ$ and $W_0 = 1{\cdot}15$ W; then, taking a mean value of $W_0 = 1{\cdot}14$ W, we have—

$$\text{efficiency} = \frac{\tan \alpha}{\tan(\alpha + 1{\cdot}14\phi)}$$

In the case of an ordinary bolt and nut, the radius at which the friction acts between the nut and the washer is about $1\frac{1}{2}$ times that of the thread, and, taking the same coefficient of friction for both, we have—

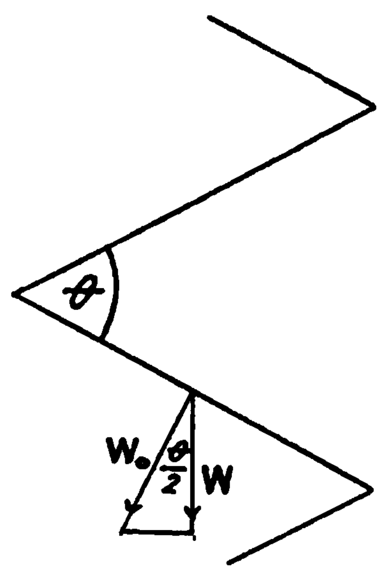


FIG. 233.

$$\left.\begin{array}{r}\text{efficiency}\\ \text{of a bolt}\\ \text{and nut}\end{array}\right\} = \frac{\tan \alpha}{\tan (\alpha + 2 \cdot 64\phi)} \text{ (approx.)}$$

The following table may be useful in showing roughly the efficiency of screws. In several cases they have been checked by experiments, and found to be fair average values; the efficiency varies greatly with the amount of lubrication.

TABLE OF APPROXIMATE EFFICIENCIES OF SCREW THREADS.

Angle of thread α.	Efficiency per cent. when no friction between nut and washer or a thrust collar.		Efficiency per cent. allowing for friction between nut and washer or a thrust collar.	
	Sq. thread.	V-thread.	Sq. thread.	V-thread.
2°	19	17	11	8
3°	26	23	14	12
4°	32	28	17	16
5°	36	33	21	20
10°	55	52	36	29
20°	67	63	48	42
$45 - \frac{\phi}{2}$	79	75	52	44

In the above table ϕ has been taken as $8 \cdot 5°$, or $\mu = 0 \cdot 15$.

Rolling Friction.—When a wheel rolls on a yielding material that readily takes a permanent deformation, the resistance is due to the fact that the wheel sinks in and makes a rut. The greater the weight W carried by the wheel, the deeper will be the rut, and consequently the greater will be the resistance to rolling.

When the wheel is pulled along, it is equivalent to constantly mounting an obstacle at A; then we have—

$$P \,.\, \overline{BA} = W \,.\, \overline{AC}$$

$$\text{or } P = \frac{W \,.\, \overline{AC}}{\overline{BA}}$$

Let AC = K;

$$\text{Then } P = \frac{W \,.\, K}{R - h}$$

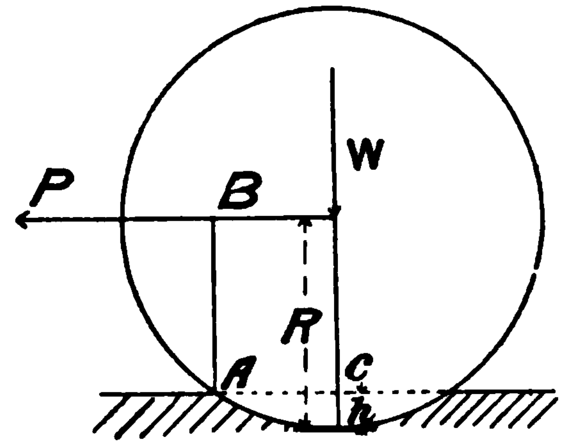


FIG. 234.

But h is usually small compared with R; hence we may write—

$$P = \frac{WK}{R} \text{ (nearly)}$$

P and W, also K and R, must be measured in the same units, or the value of K corrected accordingly. The above relation holds fairly well in practice, but the state of our knowledge of rolling resistance is in a very unsatisfactory state.

VALUES OF K.

	K (inches).
Iron or steel wheels on iron or steel rails	0·007–0·015
,, ,, wood	0·06–0·10
,, ,, macadam	0·05–0·20
,, ,, soft ground	3–5
Pneumatic tyres on good road or asphalte	0·02–0·022
,, ,, heavy mud	0·04–0·06
Solid indiarubber tyres on good road or asphalte ...	0·04
,, ,, heavy mud	0·09–0·11

Some years ago Professor Osborne Reynolds investigated the action of rollers passing over elastic materials, and showed clearly that when a wheel rolls on, say, an indiarubber road, it sinks in and compresses the rubber immediately under it, but forces out the rubber in front and behind it, as shown in the sketch. That forced up in the front slides on the surface of the wheel in just the reverse direction to the motion of the wheel, and so hinders its progress. Likewise, as the wheel leaves the heap behind it, the rubber returns to its original place, and again slips on the wheel in the reverse direction to

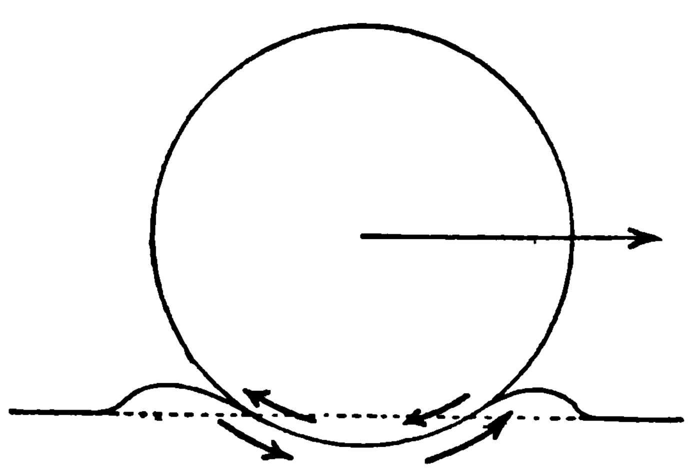


FIG. 235.

its motion. Thus the resistance to rolling is in reality due to the sliding of the two surfaces. On account of the stretching of the path over which the wheel rolls, the actual distance rolled over is greater than the horizontal distance travelled by the wheel.

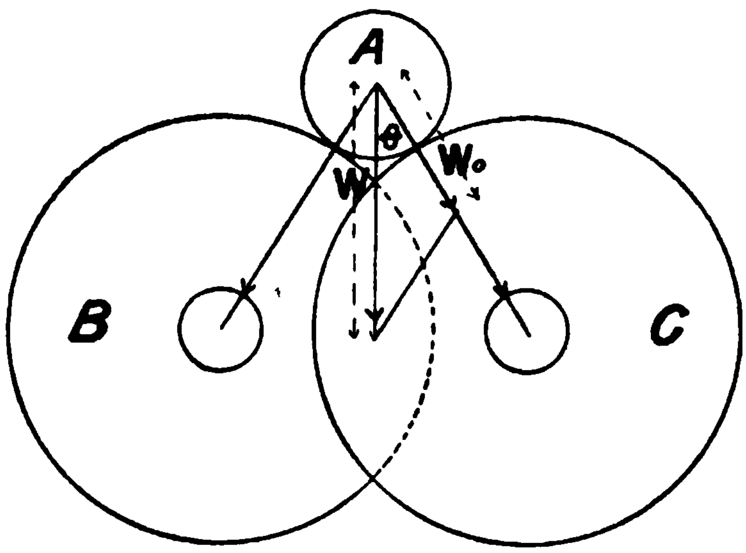


FIG. 236.

Antifriction Wheels.—In order to reduce the friction on an axle it is sometimes mounted on antifriction wheels, as shown. A is the axle in question, B and C are the antifriction wheels. If W be the load on the axle, the load on each antifriction wheel bearing will be—

$$W_0 = \frac{W}{2 \cos \theta}. \text{ and the load on both } \frac{W}{\cos \theta}$$

Let R_a = the radius of the main axle;
R = ,, ,, antifriction wheel;
r = ,, ,, axle of the antifriction wheel.

Then, neglecting the rolling resistance as being very small, we have—

The frictional resistance on the surface of the *antifriction wheel axles* $\Big\} = \mu 2 W_0 = \dfrac{\mu W}{\cos \theta}$

and the frictional resistance referred to the surface of the antifriction wheels, or the surface of the main axle $\Big\} = \dfrac{\mu W r}{R \cdot \cos \theta}$

If the main axle were running in plain bearings, the resistance would be μW; hence—

$$\frac{\text{friction with plain bearings}}{\text{friction with antifriction wheels}} = \frac{\mu W}{\dfrac{\mu W r}{R \cdot \cos \theta}}$$

$$= \frac{R \cdot \cos \theta}{r} = \frac{0{\cdot}87R}{r}$$

when θ has its usual value of 30°.

Of course, two such sets of wheels are required for mounting an axle, and unless the wheel axles are perfectly parallel, and the wheels true and of the same size, a great deal of

Longitudinal Section *on Line A-B* *on Centre Line*

FIG. 237.[1]

trouble will be experienced with the main axle travelling sideways. The author has had to use ball thrust bearings to prevent this lateral motion.

Roller Bearings.—The shaft rolls on hard steel rollers, which are kept apart and kept square by the gun-metal cages shown in section (Fig. 237). The outer casing is of cast iron. The friction is about the same as with a well-fitted ball bearing, but the rollers will, of course, stand very much heavier loads than balls. The wear is exceedingly small even when running without any oil. They are being largely used for railway rolling stock.

Ball Bearings.—In the early days of roller bearings the end friction of the rollers was a constant source of trouble; in order to avoid this, balls were substituted for rollers, the ball races being either **V** or rounded grooves made of hardened steel ground to a perfectly true surface.

In some experiments by the author, he found that the friction of ball bearings held an intermediate position between the friction of a plain bearing with syphon lubrication, and that with bath lubrication; but on starting from rest the friction of the ball bearing was in every instance far lower, often less than one-half, of even that of a bearing with bath lubrication. The same experiments also appeared to show that the friction of a ball bearing—

[1] Reproduced by the kind permission of the Roller Bearings Company, 1, Delahay Street, Westminster, S.W.

(1) Varies directly as the load.
(2) Is independent of the speed.
(3) Is independent of the temperature.

The main point to be watched when designing ball-bearings is to get a true rolling motion of the balls without any grinding.

In Fig. 238 is shown a section of a ball thrust bearing. If the sides of the ball races were made of the same slope the balls would grind, because the circumference of the ball race at *a* moves faster than that of *b*. In order that there may be no grinding, the circle on which *a* rolls must be greater

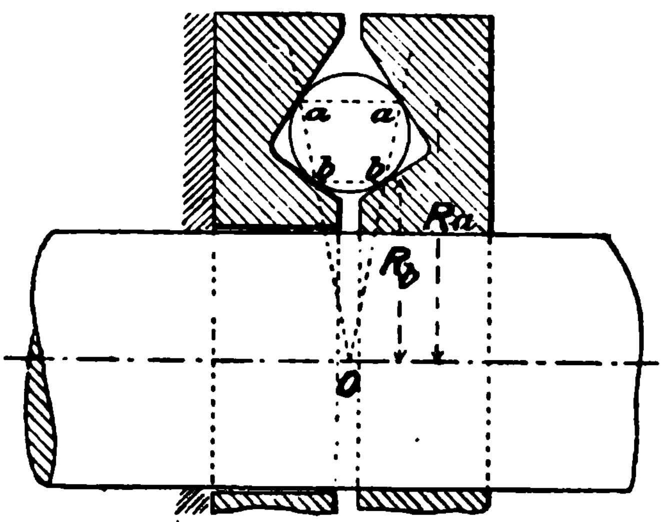


FIG. 238.

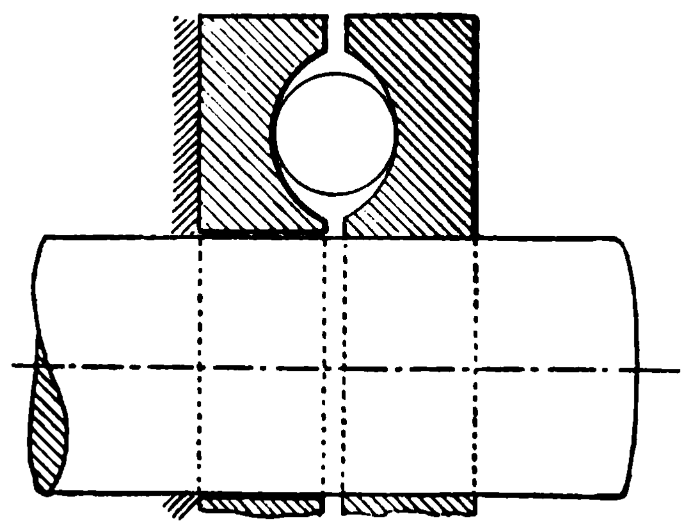


FIG. 239.

than that on which *b* rolls in the ratio of their distances from the centre of the shaft, or $\frac{aa}{bb} = \frac{Ra}{Rb}$, which is secured by the construction shown, since the triangles O*aa* and O*bb* are similar.

Although this form of bearing is right in principle, it is not found to work well in practice, probably because the exact conditions are upset when any wear takes place. From some very severe trials, the author has come to the conclusion that the form shown in Fig. 239 is in every respect better,[1] and finds that his experience quite accords with that of others who have largely used ball bearings.

For much valuable matter on this question, the reader is referred to Sharpe's book on "Bicycles and Tricycles."

Friction of Lubricated Surfaces.—The laws which appear to express the behaviour of well-lubricated surfaces are almost the reverse of those of dry surfaces. For the sake of comparison we tabulate them below, side by side:—

[1] Sometimes a gun-metal cage to keep the balls in position is found necessary.

Dry Surfaces.	*Lubricated Surfaces.*
The frictional resistance is nearly proportional to the normal pressure between the two surfaces.	1. The frictional resistance is almost independent of the pressure with bath lubrication, and approaches the behaviour of dry surfaces as the lubrication becomes more meagre.
The frictional resistance is nearly independent of the speed for low pressures.	2. The frictional resistance varies directly as the speed for low pressures. But for high pressures the friction is very great at low velocities, becoming a minimum at about 100 ft. per minute, and afterwards increases approximately as the square root of the speed. N.B.—The friction of liquids varies as the *square*, not as the *square root* of the speed, hence the friction of a well-lubricated bearing is not merely that of the lubricant.
The frictional resistance is not greatly affected by the temperature.	3. The frictional resistance depends more upon the temperature than on any other condition—partly due to the variation in the viscosity of the oil, and partly to the fact that the diameter of the bearing increases with a rise of temperature more rapidly than the diameter of the shaft, and thereby relieves the bearing of side pressure.
The frictional resistance depends largely upon the nature of the material of which the rubbing surfaces are composed.	4. The frictional resistance with a flooded bearing depends but slightly upon the nature of the material of which the surfaces are composed, but as the lubrication becomes meagre, the friction follows much the same laws as in the case of dry surfaces.
The friction of rest is slightly greater than the friction of motion.	5. The friction of rest is enormously greater than the friction of motion, especially if thin lubricants be used, probably due to them being squeezed out when standing.
When the pressures between the surfaces become excessive, seizing occurs.	6. When the pressures between the surfaces become excessive, which is at a much higher pressure than with dry surfaces, the lubricant is squeezed out and seizing occurs. The pressure at which this occurs depends upon the viscosity of the lubricant.

Dry Surfaces.

The frictional resistance is greatest at first, and rapidly decreases with the time after the two surfaces are brought together, probably due to the polishing of the surfaces.

The frictional resistance is always greater immediately after reversal of direction of sliding.

Lubricated Surfaces.

7. The frictional resistance is least at first, and rapidly increases with the time after the two surfaces are brought together, probably due to the partial squeezing out of the lubricant.

8. Same as in the case of dry surfaces.

We will now give the results of a few experiments in support of the laws of lubricated surfaces given above.

1. The curves given in Fig. 240 were taken from diagrams autographically drawn by the author's own friction-testing

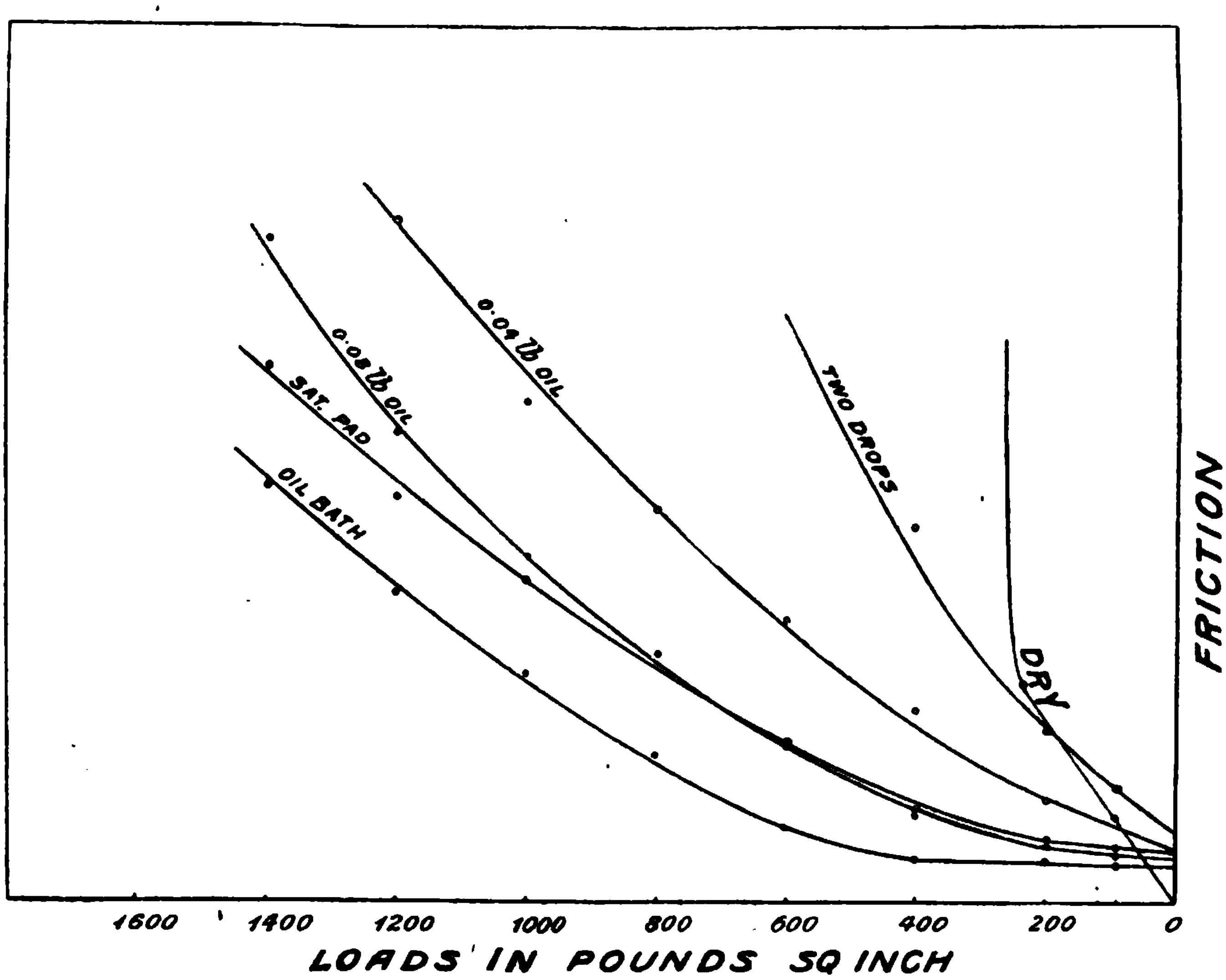


FIG. 240.

machine. They speak for themselves. The weight of pad used in each case was 0·037 lb.

It is interesting to note that the friction of a dry bearing is actually less at very low loads than the lubricated bearings, on account of the viscosity of the oil being approximately constant at all loads. After about 400 lbs. square inch the oil appears to get squeezed out.

2. The authorities, with the conditions for the various

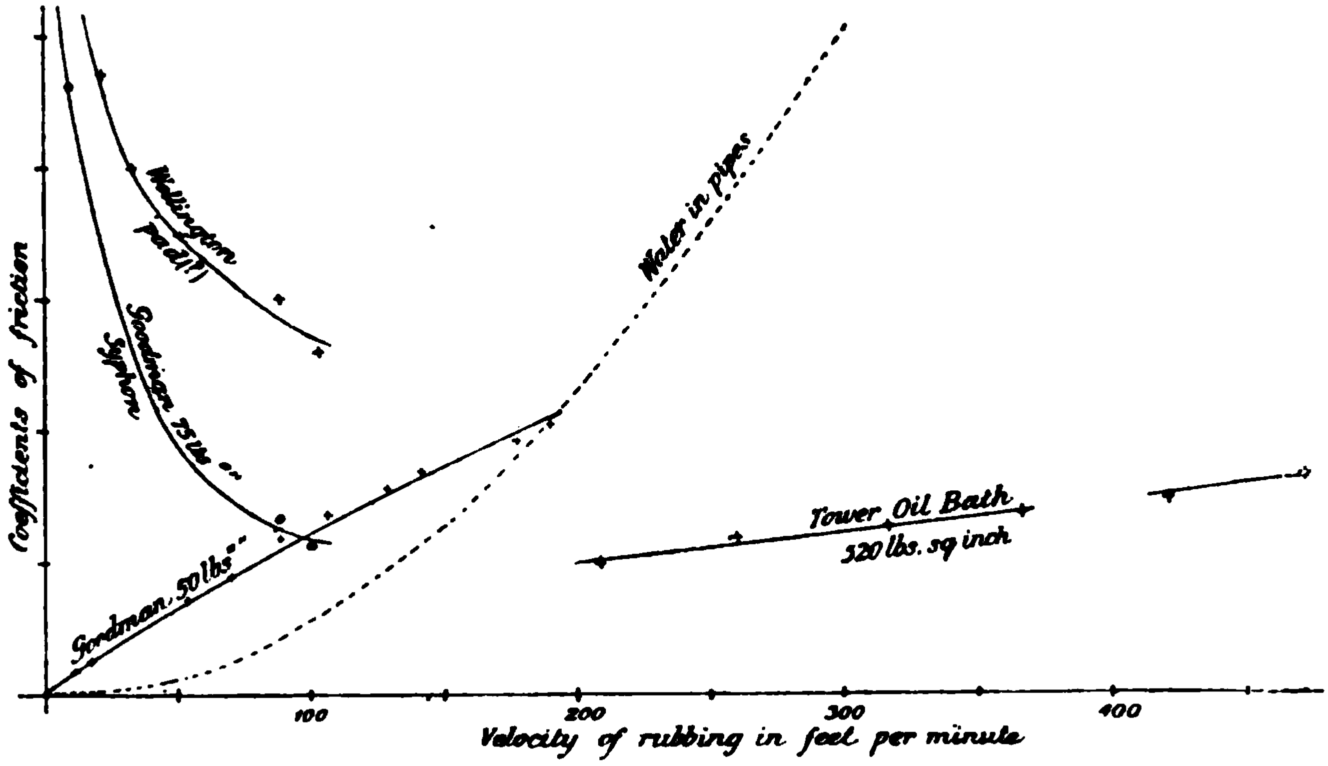


FIG. 241.

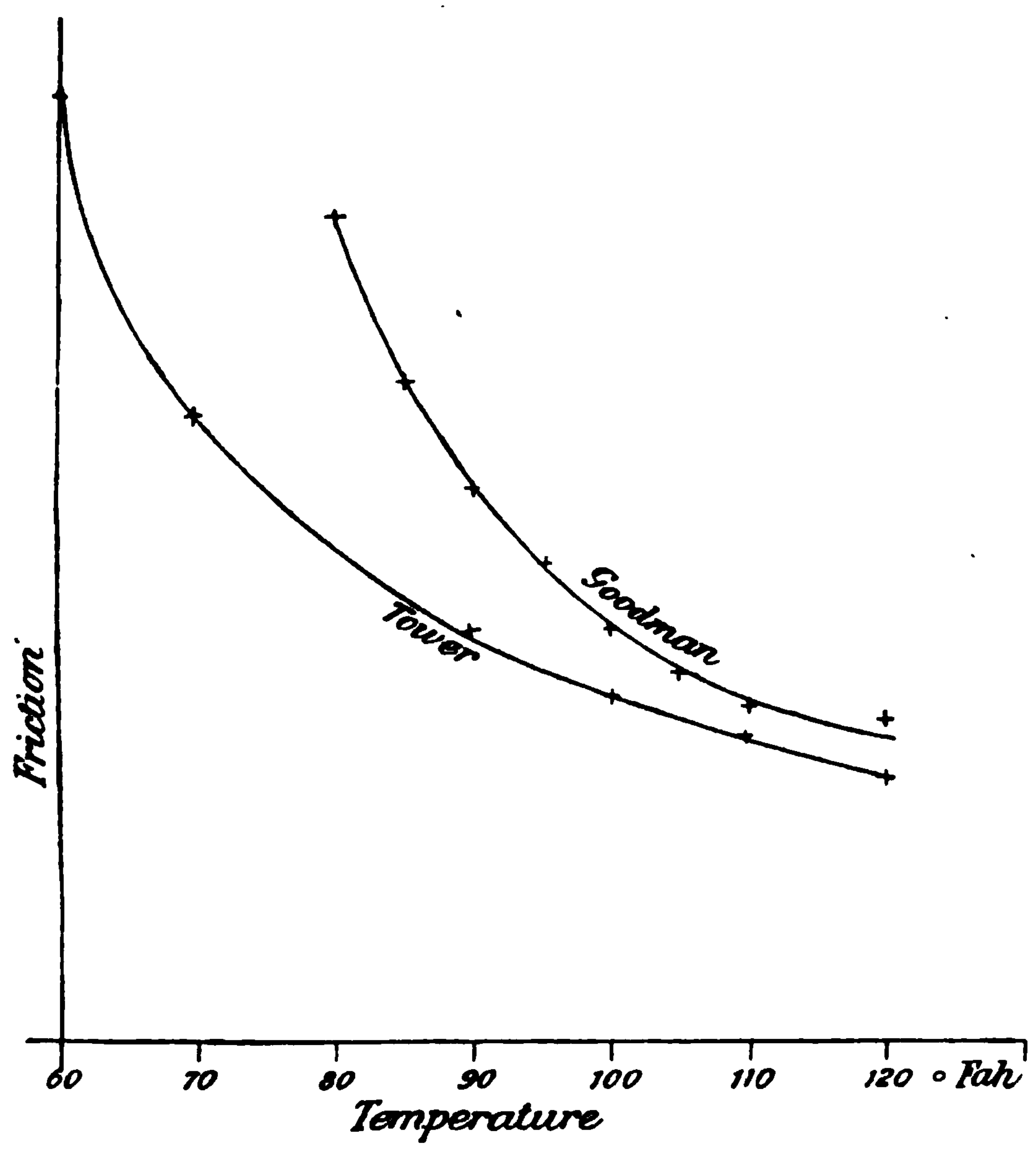


FIG. 242.

experiments, are given on the curves. Also one for water

flowing through a pipe, but not to the same scale of friction as the others.

3. The two typical curves selected will serve to make this point clear.

4. Mr. Tower has shown that in the case of a flooded bearing there is no metallic contact between the shaft and bearing; it is therefore perfectly evident that under such circumstances the material of which the bearing is composed makes no difference to the friction. When the author first began to experiment on the relative friction of antifriction metals, he used profuse lubrication, and was quite unable to detect the slightest difference in the friction; but on using the smallest amount of oil consistent with security against seizing, he was able to detect a very great deal of difference in the friction. In the table below, the two metals A and B only differed in composition by changing one ingredient, amounting to 0·23 per cent. of the whole.

Load in lbs. sq. inch	150	250	350	450	750	950
Coefficient of friction A	0·0143	0·0112	0·0091	0·0082	0·0075	0·0083
Coefficient of friction B	0·0083	0·0062	0·0054	0·0050	0·0045	0·0047

5. The following tests by Thurston will show how much greater is the friction of rest than of motion :—

Load in lbs. sq. inch.	50	100	250	500	750	1000
Coefficient of friction, 150 ft. min.	0·013	0·008	0·005	0·004	0·0043	0·009
Coefficient of friction, At instant of starting	0·07	0·135	0·14	0·15	0·185	0·18

Oil used, sperm.

6. Experiments by Tower and others show that a steel shaft in a gun-metal bearing seizes at about 600 lbs. square inch under steady running, whereas when dry the same materials seize at about 80 lbs. square inch. The author finds that the seizing pressure increases as the viscosity of the oil increases.

7. This diagram is one of many drawn autographically on the author's machine. The lever which applies the load on the bearing was lifted, and the machine allowed to run with only the weight of the bearing itself upon it; the lever

was then suddenly dropped, the friction being recorded automatically.

An indirect proof of this statement is to be found in the case of connecting-rod ends, and on pins on which the load

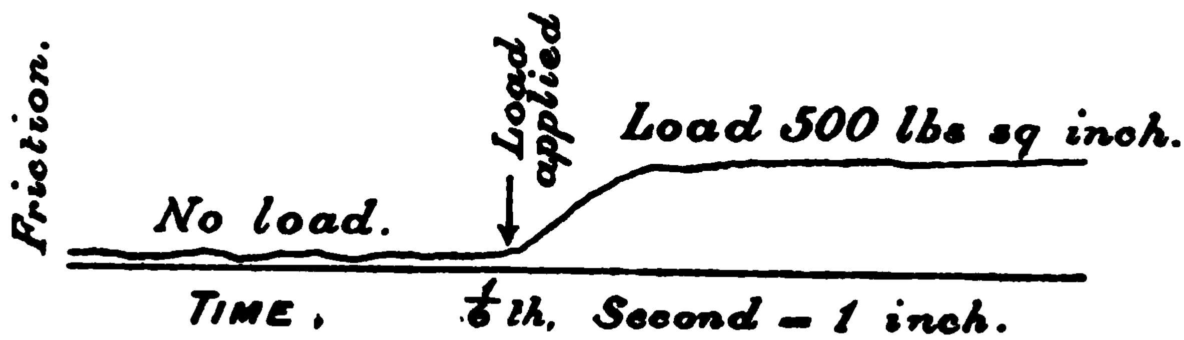


FIG. 243.

is constantly reversed; at each stroke the oil is squeezed away from the pressure side of the pin to the other side. Then, when the pressure is reversed, there is a large supply of oil between the bearing and the pin, which gradually flows to the other side. Hence at first the bearing is floating on oil, and the friction is consequently very small; as the oil flows away, the friction increases. This is the reason why a much higher bearing pressure may be allowed in the case of a connecting-rod end than in a constantly revolving bearing.

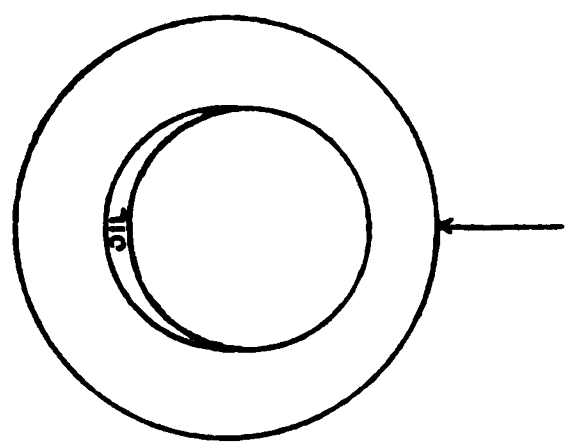


FIG. 244.

8. In friction-testing machines it is always found that the temperature and the friction of a bearing is higher after reversal of direction, but in the course of a few hours it gets back to the normal again. Some metals, however, appear to have a grain, as the friction is always much greater when running one way than when running the other way.

Nominal Area of Bearing.—The pressure on a cylindrical bearing varies from point to point; it is a maximum at the crown, and is least at the two sides. Calculations as to the distribution of pressure are possible, but the assumptions usually made are most unwarrantable. For all practical purposes, the pressure is assumed to be evenly distributed over the projected area of the bearing. Thus, if w be the width of the bearing across the chord, and l the length of the

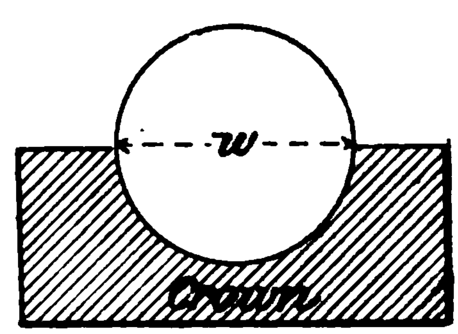


FIG. 245.

bearing, the nominal area is wl, and the nominal pressure per square inch is $\frac{W}{wl}$, where W is the total load on the bearing.

Beauchamp Tower's Experiments.—These experiments were carried out for a research committee of the Institution of Mechanical Engineers, and deservedly hold the highest place amongst friction experiments as regards accuracy. The reader is referred to the Reports for full details in the Institution *Proceedings*, 1885.

Most of the experiments were carried out with oil-bath lubrication, on account of the difficulty of getting regular

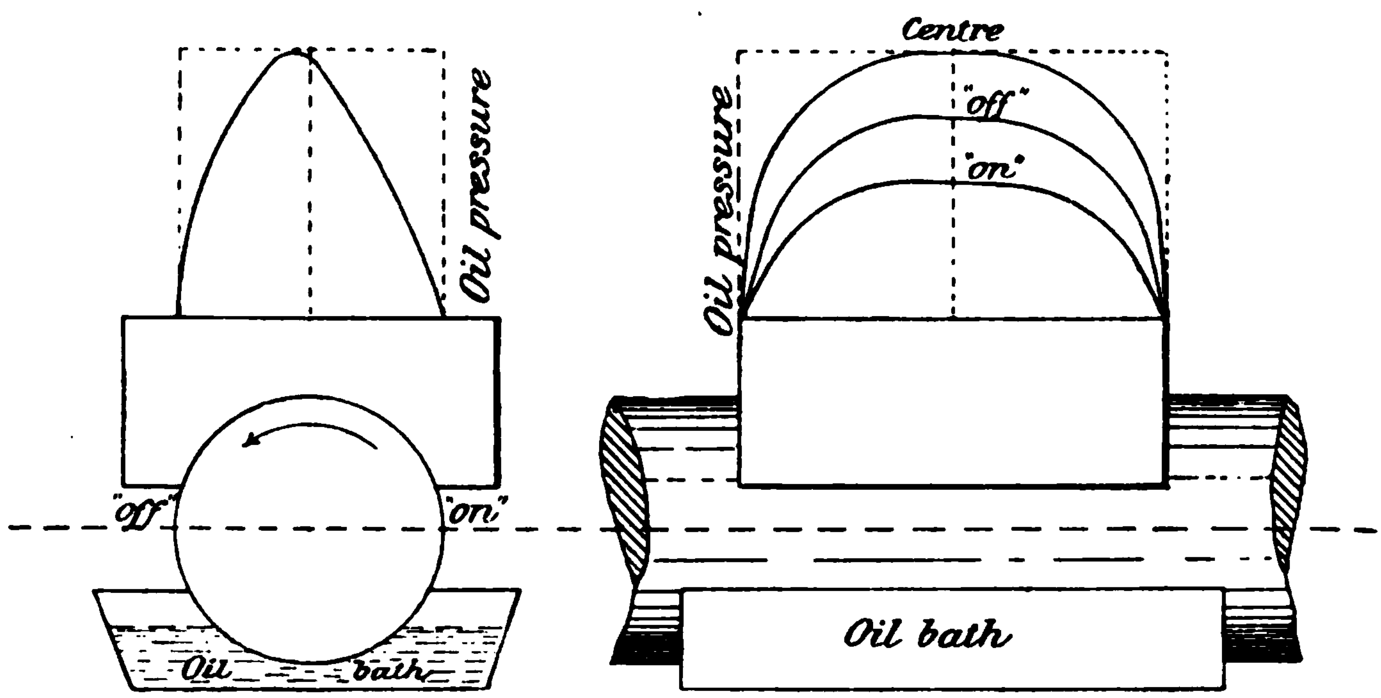


FIG. 246.

lubrication by any other system. It was found that the bearing was completely oil-borne, and that the oil pressure varied as shown in the curves, the pressure being greatest

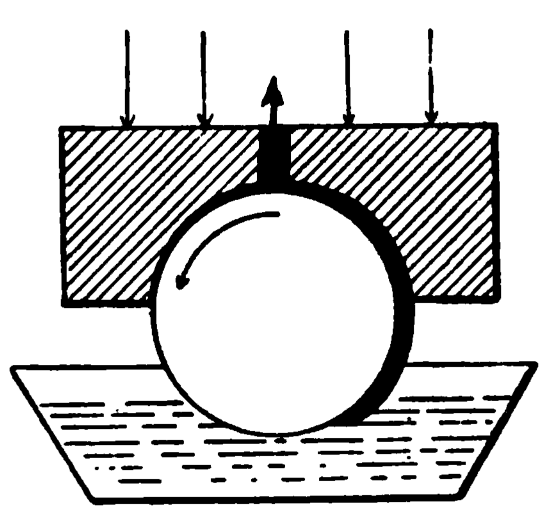


FIG. 247.

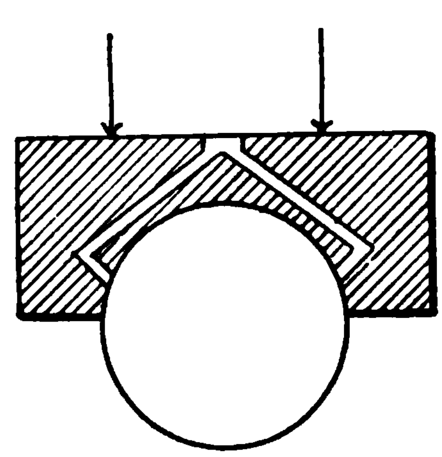


FIG. 248.

on the "off" side. In this connection Mr. Tower shows that it is useless—ay, worse than useless—to drill an oil-hole on the

resultant line of pressure of a bearing, for not only cannot oil possibly be fed to the bearing by such means, but that oil is collected from other sources and forced *out* of the hole (Fig. 247), thus robbing the bearings of oil at exactly the spot where it is most required. If oil-holes are used, they must communicate with a part of the bearing where there is little or no pressure (Fig. 248). Other of Mr. Tower's experiments are referred to in preceding and succeeding paragraphs.

Professor Osborne Reynolds' Investigations.—A theoretical treatment of the friction of a flooded bearing has been investigated by Professor Osborne Reynolds, a full account of which will be found in the *Philosophical Transactions*. In this investigation, he has shown a complete agreement between theory and experiment as regards the total frictional resistance of a flooded bearing, the distribution of oil pressure, and the thickness of the oil film, besides many other points of the greatest interest. Professor Petroff, of St. Petersburg, has also done very similar work, and has, moreover, shown both experimentally and theoretically how the thickness of the oil film varies with the viscosity of the oil and with the pressure on the bearing.

Goodman's Experiments.—The author, shortly after the results of Mr. Tower's experiments were published, repeated his experiments on a much larger machine belonging to the L. B. & S. C. Railway Company; he further found that the oil pressure could only be registered when the bearing was flooded; if a sponge saturated with oil were applied to the bearing, the pressure was immediately shown on the gauge, but as the oil ran away and the supply fell off, so the pressure fell.

In another case a bearing was provided with an oil-hole on the resultant line of pressure, to which a screw-down valve was attached. When the oil-hole was open the friction on the bearing was very nearly 25 per cent. greater than when it was closed and the oil thereby prevented from escaping.

Another bearing was fitted with a micrometer screw for the purpose of measuring the thickness of the oil film; in one instance, in which the conditions were similar to those assumed by Professor Reynolds, the thickness by measurement was found to be 0·0004 inch, and by his calculation 0·0006 inch. By the same appliance the author found that the thickness was greater on the "on" side than on the "off" side of the bearing. The wear always takes place where the film is thinnest, *i.e.* on the "off" side of the bearing—exactly the reverse of what

would be expected if the shaft were regarded as a roller, and the bearing as being rolled forwards. When white metal bearings are tested to destruction, the metal always begins to fuse on the "off" side first, and bearings originally 1 inch thick are frequently only 0·9 inch thick at *a* after a severe test. The figure shows roughly how the bearing wears.

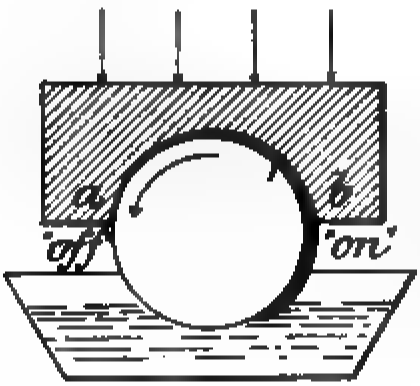


FIG. 249.

The side on which the wear takes place depends, however, upon the arc of the bearing in contact with the shaft. When the arc subtends an angle greater than about 90°, the wear is on the "off" side; if less than 90°, on the "on" side. This wear was measured thus: The four screws

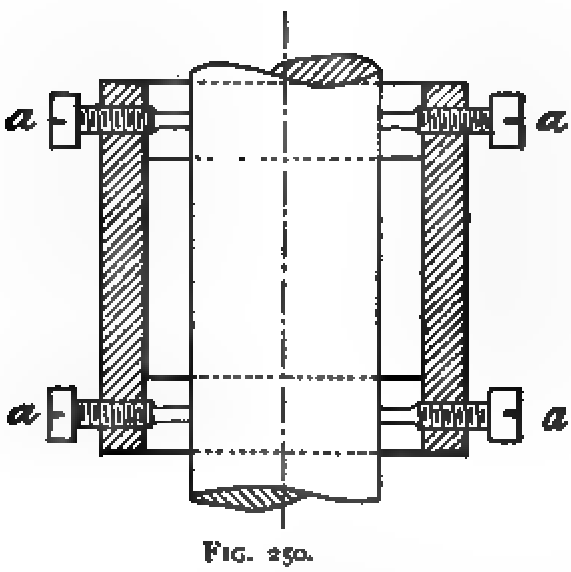


FIG. 250.

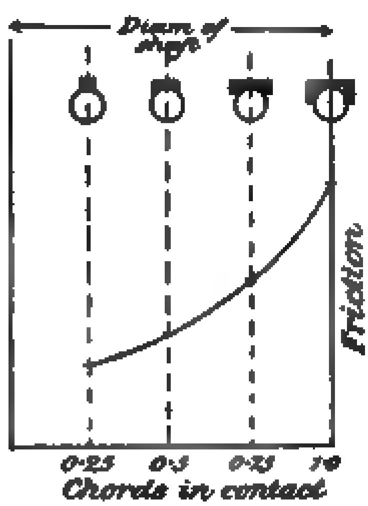


FIG. 251.

a, *a*, *a*, *a* were fitted to an overhanging lip on the bearing as shown. They were composed of soft brass. Before commencing a run, they were all tightened up to just touch the shaft; on removing the bearing after some weeks' running, it was seen at once which screws had been bearing and which were free.

Another set of experiments were made in 1885, to ascertain the effect of cutting away the sides of a bearing. The bearings experimented upon were semicircular to begin with, and the sides were afterwards cut away step by step till the width of the bearing was only $\frac{1}{4}$ *d*. The effect of removing the sides is shown in Fig. 251.

The relation may be expressed by the following empirical formula :—

Let R = frictional resistance when width of bearing = W ;
R_1 = " " " " = W_1 ;
D = diameter of journal.

$$\text{Then } R_1 = R\left(4{\cdot}7^{\frac{W_1 - W}{D}}\right)$$

Methods of Lubricating.—In some instances a small

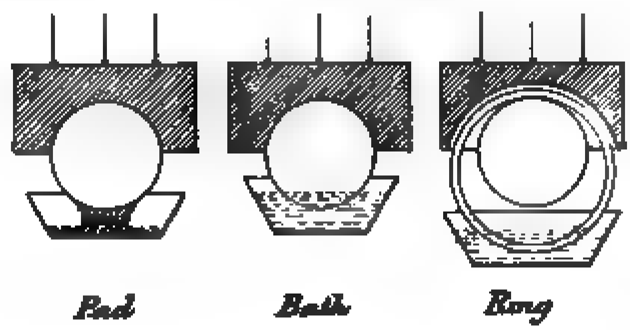


Fig. 252.

Fig. 253.—Collar bearing

Fig. 254.—Pivot or footstep.

force-pump is used to force the oil into the bearing; it then becomes equivalent to bath lubrication. An excellent instance of this is the Belliss engine, in which the oil is forced under pressure into every bearing. These engines have been known to run a whole year without stopping, with an exceedingly small amount of friction.

RELATIVE EFFICIENCY OF DIFFERENT SYSTEMS OF LUBRICATION.

Mode of lubrication.	Tower.	Goodman.
Bath	1·00	1·00
Saturated pad ...	—	1·32
Ordinary pad ...	6·48	2·21
Syphon	7·06	4·23

Seizing of Bearings.—It is well known that when a bearing is excessively loaded, the lubricant is squeezed out, and the friction takes place between metal and metal; the two surfaces then appear to weld themselves together, and if the bearing be forced round, small pieces are torn out of both surfaces. The load at which this occurs depends much upon the initial smoothness of the surfaces and upon the nature of the material, but chiefly upon the viscosity of the oil. If only the viscosity can be kept up by artificially keeping the bearing cool, by water-circulation or otherwise, the surfaces will not

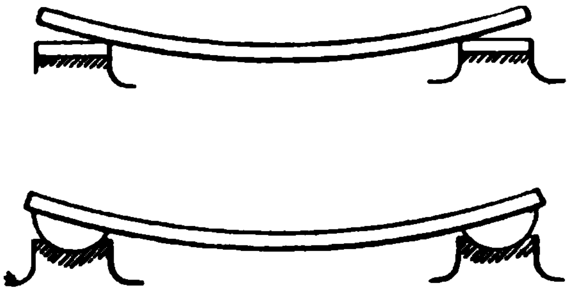


FIG. 255.

seize until the pressure becomes enormous. The author has had a bearing running for weeks under a load of *two tons per square inch* at a surface velocity of 230 feet per minute with pad lubrication, temperature being artificially kept at 110° Fahr. by circulating water through the axle.[1]

[1] In another instance nearly *four* tons per square inch for several hours.

Seizing not unfrequently occurs through unintentional high pressures on the edges of bearings. A very small amount of spring will cause a shaft to bear on practically the edge of the bearing (Fig. 255), and thereby to set up a very intense pressure. This can be readily avoided by using spherical-seated bearings as shown. For several examples of such bearings the reader is referred to Unwin's "Elements of Machine Design."

Seizing is very rare indeed with soft white metal bearings; this is probably due to the metal flowing and adjusting itself when any uneven pressure comes upon it. This flowing action is seen clearly in Fig. 256, which is from photographs. The lower portion shows the bearing before it was tested in a friction-testing machine, and the upper portion after it was tested. The metal began to flow at a temperature of 370° Fahr., under a pressure of 2000 lbs. per square inch; surface speed, 2094 feet per minute.

Fig. 256.

Area of Bearing Surfaces.—From our remarks on seizing, it will be evident that the safe working pressure for revolving bearings largely depends upon their temperature and the lubricant that is used. If the temperature rise abnormally, the viscosity of the oil is so reduced that it gets squeezed out. The temperature that a bearing attains to depends (1) on the heat generated; (2) on the means for conducting away the heat.

Let S = surface speed in feet per minute;
W = load on the bearing in pounds;
t = number of thermal units conducted away per square inch of bearing per minute.

Then—

$$\text{Work done per minute in foot-pounds} = \mu WS$$

$$\text{thermal units generated per minute} = \frac{\mu WS}{772}$$

$$\text{nominal area of bearing surface A} = \frac{\mu WS}{772t}$$

The following may be taken as fair averages:—

VALUES OF μ AND t.

Method of lubrication.	Value of μ.
Bath	0·004
Pad	0·012
Syphon	0·020

Conditions of running.	Values of t.	
	Crank and other pins.	Continuous running bearing.
Exposed to currents of cold air or other means of cooling, as in locomotive or car axles	4–7	1–1·5
In tolerably cool places, as in marine and stationary engines	0·75–1	0·3–0·5
In hot places and where heat is not readily conducted away	0·4–0·5	0·1–0·3

After arriving at the area by the method given above, it should be checked to see that the pressure is not excessive.

Bearing.	Maximum permissible pressure per sq. inch.
Crank-pins.—Locomotive	1500
Marine and stationary	600
Shearing machines	3000
Gudgeon pins.—Locomotive	2000
Marine and stationary	800
Railway car axles	350
Ordinary pedestals.—Gun-metal	200
Good white metal	500
Collar and thrust bearings.—Gun-metal	80
Good white metal	200
Lignum vitæ	50
Slide blocks.—Cast iron or gun-metal	80
Good white metal	250

Work absorbed in Revolving Bearings.

Let W = total load on bearing in pounds;
D = diameter of bearing in inches;
N = number of revolutions per minute.

For Cylindrical Bearings.—

$$\text{Work done per minute in foot-pounds} = \frac{\mu W \pi D N}{12}$$

$$\text{horse-power absorbed} = \frac{W \pi D N \mu}{12 \times 33{,}000}$$

$$= \frac{\mu W D N}{126{,}000}$$

An extremely convenient rough-and-ready estimate of the work absorbed by a bearing is to assume that the frictional resistance on the surface of a bearing is 3 lbs. per square inch for ordinary lubrication, 2 lbs. for pad, 1 lb. for bath, the surface being reckoned on the nominal area.

Flat Pivot.—If the thrust be evenly distributed over the whole surface, the intensity of pressure is—

$$p = \frac{W}{\pi R^2}$$

$$\text{pressure on an elementary ring} = 2\pi r p \,.\, dr$$

$$\left.\begin{array}{r}\text{moment of friction on an}\\ \text{elementary ring}\end{array}\right\} = 2\pi r^2 \mu p \,.\, dr$$

$$\left.\begin{array}{r}\text{moment of friction on}\\ \text{whole surface}\end{array}\right\} = 2\pi\mu p \int r^2 \,.\, dr$$

$$M_f = \frac{2\pi\mu p R^3}{3}$$

Substituting the value of p from above—

$$M_f = \tfrac{2}{3}\mu W R$$

$$\text{work done per minute in foot-pounds} = \frac{\mu W D N}{5{\cdot}73}$$

$$\text{horse-power absorbed} = \frac{\mu W D N}{189{,}000}$$

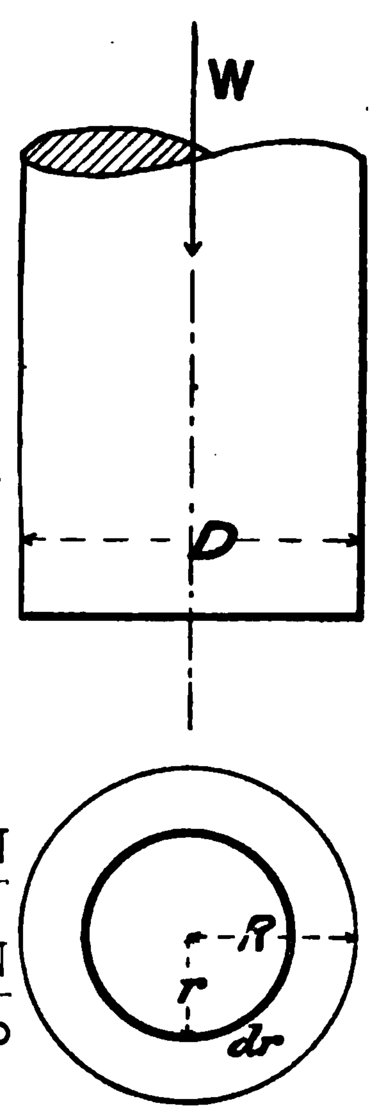


FIG. 257.

This result might have been arrived at thus: Assuming the load evenly distributed, the triangle shows the distribution of pressure, and consequently the distribution of the friction. The centre of gravity

of the triangle is then the position of the resultant friction, which therefore acts at a radius equal to $\frac{2}{3}$ radius of the pivot.

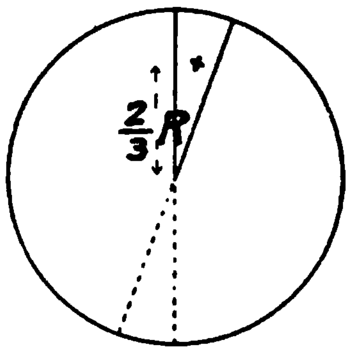


FIG. 258.

If it be assumed that the unequal wear of the pivot causes the pressure to be unevenly distributed in such a manner that the product of the normal pressure p and the velocity of rubbing V be a constant, we get a different value for M_f; the $\frac{2}{3}$ becomes $\frac{1}{2}$. It is very uncertain, however, which is the true value. The same remark also applies to the two following paragraphs.

Collar Bearing (Fig. 260).—By similar reasoning to that given above, we get—

$$\text{Moment of friction on collar} = 2\pi\mu p \int_{r = R_2}^{r = R_1} r^2 . dr$$

$$M_f = \frac{2\mu W(R_1^3 - R_2^3)}{3(R_1^2 - R_2^2)}$$

Conical Pivot.—The intensity of pressure p all over the surface is the same, whatever may be the angle a.

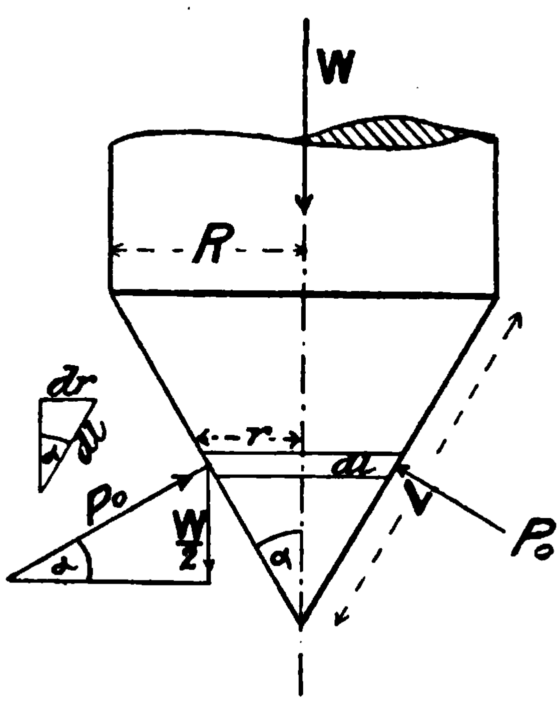


FIG. 259.

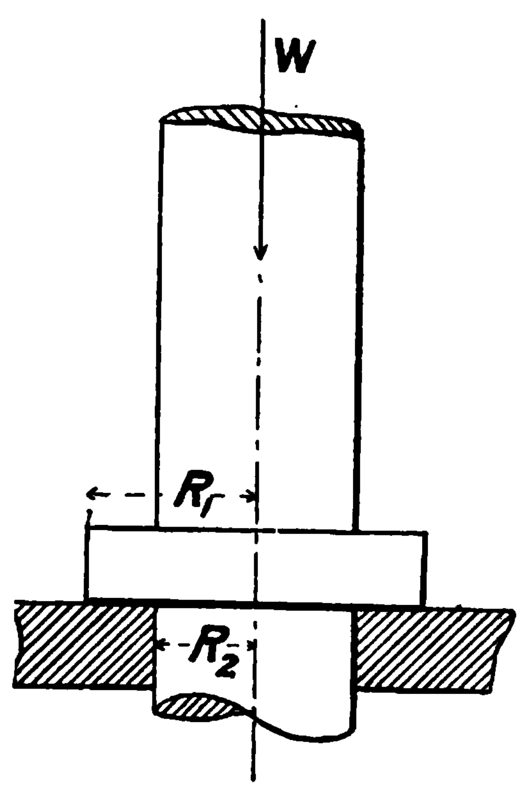


FIG. 260.

Let P_0 be the pressure acting on one half of the cone—

$$P_0 = \frac{W}{2 \sin a}$$

The area of half the surface of the cone is—

$$A = \frac{\pi RL}{2} = \frac{\pi R^2}{2 \sin \alpha}$$

$$p = \frac{P_0}{A} = \frac{W 2 \sin \alpha}{2 \sin \alpha \pi R^2} = \frac{W}{\pi R^2} = \frac{\text{weight}}{\text{projected area}}$$

Total normal pressure on any elementary ring $= 2\pi r p \,.\, dl$

moment of friction on elementary ring $= 2\pi r^2 \mu p \,.\, dl$

$\left(\text{but } dl = \dfrac{dr}{\sin \alpha}\right) \qquad = \dfrac{2\pi r^2 \mu p dr}{\sin \alpha}$

moment of friction on whole surface $= \dfrac{2\pi\mu p}{\sin \alpha}\displaystyle\int r^2 . dr$

$$M_f = \frac{2\pi\mu p R^3}{3 \sin \alpha}$$

Substituting the value of p, we have—

$$M_f = \frac{2\mu WR}{3 \sin \alpha}$$

The angle α becomes 90°, and $\sin \alpha = 1$ when the pivot becomes flat.

By similar reasoning, we get for a frustrated conical pivot—

$$M_f = \frac{2\mu W(R_1^3 - R_2^3)}{3 \sin \alpha (R_1^2 - R_2^2)}$$

Fig. 261.

Schiele's Pivot and Onion Bearing.—Conical and flat pivots often give trouble through heating, probably due to the fact that the wear is uneven, and therefore the contact between the pivot and step is imperfect, thereby giving rise to intense local pressure. The object sought in the Schiele pivot is to secure even wear all over the pivot. As the footstep wears, every point in the pivot will sink a vertical distance h, and the point a sinks to a_1. $aa_1 = h_1$. Draw ab normal to the curve at a, and ac normal to the axis. Also draw ba_1 tangential to the dotted curve at b, and ad to the full-lined curve at a; then, if h be taken as very small, ba_1 will be practically parallel to ad, and the two triangles aba_1 and acd will be practically similar, and—

$$\frac{ad}{ac} = \frac{aa_1}{ba}, \text{ or } ad = \frac{ac \times aa_1}{ba}$$

$$\text{or } ad = \frac{r \cdot h}{ba}$$

But ba is the wear of the footstep normal to the pivot, which is

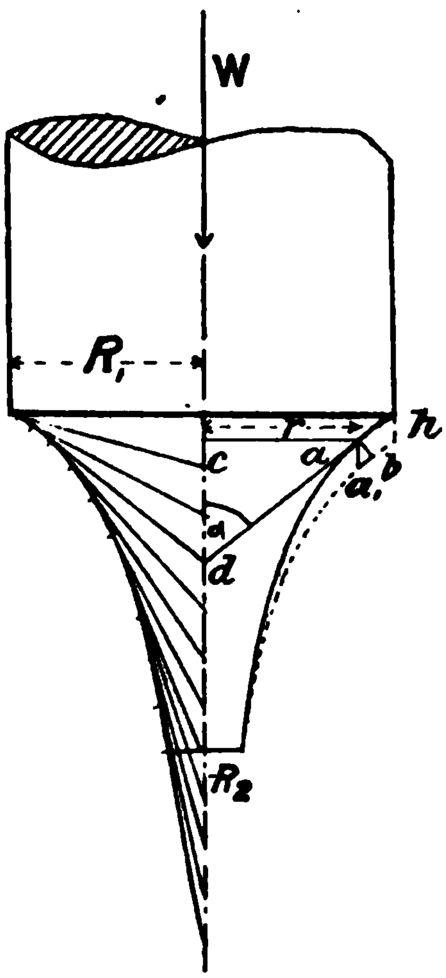


FIG. 262.

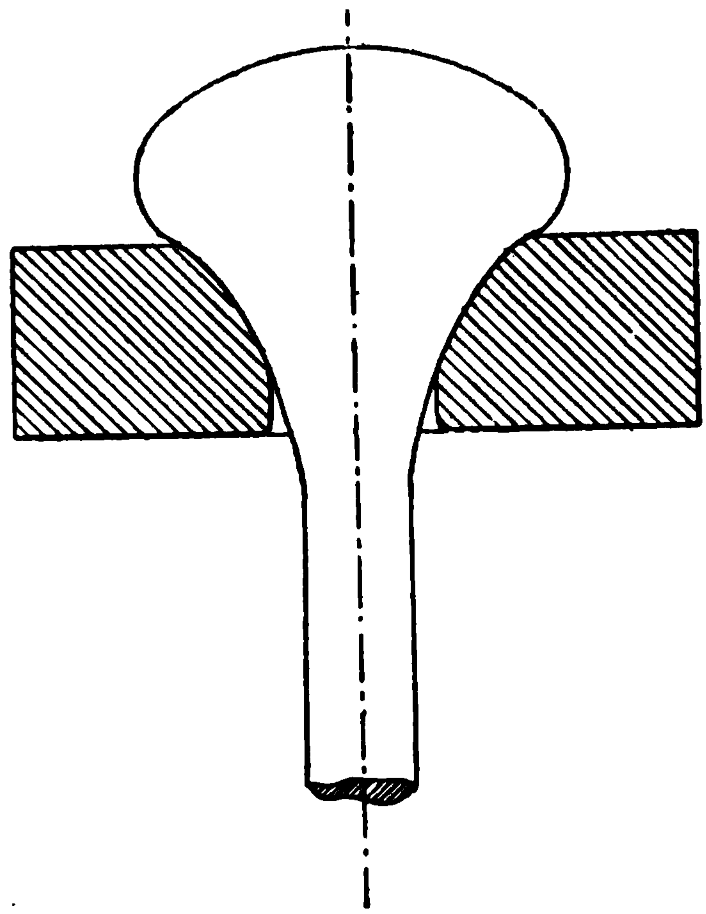


FIG. 263.

usually assumed to be proportional to the friction F between the surfaces, and to the velocity V of rubbing, hence—

$$\overline{ba} \propto \text{FV} \propto \mu p \cdot 2\pi r\text{N}$$

$$\text{or } \overline{ba} = \text{K}\mu pr$$

where K is a constant for any given speed and rate of wear; hence—

$$ad = \frac{r \cdot h}{\text{K}\mu pr} = \frac{h}{\text{K}\mu p}$$

But h is constant by hypothesis, and μ is assumed to be constant all over the pivot; p we have already proved to be constant (last paragraph); hence $\overline{ad}$, the length of the tangent to the curve, is constant; thus, if the profile of a pivot be so constructed that the length of the tangent $ad = t$ be constant, the

wear will be (nearly) even all over the pivot. Although our assumptions are not entirely justified, experience shows that such pivots do work very smoothly and well. The calculation of the friction moment is very similar to that of the conical pivot.

The normal pressure at every point is—

$$p = \frac{\text{weight}}{\text{projected area}} = \frac{W}{\pi(R_1{}^2 - R_2)^2}$$

By similar reasoning to that given for the conical pivot, we have—

$$\left.\begin{array}{l}\text{Moment of friction on an elementary}\\ \quad\text{ring of radius } r\end{array}\right\} = \frac{2\pi r^2 \mu p dr}{\sin a}$$

$$\left(\text{but } \frac{r}{\sin a} = t\right) \qquad = 2\pi t \mu p r dr$$

$$\text{and moment of friction for whole pivot} = 2\pi t \mu p \int_{r=R_2}^{r=R_1} r \,.\, dr$$

$$M_f = 2\pi t \mu p \frac{R_1{}^2 - R_2{}^2}{2}$$

$$\text{Substituting the value of } p,\ M_f = W\mu t$$

The onion bearing shown in the figure is simply a Schiele pivot with the load suspended from below.

Friction of Cup Leathers.—The resistance of a hydraulic plunger sliding through a cup leather has been investigated by Hick, Tuit, and others. The formula proposed by Hick for the friction of cup leathers does not agree well with experiments; the author has therefore recently tabulated the results of published experiments and others made in his laboratory, and finds that the following formulæ much more nearly agree with experiment:—

Let F = frictional resistance of a leather in pounds square inch of water pressure.

d = diameter of plunger in inches.

p = water-pressure in pounds square inch.

$$\text{Then } F = 0{\cdot}08p + \frac{100}{d} \text{ when in good condition}$$

$$F = 0{\cdot}08p + \frac{250}{d} \quad \text{,, } \quad \text{bad}$$

Efficiency of Machines.—In all cases of machines, the

work supplied is expended in overcoming the useful resistances for which the machine is intended, in addition to the useless or frictional resistances. Hence the work supplied must always be greater than the useful work done by the machine.

Let the work supplied to the machine be equivalent to *lowering* a weight W through a height h;

the useful work done by the machine be equivalent to *raising* a weight W_u through a height h_u;

the work done in overcoming friction be equivalent to *raising* a weight W_f through a height h_f.

Then, if there were no friction—

$$\text{Supply of energy} = \text{useful work done}$$
$$Wh = W_u h_u$$
$$\text{or } \frac{W}{W_u} = \frac{h_u}{h} = R$$

or mechanical advantage = velocity ratio

When there is friction, we have—

$$\text{Supply of energy} = \text{useful work done} + \text{work wasted in friction}$$
$$Wh = W_u h_u + W_f h_f$$

and—

$$\text{the mechanical efficiency} = \frac{\text{useful work done}}{\text{total work done}}$$
$$\text{or} = \frac{\text{the work got out}}{\text{the work put in}}$$

Let η = the mechanical efficiency; then—

$$\eta = \frac{W_u h_u}{Wh}, \text{ or } \frac{W_u h_u}{W_u h_u + W_f h_f}$$

η is, of course, always less than unity. The "counter-efficiency" is $\frac{1}{\eta}$, and is always greater than unity.

Reversed Efficiency.—When a machine is reversed, for example, when a load is being lowered by lifting-tackle, the original resistance becomes the driver, and the original driver becomes the resistance; then—

$$\text{Reversed efficiency} = \frac{\text{useful work done in lifting W through } h}{\text{total work done in lowering } W_u \text{ through } h_u}$$
$$\eta_r = \frac{Wh}{W_u h_u} = \frac{W_u h_u - W_f h_f}{W_u h_u}$$

When W acts in the same direction as W_u, *i.e.* when the machine has to be assisted to lower its load, η_r takes the negative sign. In an experiment with a two-sheaved pulley block, the pull on the rope was 170 lbs. when lifting a weight of 500 lbs.; the velocity ratio in this case $R = \frac{h_u}{h} = \frac{1}{4}$.

$$\text{Then } \eta = \frac{W_u h_u}{W h} = \frac{500 \times 1}{170 \times 4} = 0{\cdot}735$$

The friction work in this case $W_f h_f$ was $170 \times 4 - 500 \times 1 = 180$ foot-lbs. Hence the reversed efficiency $\eta_r = \frac{500 - 180}{500} = 0{\cdot}64$, and in order to lower the 500 lbs. weight gently, the backward pull on the rope must be—

$$\frac{500}{4} \times 0{\cdot}64 = 80 \text{ lbs.}$$

If the 80 lbs. had been found by experiment, the reversed efficiency would have been found thus:

$$\eta_r = \frac{80 \times 4}{500 \times 1} = 0{\cdot}64$$

The reversed efficiency must always be less than unity, and may even become negative when the frictional resistance of the machine is greater than the useful resistance. In order to lower the load with such a machine, an additional force acting in the same sense as the load has to be applied; hence such a machine is self-sustaining, *i.e.* it will not run back when left to itself. The least frictional resistance necessary to ensure that it shall be self-sustaining is when $W_f h_f = W_u h_u$; then, substituting this value in the efficiency expression for forward motion, we have—

$$\eta = \frac{W_u h_u}{2 W_u h_u} = \frac{1}{2}$$

Thus, in order that a machine may be self-supporting, its efficiency cannot be over 50 per cent. This statement is not strictly accurate, because the frictional resistance varies somewhat with the forces transmitted, and consequently is smaller when lowering than when raising the load; the error is, however, rarely taken into account in practical considerations of efficiency (see Appendix).

This self-supporting property of a machine is, for many purposes, highly convenient, especially in hand-lifting tackle, such as screw jacks, Weston pulley blocks, etc.

Combined Efficiency of a Series of Mechanisms.—If in any machine the power is transmitted through a series of simple mechanisms, the efficiency of each being η_1, η_2, η_3, etc., the efficiency of the whole machine will be—

$$\eta = \eta_1 \times \eta_2 \times \eta_3, \text{ etc.}$$

If the power be transmitted through n, mechanisms of the same kind, each having an efficiency η_1, the efficiency of the whole series will be approximately—

$$\eta = \eta_1^n$$

Hence, knowing the efficiency of various simple mechanisms, it becomes a simple matter to calculate with a fair degree of accuracy the efficiency of any complex machine.

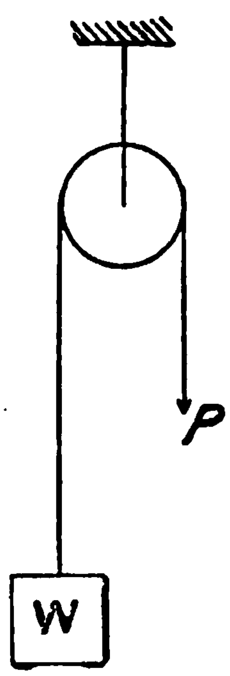


FIG. 264.

Efficiency of Various Machine Elements.

Pulleys.—In the case of a rope or chain passing over a simple pulley, the frictional resistances are due to (1) the resistance of the rope or chain to bending; (2) the friction on the axle. The first varies with the make, size, and newness of rope; the second with the lubrication. The following table gives a fairly good idea of the total efficiency at or near full load of single pulleys; it includes both resistances 1 and 2:—

Diameter of rope		$\frac{3}{8}$ in.	$\frac{3}{4}$ in.	1 in.	$1\frac{1}{2}$ in.	chain.
Maximum efficiency per cent.	Clean and well oiled	96	93	91	88	95–97
	Dirty	94	91	89	86	93–96
	Clean and well oiled, with stiff new rope	—	91	—	—	—

These figures are fair averages of a large number of experiments. The diameter of the pulley varied from 8 to 12 times the diameter of the rope, and the diameter of the pins from $\frac{1}{2}$ inch to $1\frac{1}{4}$ inch.

It is useless to attempt to calculate the efficiency with any great degree of accuracy.

Pulley Blocks.—When a number of pulleys are combined for hoisting tackle, the efficiency of the whole may be calculated approximately from the known efficiency of the single pulley. The efficiency of a single pulley does not vary greatly with the load unless it is absurdly low, hence we may assume that the efficiency of each is the same. Then, if the rope passes over n pulleys, each having an efficiency η_1, we have the efficiency of the whole—

$$\eta = \eta_1^n$$

The following table of efficiencies has been compiled by plotting curves for experiments made at the Yorkshire College laboratory :—

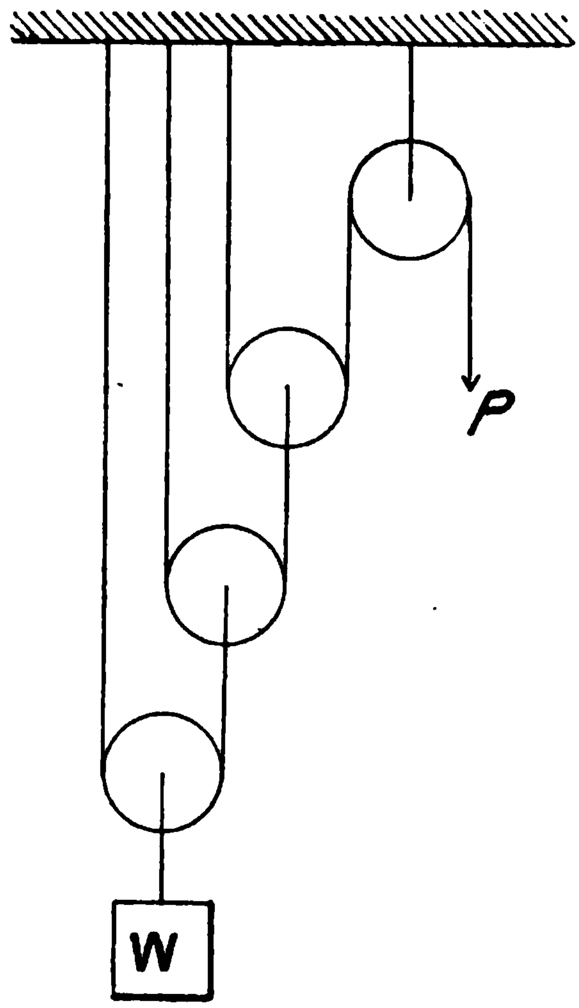


FIG. 265.

Loads in pounds.	Single pulley.		Two-sheaved.		Three-sheaved.	
	Old $\frac{3}{4}$-in. rope.	New $\frac{3}{4}$-in. rope.	Old $\frac{1}{2}$-in. rope.	New $\frac{1}{2}$-in. rope.	Old $\frac{3}{4}$-in. rope.	New $\frac{3}{4}$-in. rope.
14	94	90	—	—	—	—
28	94·5	90·5	80	75	30	24
56	95	91	84	78·5	50	35
112	96	92	86	91·5	60	41
168	—	—	87·5	93	65	44
224	—	—	89	93	69	47
280	—	—	90	94	72	50
336	—	—	—	—	74	53
448	—	—	—	—	78	56

Levers.—The efficiency of a simple lever (when used at any other than very low loads) with two pin joints varies from 94 to 97 per cent., the lower value for a short and the higher for a long lever.

When mounted on well-formed knife-edges, the efficiency is practically 100 per cent.

Toothed Gearing.—The efficiency of toothed gearing

depends on the smoothness and form of the teeth, and whether lubricated or not. Knowing the pressure on the teeth and the distance through which rubbing takes place (see p. 138), also the μ, the efficiency is readily arrived at; but the latter varies so much, even in the same pair of wheels, that it is very difficult to repeat experiments within 2 or 3 per cent., hence calculated values depending on an arbitrary choice of μ cannot have any pretence to accuracy. The following empirical formula fairly well represents average values of experiments :—

For one pair of machine-cut toothed wheels, including the friction on the axles—

$$\eta = 0 \cdot 96 - \frac{1}{0 \cdot 025N}$$

for rough unfinished teeth—

$$\eta = 0 \cdot 90 - \frac{1}{0 \cdot 025N}$$

Where N is the number of teeth in the smallest wheel.

When there are several wheels in one train, let $n =$ the number of pairs of wheels in gear;

$$\text{Efficiency of train } \eta_1 = \eta^n$$

The efficiency increases slightly with the velocity of the pitch lines (see *Engineering*, vol. xli. pp. 285, 363, 581; also Kennedy's "Mechanics of Machinery," p. 579).

Velocity of pitch line in feet per minute	10	50	100	150	200
Efficiency	0·940	0·972	0·980	0·984	0·986

Screw and Worm Gearing.—We have already shown

Fig. 266.

how to arrive at the efficiency of screws and worms when the

coefficient of friction is known. The following table is taken from the source mentioned above:—

Velocity of pitch line in feet per minute	10	50	100	150	200
	Efficiency per cent.				
Angle of thread α, 45°	87	94	95	96	97
" " 30°	82	90	93	94	95
" " 20°	75	86	90	92	92
" " 15°	70	82	87	89	90
" " 10°	62	76	82	85	86
" " 7°	53	69	76	80	81
" " 5°	45	62	70	74	76

The figure shows an ordinary single worm and wheel. As the angle α increases, the worm is made with more than one thread; the worm and wheel is then known as screw gearing. For details the reader should refer to Unwin's "Elements of Machine Design."

Friction of Slides.—A slide is generally proportioned so that its area bears some relation to the load, hence when the load and coefficient of friction are unknown, the resistance to sliding may be assumed to be proportional to the area; when not unduly tightened, the resistance may be taken as about 3 lbs. per square inch.

Friction of Shafting.—A 2-inch diameter shaft running at 100 revolutions per minute requires about 1 horse-power per 100 feet when there are no belts on the pulleys. The horse-power increases directly as the speed and approximately as the cube of the diameter.

Belt and Rope Transmission.—The efficiency of belt

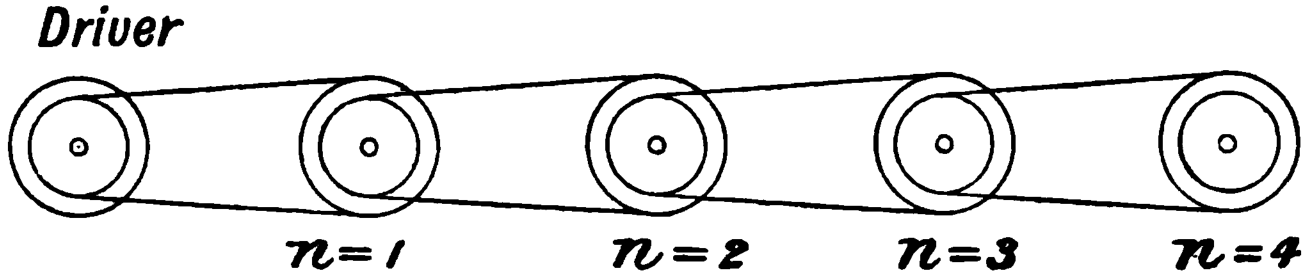


FIG. 267.

and rope transmission for each pair of pulleys is from 95 to 96 per cent., including the friction on the bearings; hence, if

there are n sets of ropes or belts each having an efficiency η, the efficiency of the whole will be, approximately—

$$\eta_1 = \eta^n$$

Experiments by the author on a large number of belts show that the work wasted by belts due to resistance to bending over pulleys, creeping, etc., varies from 16 to 21 foot-lbs. per square foot of belt passed over the pulleys.

Mechanical Efficiency of Steam-engines. — The work absorbed in overcoming the friction of a steam-engine is roughly constant at all powers; it increases slightly as the power increases. A full investigation of the question has been made by Professor Thurston, who finds that the friction is distributed as follows:—

Main bearings	35–47 per cent.
Piston and rod	21–33 ,,
Crank-pin	5–7 ,,
Cross-head and gudgeon pin	4–5 ,,
Valve and rod ... 2·5 balanced, 22 unbalanced	,,
Eccentric strap	4–5 ,,
Link and eccentric	9 ,,

The following instances may be of interest in illustrating the approximate constancy of the friction at all powers:—

EXPERIMENTAL ENGINE, UNIVERSITY COLLEGE, LONDON.

Syphon Lubrication.

I.H.P.	2·75	9·25	10·23	11·14	12·34	13·95	14·29
B.H.P.	0·0	5·63	7·50	7·66	9·09	11·09	11·25
Friction H.P. ...	2·75	3·62	2·73	3·48	3·25	2·86	3·04

EXPERIMENTAL ENGINE, YORKSHIRE COLLEGE, LEEDS.

Syphon and Pad Lubrication.

I.H.P. ...	2·48	5·16	6·83	8·30	11·50	13·84	17·02	22·30
B.H.P. ...	0·0	2·35	3·94	5·61	8·70	10·82	13·89	19·09
Friction H.P.	2·48	2·81	2·89	2·69	2·80	3·02	3·13	3·21

BELLISS ENGINE, BATH (FORCED) LUBRICATION.
(See *Proc. I.M.E.*, 1897.)

I.H.P.	49·8	102·7	147·1	193·6	217·5
B.H.P.	44·5	97·0	140·6	186·0	209·5
Friction H.P. ...	5·3	5·7	6·5	7·6	8·0

MECHANICAL EFFICIENCY PER CENT. OF VARIOUS MACHINES.

From experiments in all cases with more than quarter full load.

Weston pulley block (½ ton)	20–25
Epicycloidal pulley block	40–45
One-ton steam hoists or windlasses	50–70
Hydraulic windlass	60–80
,, jack	80–90
Cranes (steam)	60–70
Travelling overhead cranes	30–50
Locomotives $\frac{\text{draw bar H.P.}}{\text{I.H.P.}}$	65–75
Two-ton testing-machine, worm and wheel, screw and nut, slide, two collars	2–3
Screw displacer—hydraulic pump and testing-machine, two cup leathers, toothed-gearing four contacts, three shafts (bearing area, 48 sq. inches), area of flat sides, 18 sq. inches, two screws and nuts	2–3

Lancashire Cotton Mills (see *Proceedings of the Manchester Association of Engineers*, 1892).	About 1000 H.P. engines, spur-gearing, and engine friction	74
	Rope drives	70
	Belt ,, ...	71
	Direct (400-H.P. engines)...	76

For much valuable information on the subject of friction, the reader is referred to the Cantor Lectures on Friction, delivered at the Society of Arts by Professor Hele-Shaw, LL.D.

Belts.

Coil Friction.—Let the pulley in Fig. 268 be fixed, and a belt or rope pass round a portion of it as shown. The weight W produces a tension T_1; in order to raise the weight W, the tension T_2 must be greater than T_1 by the amount of friction between the belt and the pulley.

Let F = frictional resistance of the belt;

p = normal pressure between belt and pulley at any point.

Then, if μ = coefficient of friction—

$$F = T_2 - T_1 = \Sigma\mu p$$

Let the angle α embraced by the belt be divided into a great number, say n, parts, so that $\frac{\alpha}{n}$ is very small; then the tension on both sides of this very small angle is practically the

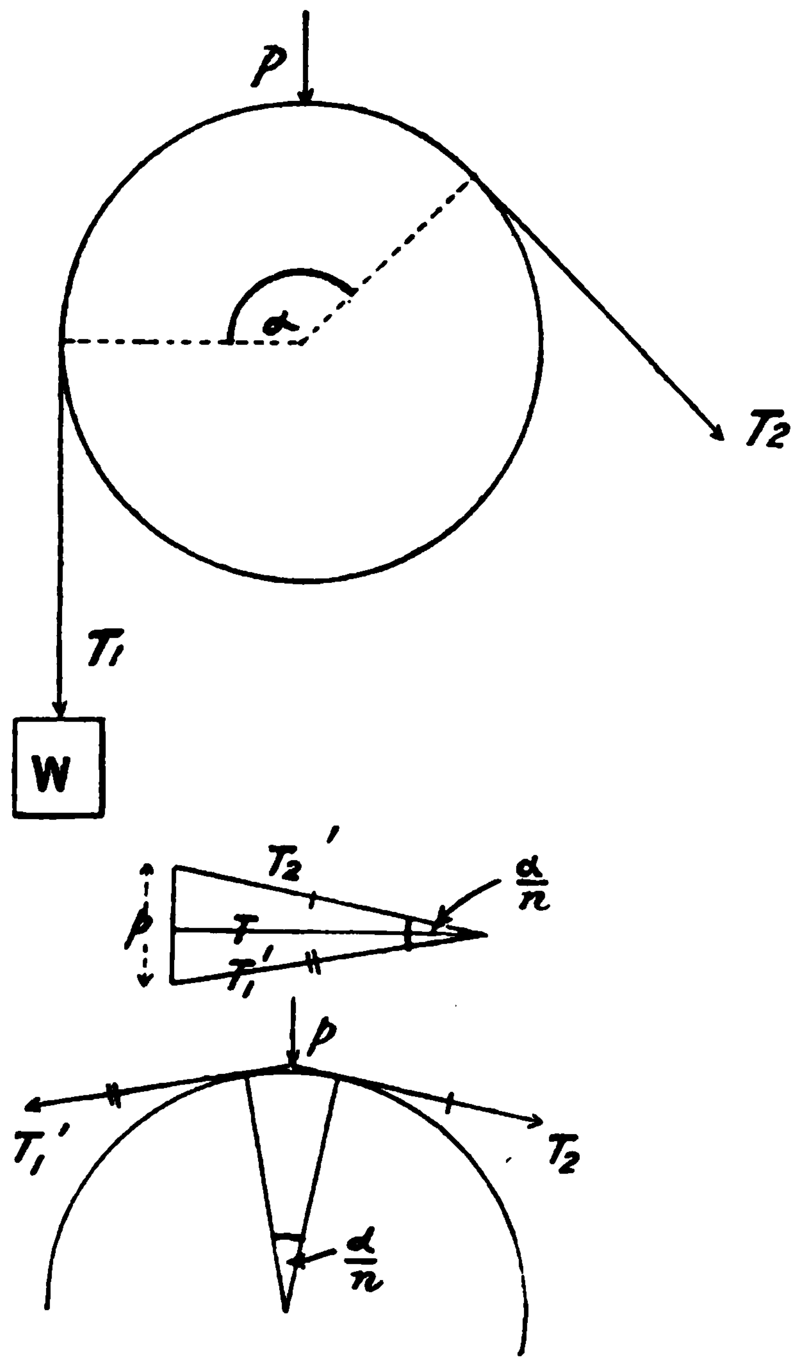


FIG. 268.

same. Let it be T; then, expressing α in circular measure, we have—

$$p = \mathrm{T}\frac{\alpha}{n}$$

The friction at any point is (neglecting the stiffness of the belt)—

$$\mu p = \mu \mathrm{T}\frac{\alpha}{n} = \mathrm{T}_2' - \mathrm{T}_1'$$

But we may write $\frac{\alpha}{n}$ as $\delta\alpha$; also $\mathrm{T}_2' - \mathrm{T}_1'$ as $\delta\mathrm{T}$. Then—

$$\mu \mathrm{T} \,.\, \delta\alpha = \delta \mathrm{T}$$

which in the limit becomes—

$$\mu T \,.\, da = dT$$

$$\frac{dT}{T} = \mu \,.\, da$$

We now require the sum of all these small tensions expressed in terms of the angle embraced by the belt:

$$\int_{T_1}^{T_2} \frac{dT}{T} = \mu \int_0^a da$$

$$\log_e T_2 - \log_e T_1 = \mu a$$

$$\log_e \frac{T_2}{T_1} = \mu a$$

$$\frac{T_2}{T_1} = e^{\pm \mu a}$$

$$\text{or } \log \left(\frac{T_2}{T_1}\right) = \pm\, 0{\cdot}4343 \mu a$$

where $e = 2{\cdot}72$, the base of the system of natural logarithms, and $\log e = 0{\cdot}4343$.

When W is being raised, the + sign is used in the index, and when lowered, the − sign. The value of μ for leather, cotton, or hemp rope on cast iron is from 0·2 to 0·4, and for wire rope 0·5.

If a wide belt or plaited rope be used as an absorption dynamometer, and be thoroughly smeared with tallow or other thick grease, the resistance will be greatly increased, due to the shearing of the film of grease between the wheel and the rope. By this means the author has frequently obtained an apparent value of μ of over 1—a result, of course, quite impossible with perfectly dry surfaces.

Power transmitted by Belts.—Generally speaking, the power that can be transmitted by a belt is limited by the friction between the belt and the pulley. When excessively loaded, a belt usually slips rather than breaks, hence the friction is a very important factor in deciding upon the power that can be transmitted. When the belt is just on the point of slipping, we have—

$$\text{Horse-power transmitted} = \frac{FV}{33{,}000} = \frac{(T_2 - T_1)V}{33{,}000}$$

$$= \frac{T_2\left(1 - \frac{1}{e^{\mu a}}\right)V}{33{,}000}$$

Where the friction F is expressed in pounds, and V = velocity in feet per minute. Substituting the value of e, and putting $\mu = 0.4$ and $\alpha = 3.14$ (180°), we have—

$$\text{H.P.} = \frac{0.72 T_2 V}{33,000}$$

For single ply belting T_2 may be taken as about 80 lbs. per inch of width, allowing for the laced joints, etc.

Let w = width of belt;

$$\text{Then } T_2 = 80w$$
$$\text{and H.P.} = \frac{0.72 \times 80wV}{33,000} = \frac{wV}{600}$$

for single ply belting;

$$\text{and H.P.} = \frac{wV}{300}$$

for double ply belting.

The number of square feet of belt passing over the pulleys per minute is $\frac{wV}{12}$.

Hence the number of square feet of belt required per minute per horse-power is—

$$\frac{\frac{wV}{12}}{\frac{wV}{600}} = 50 \text{ square feet per minute for single ply, and } 25 \text{ square feet per minute for double ply}$$

This will be found to be an extremely convenient expression for committal to memory.

Centrifugal Action on Belts.—In Chapter VI., we showed that the two halves of a flywheel rim tended to fly apart due to the centrifugal force acting on them; in precisely the same manner a tension is set up in that portion of a belt wrapped round a pulley. On p. 160, we showed that the stress due to centrifugal force was—

$$f = \frac{W_r V_w^2}{g}$$

where W_r is the weight of 1 foot of belting 1 square inch in section. $W_r = 0.43$ lb., and V_w = the velocity in feet per *second;* hence—

$$f = \frac{0.43 V_w^2}{32.2} = \frac{V_w^2}{75}$$

hence the tension T_2 in a belt is increased by centrifugal force to—

$$T_2 + \frac{V_w^2}{75}$$

or, putting the velocity in feet per minute V, the total tension is—

$$T_2 + \frac{V^2}{270{,}000}$$

Hence when a belt runs at high velocities, say 1 mile per minute, the stress is more than doubled by centrifugal action, and the width of belt should be increased accordingly.

The accompanying figure (Fig. 269), showing the stretch of a belt due to centrifugal tension, is from a photograph of an indiarubber belt running at a very high speed; for comparison the belt is also shown stationary. The author is indebted to his colleague Dr. Stroud for the photograph, taken in the physics laboratory at the Yorkshire College.

Creeping of Belts.—The material on the tight side of a belt is necessarily stretched more than that on the slack side, hence a driving pulley always receives a greater length of belt than it gives out; in order to compensate for this, the belt creeps as it passes over the pulley.

Let l = unstretched length of belt passing over the pulleys in feet per minute

l_2 = stretched length on the T_2 side;

l_1 = " " " T_1 "

N_1 = revolutions per minute of driven pulley;

N_2 = " " " driving "

d_1 = diameter of driven pulley } measured to the middle

d_2 = " driving " } of the belt;

x = stretch of belt in feet;

E = Young's modulus;

f_1 and f_2 = stresses corresponding to T_1 and T_2.

$$\text{Then } x = \frac{f_2 l}{E}$$

$$l_2 = l + x = l\left(1 + \frac{f_2}{E}\right) = \pi d_2 N_2$$

$$l_1 = l\left(1 + \frac{f_1}{E}\right) = \pi d_1 N_1$$

$$\frac{N_1}{N_2} = \frac{l\left(1 + \frac{f_1}{E}\right)d_2}{l\left(1 + \frac{f_2}{E}\right)d_1}$$

$$\frac{N_1}{N_2} = \frac{(E + f_1)d_2}{(E + f_2)d_1}$$

Fig. 269.

If there were no creeping, we should have—

$$\frac{N_1}{N_2} = \frac{d_2}{d_1}$$

E = from 8,000 to 10,000 lbs. per square inch. Taking $T_2 = 80$ lbs. per inch of width, and the thickness as 0·22 inch, we have—

$$f_2 = \frac{80}{0 \cdot 22} = 364 \text{ lbs. per square inch}$$

$$\text{and } T_1 = \frac{80}{2 \cdot 72^{0 \cdot 4 \times 3 \cdot 14}} = \frac{80}{3 \cdot 5} = 23 \text{ lbs. per inch width}$$

$$f_1 = \frac{23}{0 \cdot 22} = 104 \text{ lbs. per square inch}$$

$$\text{Hence } \frac{E + f_1}{E + f_2} = \frac{10,000 + 104}{10,000 + 364} = 0 \cdot 975$$

or the belt under these conditions creeps or slips 2·5 per cent.

When a belt transmits power, however small, there must be some slip or creep.

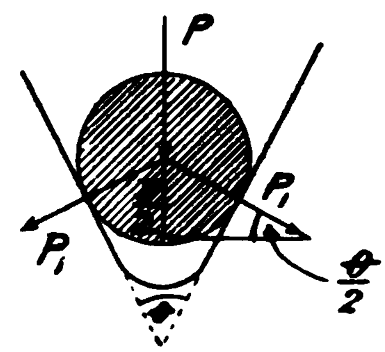


FIG. 270.

Rope Driving.—When a rope does not bottom in a grooved pulley, it wedges itself in, and the normal pressure is thereby increased to—

$$P_1 = \frac{P}{\sin \frac{\theta}{2}}$$

The angle θ is usually about 45°, hence $P_1 = 2 \cdot 6P$.

The most convenient way of dealing with this increased pressure is to use a false coefficient 2·6 times its true value. Taking $\mu = 0 \cdot 3$ for a rope on cast iron, the false μ for a grooved pulley becomes $2 \cdot 6 \times 0 \cdot 3 = 0 \cdot 72$.

The value of $e^{\mu a}$ now becomes 9·5 when the rope embraces half the pulley. The factor of safety on driving-ropes is very large, often amounting to about 80, to allow for defective splicing, and to prevent undue stretching. The working strength in pounds may be taken as $10c^2$, where c is the circumference in inches.

Then, by similar reasoning to that given for belts, we get for the horse-power that may be transmitted *per rope*—

$$\text{H.P.} = \frac{c^2 V}{3700}, \text{ or } \frac{d^2 V}{370}$$

where d = diameter of rope in inches.

CHAPTER VIII.

STRESS, STRAIN, AND ELASTICITY.

Stress.—If, on any number of sections being made in a body, it is found that there is no tendency for any one part of it to move relatively to any other part, that body is said to be in a *state of ease;* but when one part tends to move relatively to the other parts, we know that the body is acted upon by a system of equal and opposite forces, and the body is said to be in a *state of stress.* Thus, if, on making a series of saw-cuts in a plate of metal, the cuts were found to open or close before the saw had got right through, we should know that the plate was in a state of stress, because the one part tends to move relatively to the other. The stress might be due either to external forces acting on the plate, or to internal initial stresses in the material, such as is often found in badly designed castings.

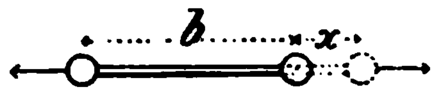
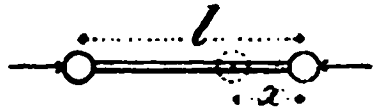
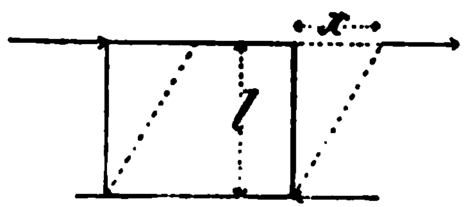
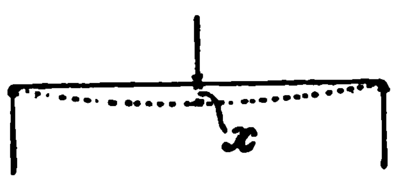
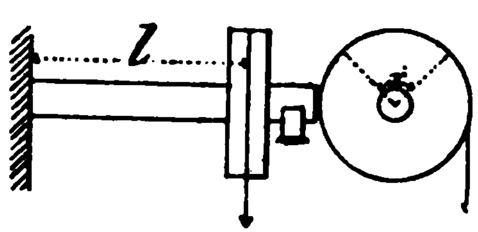
Intensity of Stress.—The intensity of direct stress on any given section of a body is the total force acting normal to the section divided by the area of the section over which it is distributed; or, in other words, it is the amount of force per unit area.

$$\left.\begin{array}{r}\text{Intensity of stress in}\\ \text{pounds per sq. inch}\end{array}\right\} = \frac{\text{the given force in pounds}}{\text{area of the section over which the force acts in sq. inches}}$$

The conditions which have to be fulfilled in order that the intensity of stress may be the same at all parts of the section are dealt with in Chapter XIII.

Strain.—The strain of a body is the change of form or dimensions that it undergoes when placed in a state of stress. No bodies are absolutely rigid; they all yield, or are strained more or less, when subjected to stress, however small in amount.

The various kinds of stresses and strains that we shall consider are given below in tabular form.

	Nature of load.	The external forces act.	Kind of *stress*.	Kind of *strain*.	When the ratios below are equal to unity, the strain is termed "unit strain."
			Produced in the direction of the applied forces.		
 FIG. 271.	Pull	*From* one another in the same straight line.	Tensile stress or tension.	Extension or stretch (x)	$\frac{\text{Extension}}{\text{Original length}} = \frac{x}{l}$
 FIG. 272.	Push or thrust	*Towards* one another in the same straight line.	Compressive stress or pressure.	Compression or contraction (x)	$\frac{\text{Compression}}{\text{Original length}} = \frac{x}{l}$
 FIG. 273.	Shear	*Parallel* to one another.	Shearing stress.	Shear or slide (x) or distortion	$\frac{\text{Shear}}{\text{Length}} = \frac{x}{l}$
 FIG. 274.	Bending	*Parallel* to one another.	Bending stress.	Deflection (x)	
 FIG. 275.	Twisting	In planes *parallel* with one another.	Torsion or shearing stress	Twist, usually measured in degrees.	$\frac{\text{Surface twist}}{\text{Length}} = \frac{x}{l}$

Elasticity.—A body is said to be elastic when the strain entirely disappears on the removal of the stress that produced it. Very few materials can be said to be perfectly elastic except for very low stresses, but a great many are approximately so over a wide range of stress.

Permanent Set.—That part of the strain that does not entirely disappear on the removal of the stress is termed "permanent set."

Elastic Limit.—The stress at which a marked permanent set occurs is termed the elastic limit of the material. We use the word *marked* because, if very delicate measuring instruments be used, very slight sets can be detected with low stresses far below that usually associated with the elastic limit. In elastic materials the strain is usually proportional to the stress; but this is not the case in all materials that fulfil the conditions of elasticity laid down above. Hence there is an objection to the definition that the elastic limit is that point at which the strain ceases to be proportional to the stress.

Plasticity.—If none of the strain disappears on the removal of the stress, the body is said to be plastic. Such bodies as soft clay and wax are almost perfectly plastic.

Ductility.—If only a small part of the strain be elastic, but the greater part be permanent after the removal of the stress, the material is said to be ductile. Soft wrought iron, mild steel, copper, and other materials, pass through such a stage before becoming plastic.

Brittleness.—When a material breaks with a very low stress and stretches but a very small amount before fracture, it is termed a brittle material.

Behaviour of Materials subjected to Tension.

Ductile Materials.—If a bar of ductile metal, such as wrought iron or mild steel, be subjected to a low tensile stress, it will stretch a certain amount, depending on the material; and if the stress be doubled, the stretch will also be doubled, or the stretch will be proportional to the stress (within very narrow limits). Up to this point, if the bar be relieved of stress, it will return to its original length, *i.e.* the bar is elastic; but if the stress be gradually increased, a point will be reached when the stretch will increase much more rapidly than the stress; and if the bar be relieved of stress, it will not return to its original length—in other words, it has taken a "permanent set." The stress at which this occurs is, as will be seen from our definition above, the *elastic limit* of the material.

Let the stress be still further increased. Very shortly a

point will be reached when the strain will (in good wrought iron and mild steel) suddenly increase to 10 or 20 times its previous amount. This point is termed the *yield point* of the material, and is always quite near the elastic limit. For all commercial purposes, the elastic limit is taken as being the same as the yield point. Just before the elastic limit was reached, while the bar was still elastic, the stretch would only be about $\frac{1}{1000}$ of the length of the bar; but when the yield point is reached, the stretch would amount to $\frac{1}{100}$, or $\frac{1}{50}$ of the length of the bar.

As the stress is increased beyond the yield point, the strain continues to increase much more rapidly than before, and the material becomes more and more ductile; and if the stress be now removed, almost the whole of the strain will be found to be permanent. But still a careful measurement will show that a very small amount of the strain is still elastic.

Just before the maximum stress is reached, the material appears to be nearly perfectly plastic. It keeps on stretching without any increase in the load. Up to this point the strain on the bar has been evenly distributed (approximately) along its whole length; but very shortly after the plastic state has been reached the bar extends locally, and "stricture" commences, *i.e.* a local reduction in the diameter occurs, which is followed almost immediately by the fracture of the bar. The extension before stricture occurs is termed the "proportional" extension, and that after fracture the "final" extension, which is known simply as the "extension" in commercial testing. We shall return to this point later on.

The stress-strain diagram given in Fig. 276 will illustrate clearly the points mentioned above.

Brittle Materials.—Brittle materials at first behave in a similar manner to ductile materials, but have no marked elastic limit or yield point. They break off short, and have no ductile or plastic stage.

Extension of Ductile Materials.—We pointed out above that the final extension of a ductile bar consisted of two parts—(1) An extension evenly distributed along the whole length of the bar, the total amount of which is consequently proportional to the length of the bar; (2) A local extension at fracture, which is very much greater per unit length than the distributed or proportional extension, and is independent (nearly so) of the length of the bar. Hence, on a short bar the local extension is a very much greater proportion of the whole than on a long bar. Consequently, if two bars of the same material

but of different lengths be taken, the percentage of extension on the short bar will be much greater than on the long bar.

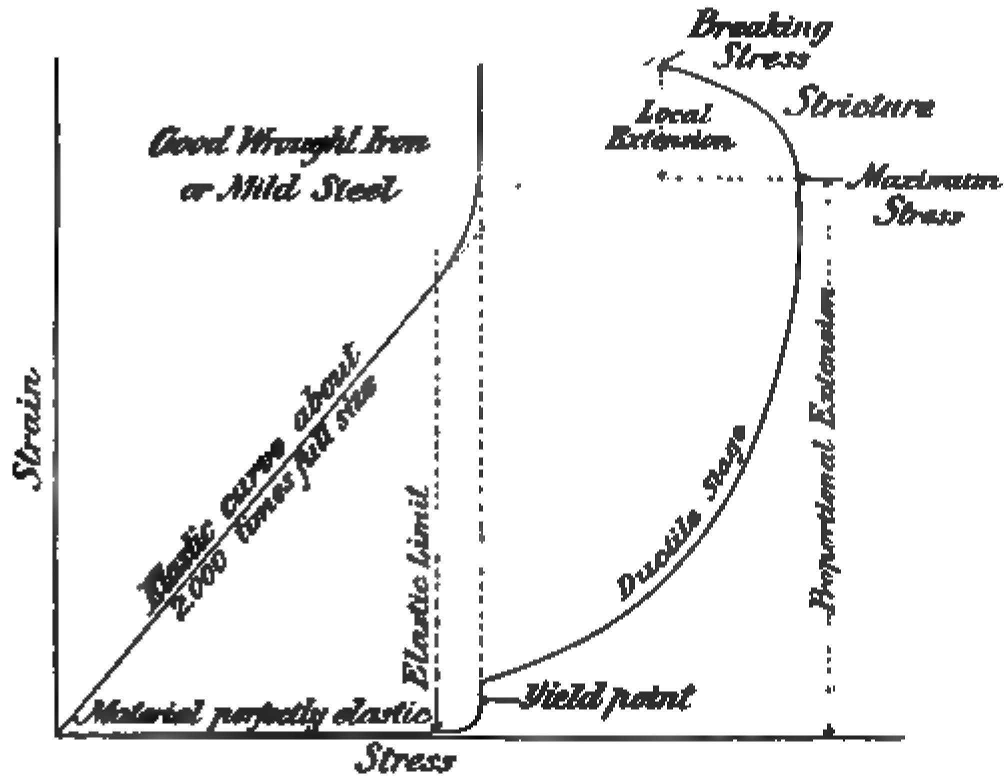


FIG. 276.

The following results were obtained from a bar of Lowmoor iron :—

FIG. 277.

The local extension in this bar was 54 per cent. on 2 inches.

The final extensions reckoned on various lengths, each including the fracture, were as follows :—

Length	10″	8″	6″	4″	2″
Percentage of extension ...	22	24·5	34	41	54

(See a paper by Mr. Kicksteed in "Industries," Sept. 26, 1890.)

Hence it will be seen that the length on which the percentage of extension is measured must always be stated. The simplest way of obtaining comparative results for specimens of various lengths is to always mark them out in inches throughout their whole length, and state the percentage of extension on the 2 inches at fracture as well as on the total length of the bar. A better method would be to make all test specimens of similar form, *i.e.* the diameter a fixed proportion of the length; but any one acquainted with commercial testing knows how impracticable such a suggestion is.

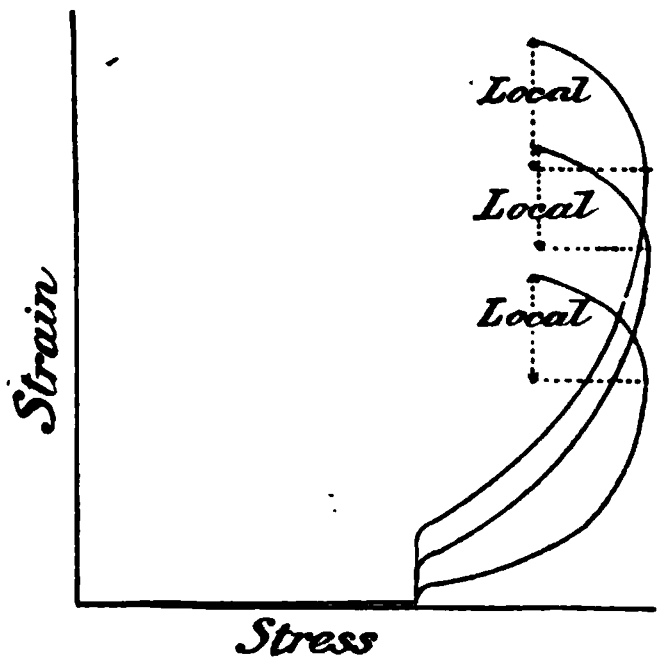


FIG. 278.

Load-strain diagrams taken from bars of similar material, but of different lengths, are somewhat as shown in Fig. 278.

If L = original length of a test bar between the datum points;
L_1 = stretched length of a test bar between the datum points.

Then $L_1 - L = x$, the extension

The percentage of extension is—

$$\frac{L_1 - L}{L} \times 100 = \frac{100x}{L}$$

In Fig. 279 we show some typical fractures of materials tested in tension.

Reduction in Area of Ductile Materials.—The volume of a test bar remains constant within exceedingly small limits, however much it may be strained; hence, as it extends the sectional area of the bar is necessarily reduced. The reduction in area is considered by some authorities to be the best measure of the ductility of the material.

Let A = the original sectional area of the bar;
A_1 = the final area at the fracture.

Then the percentage of reduction in area is $\frac{A - A_1}{A} \times 100$

If a bar remained parallel right up to the breaking point,

Gun metal. Hard steel. Soft steel. Copper Delta metal.

FIG. 279.

as some materials approximately do, the reduction in area can be calculated from the extension, thus :

The volume of the bar remains constant; hence—

$$LA = L_1A_1, \text{ or } A_1 = \frac{LA}{L_1}$$

and the reduction in area is—

$$\frac{A - A_1}{A}$$

Then, substituting the value of A_1, we have—

$$\frac{L_1A - LA}{L_1A} = \frac{L_1 - L}{L_1} = \frac{x}{L_1}$$

Thus the reduction in area in the case of a test bar which remains parallel is equal to the extension on the bar *calculated on the stretched length.* This method should never be used for calculating the reduction in area, but it is often a useful check. The published account of some tests of steel bars gave the following results :—

Length of bar, 2 inches ; extension, 6·0 per cent. ; reduction in area, 4·9 per cent. ;

Then x in this case was $\frac{6 \times 2}{100} = 0{\cdot}12$ inch

$$\text{and } L_1 = 2{\cdot}12 \text{ inches}$$

$$\text{Reduction in area} = \frac{0{\cdot}12 + 100}{2{\cdot}12} = 5{\cdot}66 \text{ per cent.}$$

Thus there is probably an error in measurement in getting the 4·9 per cent., for the reduction in area could not have been less than 5·66 per cent. unless there had been a hard place in the metal, which is improbable in the present instance.

Real and Nominal Stress in Tension.—It is usual to calculate the breaking stress on a test bar by dividing the breaking load by the area of the original section. This method, though convenient and always adopted for commercial purposes, is not strictly accurate, on account of the reduction of the area as the bar extends.

Using the same notation as before for the lengths and areas—

Let W = the load on the bar at any instant;

S = the nominal stress on the bar, viz. $\frac{W}{A}$;

S_1 = the real stress on the bar, viz. $\frac{W}{A_1}$.

Then, as the volume of the bar remains constant—

$$LA = L_1A_1, \text{ and } \frac{L_1}{L} = \frac{A}{A_1}$$

$$\text{and } \frac{S_1}{S} = \frac{\frac{W}{A_1}}{\frac{W}{A}} = \frac{A}{A_1} = \frac{L_1}{L}$$

$$\text{or the real stress } S_1 = \frac{SL_1}{L}$$

The diagram of real stress may be conveniently constructed as in Fig. 280 from the ordinary stress-strain diagram.

The construction for one point only is given. The length L of the specimen is set off along the strain axis, and the stress ordinate *de* is projected on to the stress axis, viz. *ao*. The line *ba* is then drawn to meet *cd* produced in *c*, which

gives us one point on the curve of real stress. For by similar triangles we have—

$$\frac{S_1}{S} = \frac{L_1}{L}, \text{ or } S_1 = \frac{SL_1}{L}$$

which we have shown above to be the real stress.

The last part of the diagram, however, cannot be obtained

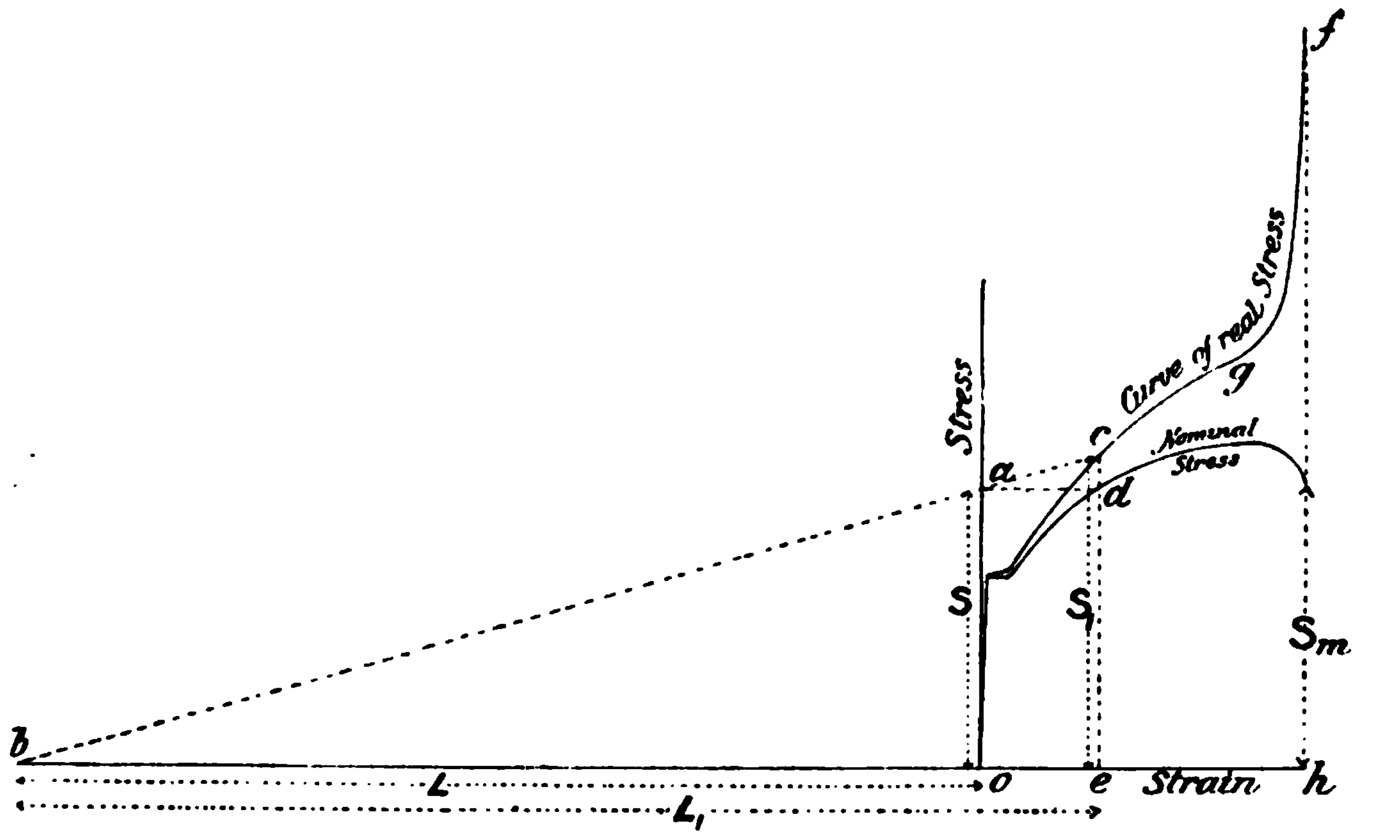


FIG. 280.

thus, as the above relation only holds as long as the bar remains parallel; but the point f can be calculated thus: The final load S_m, or nominal stress, as the case may be, can be measured off the diagram. The final area A_1 is also known;

$$\text{Then } fh = \frac{S_m}{A_1}$$

The curve from f to g has to be put in by eye.

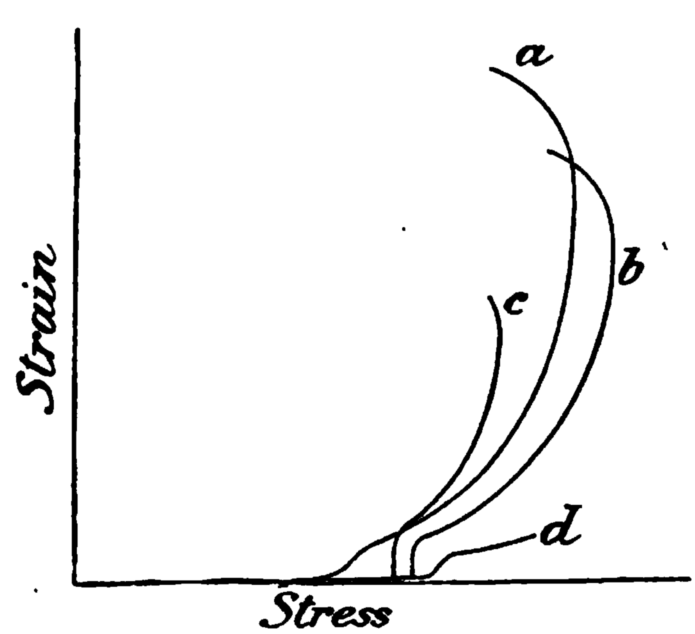


FIG. 281.—(*a*) Swedish and Yorkshire irons; (*b*) good bar iron; (*c*) fair quality of plates with the grain; (*d*) ditto across the grain.

Typical Stress. Strain Curves for Various Materials in Tension.—Most of the above curves are copies (with altered scales) of diagrams taken by the author on a

Wicksteed autographic recorder when chief assistant to Mr. W. H. Stanger, of The Broadway Testing Works, Westminster, by whose courtesy they are here reproduced.

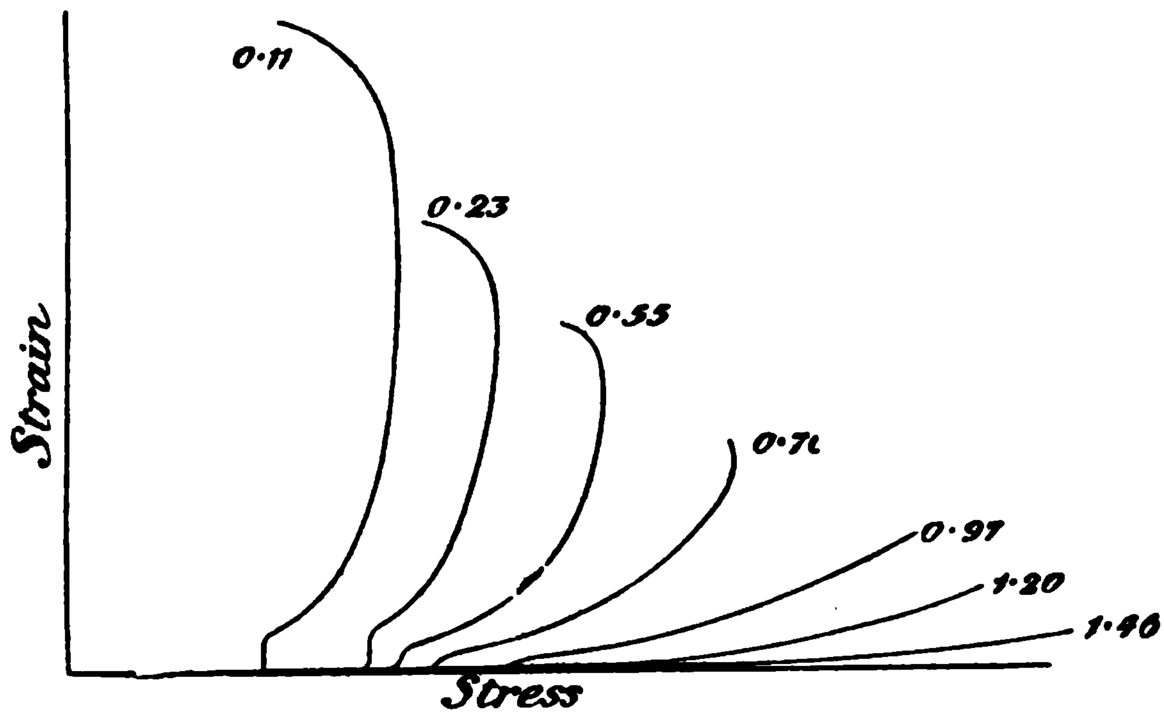


FIG. 282.—Steel containing several percentages of carbon.

The curves, showing the effects of the various percentages of carbon, have been sketched from a series of about forty curves in the possession of the author, covering a range of carbon from about 0·1 to about 1·5 per cent. Some of the curves in Fig. 283 are curiously serrated, *i.e.* the metal does not stretch regularly (these serrations are not due to errors in the recording apparatus, such as are obtained by recorders which record the faults of the operator as well as the characteristics of the material). The author finds that all alloys containing iron give a serrated diagram when cold and a smooth diagram when hot, whereas steel does the reverse. This peculiar effect, which is disputed by some, has been independently noticed by Mons. Le Chatelier.

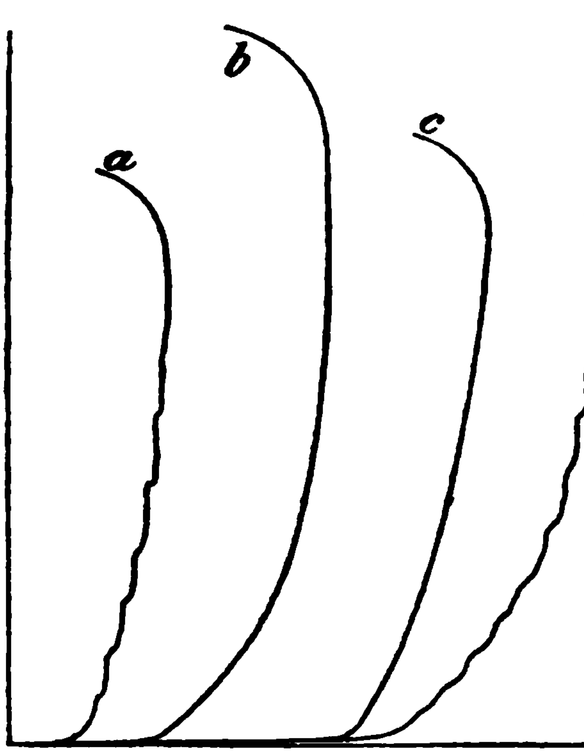


FIG. 283.—(*a*) Rolled aluminium; (*b*) rolled copper; (*c*) rolled "bull" metal, temp. 400° Fahr.; (*d*) ditto 60° Fahr. N.B.—Bull metal and delta metal behave in practically the same way in the testing-machine.

Artificial Raising of the Elastic Limit.—The form of

a stress-strain curve depends much upon the physical state of the metal, and whether the elastic limit has been artificially raised or not. It has been known for many years that if a piece of metal be loaded beyond the elastic limit, and the load be then released, the next time the material is loaded, the elastic limit will approximately coincide with the previous load. In the diagram in Fig. 284, the metal was loaded up to the point *c*, and then released; on reloading, the elastic limit occurred at the stress *cd*, whereas the original elastic limit was at the stress *ab*. Now, if in manufacture, by cold rolling, drawing, or otherwise, the limit had been thus artificially raised, the stress-strain diagram would have been *dce*.

Strength of Wire.—Wire, through having been drawn

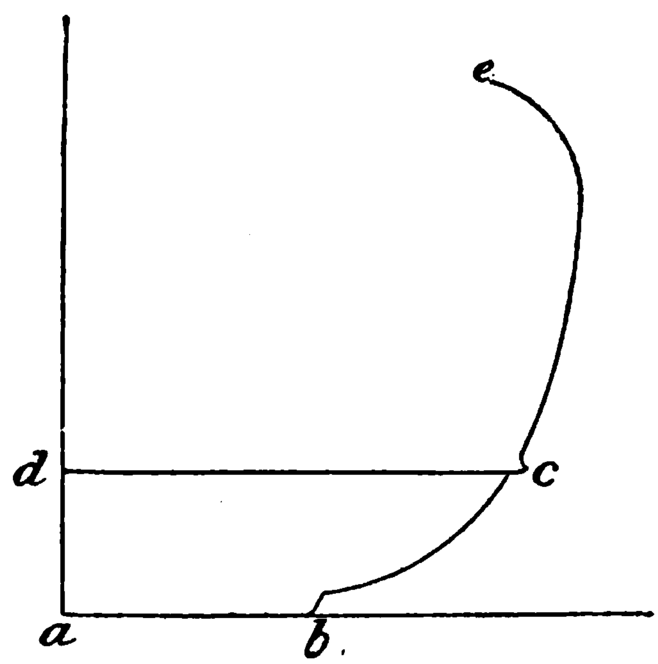


FIG. 284.

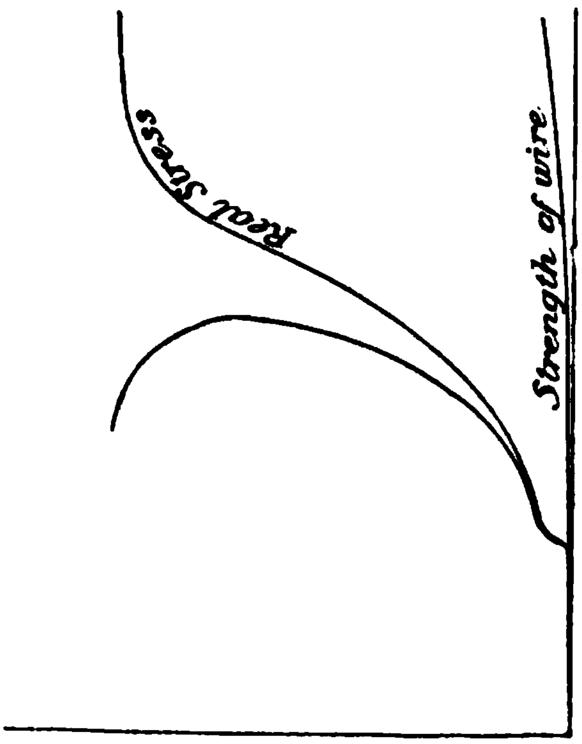


FIG. 285.

through a series of die plates, is an extreme case of metal with an artificially raised elastic limit. In hard drawn, unannealed wire, the elastic limit practically coincides with the breaking point; the metal is then in approximately the same state as the metal at the point of stricture in the test specimen. We have seen above, however, that the real stress calculated on this reduced area is greatly in excess of the strength calculated on the original area; hence, the strength of wire is correspondingly greater than the nominal strength of the metal in the billets or rods from which it was made (see Fig. 285). Annealing almost entirely restores wire to its original state.

The specific gravity of wire is slightly higher than that of the metal from which it was made, which also tends to increase its relative strength.

The strength of wire may be anything between the original strength of the metal and say 100 per cent. greater; but the higher the relative strength the smaller will be the extension.

Work done in fracturing a Bar.—Along one axis a load-strain diagram shows the resistance a bar offers to being pulled apart, and along the other the distance through which this resistance is overcome; hence the product of the two, viz. the area of the diagram, represents the amount of work done in fracturing the bar.

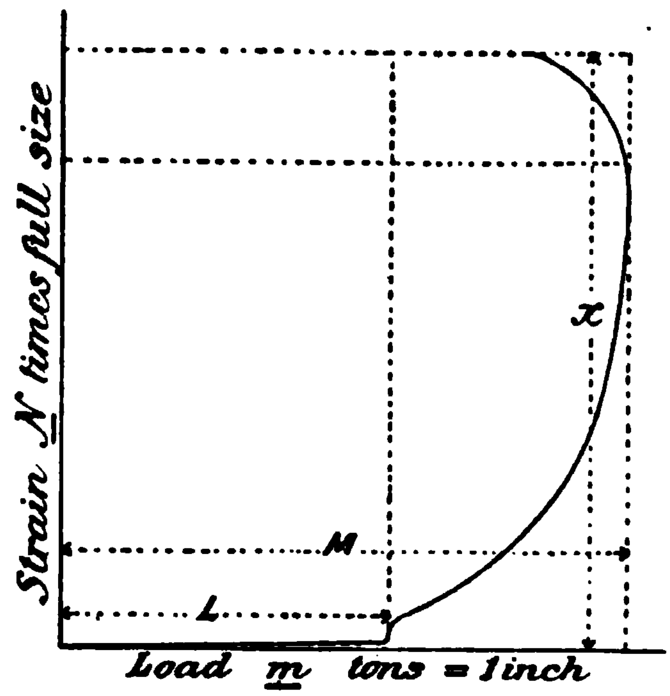


FIG. 286.

Let a = the area of the diagram in square inches;
l = the length of the bar in inches (between datum points);
A = the sectional area of the bar.

If the diagram were drawn 1 inch = 1 ton, and the strain were full size, then a would equal the work done in fracturing the bar; but, correcting for scales, we have—

$$\frac{am}{N} = \text{work done in inch-tons in fracturing the bar}$$

and $\dfrac{am}{lAN}$ = work done in inch-tons per cubic inch in fracturing the bar

Professor Kennedy has pointed out that the curve during the ductile and plastic stages is a very close approximation to a parabola. Assuming it to be so, the work done can be calculated without the aid of a diagram, thus:

Let L = the elastic limit in tons per square inch;
M = the maximum stress in tons per square inch;
x = the extension in inches.

Then the work done in inch-tons per square inch of section of bar $\left.\right\} = Lx + \frac{2}{3}(M - L)x$

$$= \frac{x}{3}(L + 2M)$$

work done in inch-tons per cubic inch $= \dfrac{x}{3l}(L + 2M)$

But $\frac{x}{l} \times 100 = e$, the percentage of extension

hence the work done in inch-tons per cubic inch $\Big\} = \frac{e}{300}(\mathrm{L} + 2\mathrm{M})$

The work done in inch-tons per cubic inch is certainly by far the best method of measuring the capacity of a given material for standing shocks and blows. Strictly speaking, in order to get comparative results from bars of various lengths, that part of the diagram where stricture occurs should be omitted, but with our present system of recording tests such a procedure would be inconvenient.

Behaviour of Materials subjected to Compression. *Ductile Materials.*—In the chapter on columns it is shown that the length very materially affects the strength of a piece of material when compressed, and for getting the true compressive strength, very short specimens have to be used in order to prevent buckling. Such short specimens, however, are inconvenient for measuring accurately the relations between the

Aluminium. Original form. Gun metal. Cast iron. Soft brass. Cast iron.

FIG. 287.

stress and the strain. Up to the elastic limit, ductile materials compress in much the same way as they do in tension, viz. the strain is proportional to the stress. At the yield point the strain does not increase so suddenly as in tension, and when the plastic stage is reached, the sectional area gradually increases and the metal spreads. With very soft homogeneous materials, this spreading goes on until the metal is squeezed to a flat disc,

and never fractures. Such materials are soft copper, or aluminium, or lead.

In fibrous materials, such as wrought iron and wood, in which the strength across the grain is much lower than with the grain, the material fails by splitting sideways, due to the lateral tension.

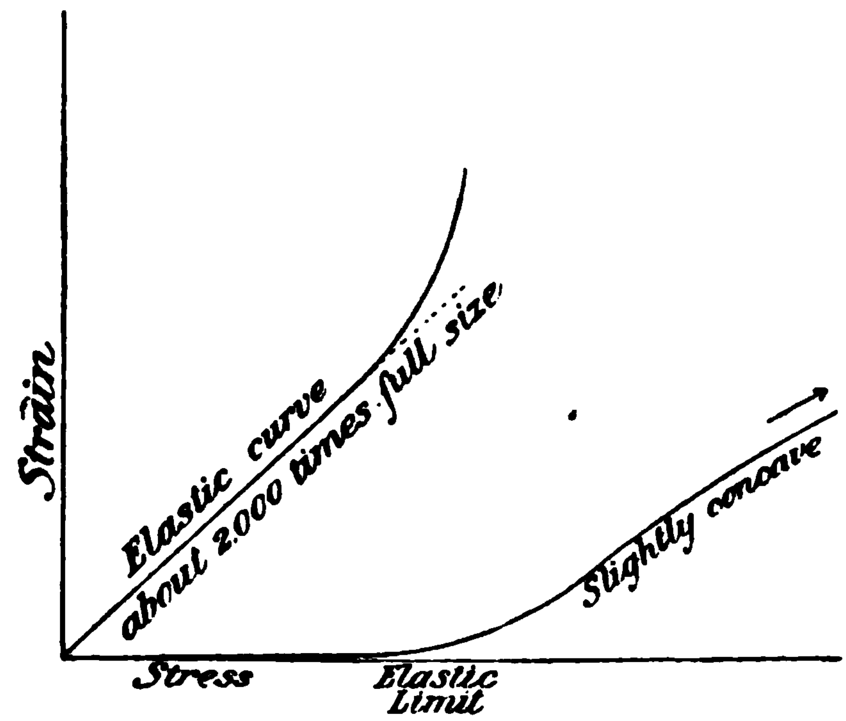


FIG. 288.

The usual form of the stress - strain curve for a ductile material is somewhat as shown in Fig. 288.

If the material reached a perfectly plastic stage, the real stress, *i.e.* $\dfrac{\text{load at any instant (W)}}{\text{sectional area at that instant } (A_1)}$, would be constant,

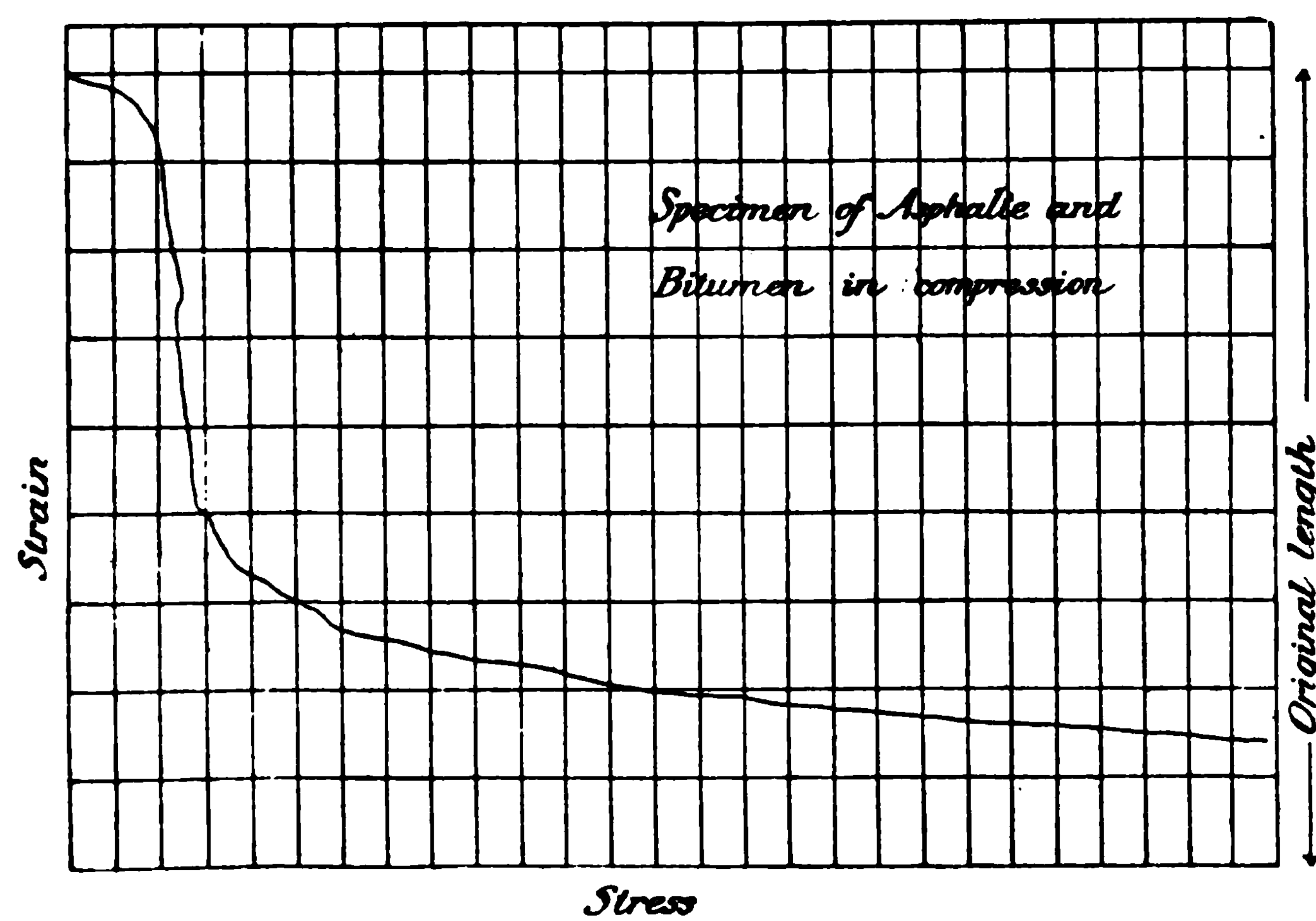


FIG. 289.

however much the material was compressed; then, using the same notation as before—

$$A_1 = \frac{lA}{l_1}$$

$$\text{and from above, } \frac{W}{A_1} = \text{constant}$$

Substituting the value of A_1 from above, we have—

$$\frac{Wl_1}{lA} = \text{constant};$$

But lA, the volume of the bar, is constant;

$$\text{hence } Wl_1 = \text{constant}$$

or the stress-strain curve during the plastic period is a hyperbola. The material never is perfectly plastic, and therefore never perfectly complies with this, but in some materials it very nearly approaches it. For example, asphalte mixed with bitumen (Fig. 289).[1] The constancy of the real stress will be more apparent when we come to draw the real stress curves.

Brittle Materials.—Brittle materials in compression, as in tension, have no marked elastic limit or plastic stage. When crushed they either split up into prisms or, if of cubical form, into pyramids, and sometimes by the one-half of the specimen shearing over the other at an angle of 45°. Such a fracture is shown in Fig. 287 (cast iron).

FIG. 290.—Asphalte.

The shearing fracture is quite what one might expect from purely theoretical reasoning. In Fig. 291 let the sectional area = A;

$$\text{then the stress on the cross-section } S = \frac{W}{A}$$

[1] Kindly supplied by W. Harry Stanger, Esq., Broadway Testing Works, Westminster.

and the stress on an oblique section *aa*, making an angle α

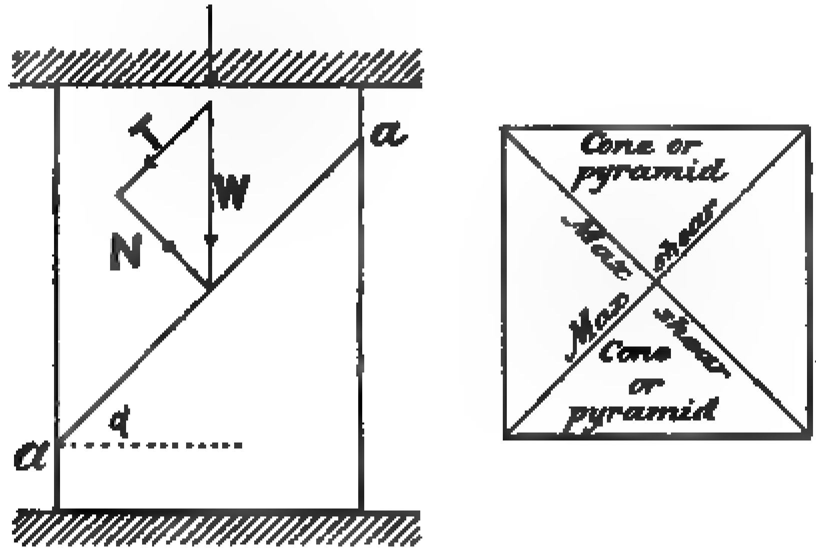


FIG. 291.

FIG. 292.—Portland cement.

with the cross-section, may be found thus: resolve W into components normal $N = W \cos \alpha$, and tangential $T = W \sin \alpha$.

$$\text{The area of the oblique section } aa = A_0 = \frac{A}{\cos \alpha}$$

$$\text{the normal stress} = \frac{N}{A_0} = \frac{W \cos \alpha}{\frac{A}{\cos \alpha}} = \frac{W \cos^2 \alpha}{A} = S \cdot \cos^2 \alpha$$

$$\left.\begin{array}{r}\text{tangential or shear-}\\\text{ing stress}\end{array}\right\} = \frac{T}{A_0} = \frac{W \sin \alpha}{\frac{A}{\cos \alpha}} = \frac{W \cos \alpha \sin \alpha}{A}$$

$$= S \cos \alpha \sin \alpha$$

If we take a section at right angles to aa, T becomes the normal component, and N the tangential, and it makes an angle of $90 - \alpha$ with the cross-section; then, by similar reasoning to the above, we have—

$$\text{The area of the oblique section} = A_0' = \frac{A}{\cos (90 - \alpha)} = \frac{A}{\sin \alpha}$$

$$\text{normal stress} = S \sin^2 \alpha$$

$$\text{tangential stress or shearing stress} = S \sin \alpha \cos \alpha$$

So that the tangential stress is the same on two oblique sections at right angles, and is greatest when $\alpha = 45°$; it is then $= S \times 0·71 \times 0·71 = 0·5$ S.

From this reasoning, we should expect compression specimens to fail by shearing along planes at right angles to one another, and in a cylindrical specimen to form two cones top and bottom, and the sides to break away in triangular sections, which in the cube become six pyramids. That this theory is fairly correct is shown by the illustrations in Figs. 290–292.

Real and Nominal Stress in Compression (Fig. 293).—In the paragraph on real and nominal stress in tension, we showed how to construct the curve of real stress from the ordinary load-strain diagram. Then, assuming that the compression specimen remains parallel (which is not quite true, as the specimens always become barrel-shaped), the same method of constructing the real stress curve serves for compression. As in the tension curve, it is evident that (see Fig. 280)—

$$\frac{S_1}{S} = \frac{L_1}{L}$$

$$\text{or } S_1 = \frac{L_1 S}{L}$$

Behaviour of Materials subjected to Shear. Nature of Shear Stress.—If an originally square plate or block be

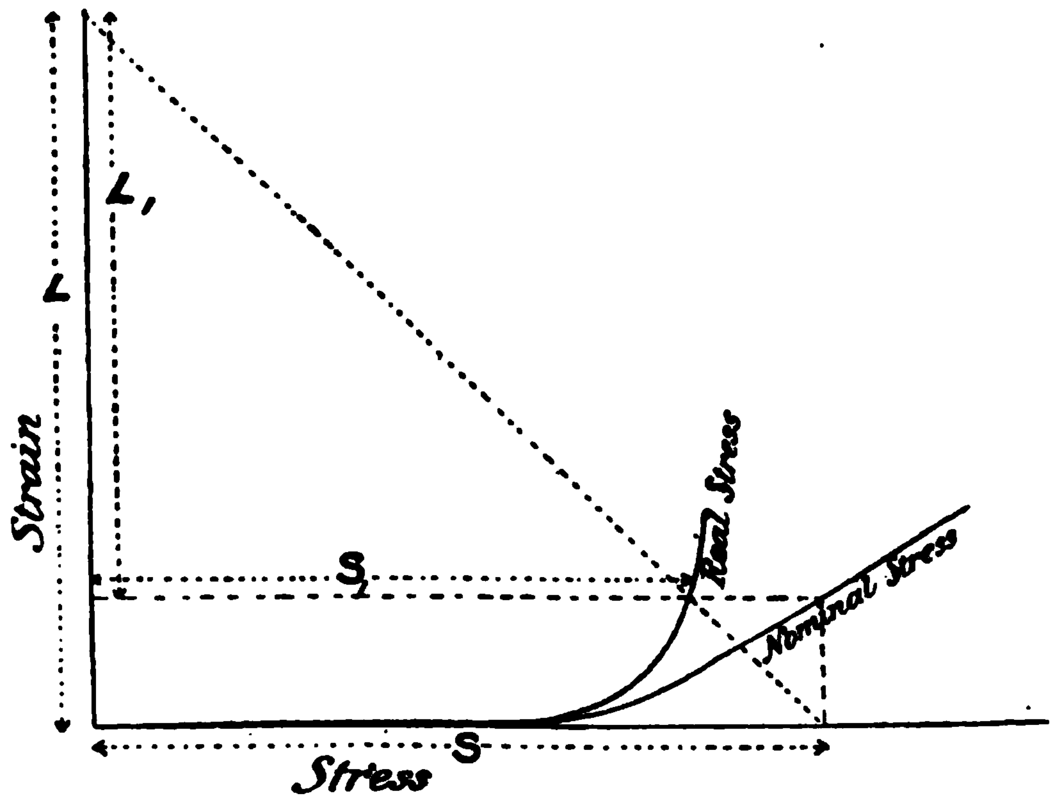


FIG. 293.

acted upon by forces p parallel to two of its opposite sides, the square will be distorted into a rhombus, as shown in Fig. 294,

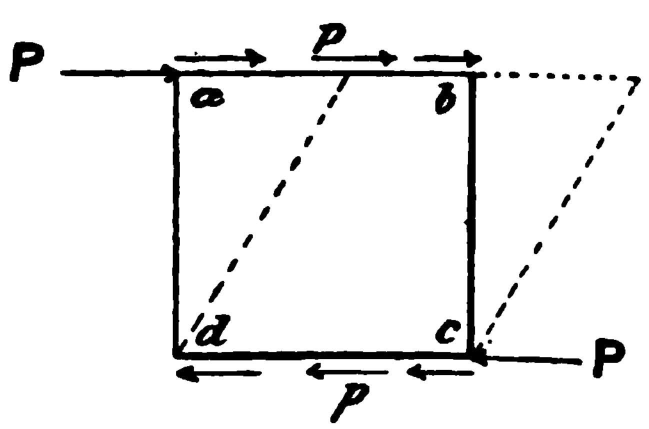


FIG. 294.

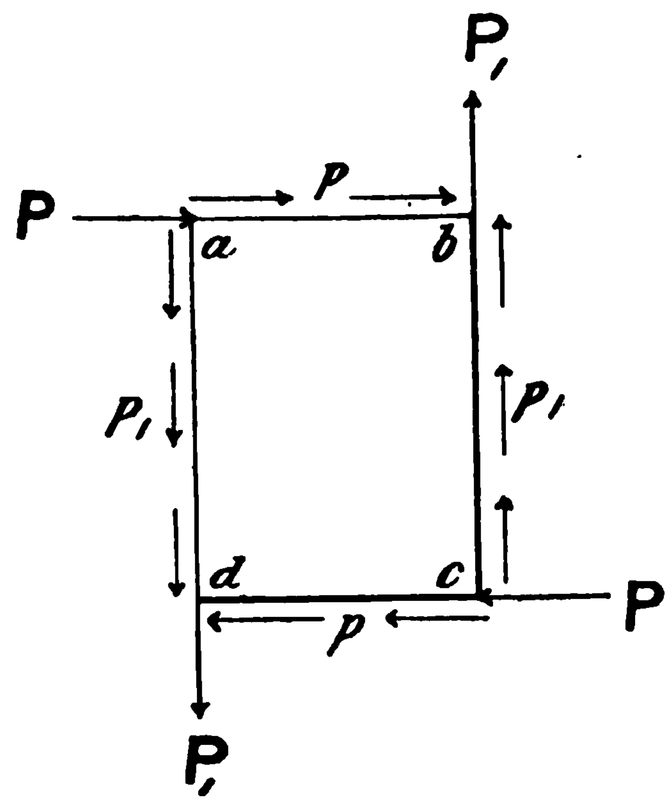


FIG. 295.

and the shearing stress will be $p = \frac{P}{ab}$, taking it to be of unit thickness. This block, however, will spin round due to the couple P . *ad* or P . *bc*, unless an equal and opposite couple be applied to the block. In order to make the following remarks

perfectly general, we will take a rectangular plate as shown in Fig. 295.

The plate is acted upon by a clockwise couple, $P . ad$, or

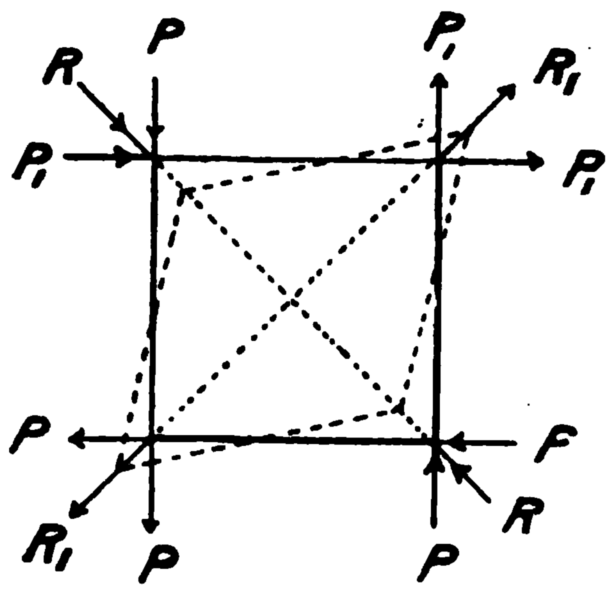


Fig. 296.

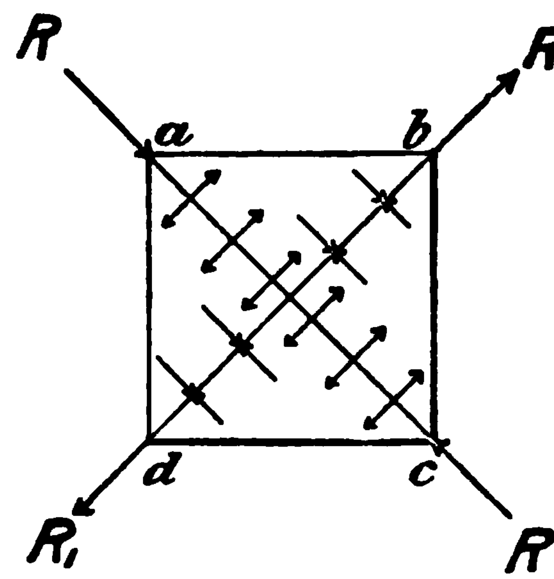


Fig. 297.

$p . ab . ad$, and a contra-clockwise couple, $P_1 . ab$ or $p_1 . ad . ab$, but these must be equal if the plate be in equilibrium;

$$\text{then } p . ab . ad = p_1 . ad . ab$$
$$\text{or } p = p_1$$

i.e. the intensity of stress on the two sides of the plate is the same.

Now, for convenience we will return to our square plate. The forces acting on the two sides P and P_1 may be resolved into forces R and R_1 acting along the diagonals as shown in Fig. 296. The effect of these forces will be to distort the square into a rhombus exactly as before. (N.B.—The rhombus in Fig. 294 is drawn in a wrong position for simplicity.) These two forces act at right angles to one another; hence we see that a shear stress consists of two equal and opposite stresses, a tension and a compression, acting at right angles to one another.

In Fig. 297 it will be seen that there is a tensile stress acting normal to one diagonal, and a compressive stress normal to the other. The one set of resultants, R, tend to pull the two triangles *abc*, *acd* apart, and the other resultants to push the two triangles *abd*, *bdc* together.

Let p_0 = the stress normal to the diagonal.

$$\text{Then } p_0 ac = p_0\sqrt{2}ab, \text{ or } p_0\sqrt{2}bc = R$$
$$\text{But } \sqrt{2}P, \text{ or } \sqrt{2}p . ab, \text{ or } \sqrt{2}p . bc = R$$
$$\text{hence } p_0 = p = p_1$$

Thus the intensity of shear stress is equal on all the four edges and on the two diagonals of a rectangular plate subjected to shear.

Ductile Materials in Shear.—When ductile materials are sheared, they pass through an elastic stage similar to that in tension and compression. If an element be slightly distorted, it will return to its original form on the removal of the stress, and during this period the strain is proportional to the stress; but after the elastic limit has been reached, the plate becomes permanently deformed, but has not any point of sudden alteration as in tension. On continuing to increase the stress, a ductile and plastic stage is reached, but as there is no alteration of area under shear, there is no stage corresponding with the stricture stage in tension.

The shearing strength of ductile materials, both at the elastic limit and at the maximum stress, is about $\frac{4}{5}$ of their tensile strength (see p. 269).

Brittle materials in shear are elastic, although somewhat imperfectly in some cases, right up to the point of fracture; they have no marked elastic limit.

It is generally stated in text-books that the shearing strength of brittle materials is much below $\frac{4}{5}$ of the tensile strength, but this is certainly an error, and has probably come about through the use of imperfect shearing tackle, which has caused double shear specimens to shear first through one section, and then through the other. In a large number of tests made in the author's laboratory, the shearing strength of cast iron has come out rather higher than the tensile stress in the ratio of 1·1 to 1.

Shear combined with Tension or Compression.—We have shown above that when a block or plate, such as *abcd*, is subjected to a shear, there will be a direct stress acting normally to the diagonal *bd*. Likewise if the two sides *ad*, *bc* are subjected to a normal stress, there will be a direct stress acting normally to the section *ef*; but when the block is subjected to both a direct stress and a shear, there will be a direct stress acting normally to a section occupying an intermediate position, such as *gh*.

Consider the stresses acting on the triangular element shown, which is reproduced from the figure above for clearness. The intensity of the shear stress on the two edges will be equal (see p. 258). Hence—

The total shear stress on the face $gi = p \, . \, gi = P_1$

" " " $hi = p \, . \, hi = P$

" direct " $gi = t \, . \, gi = T$

Let the resultant stress on the face *gh*, which we are about

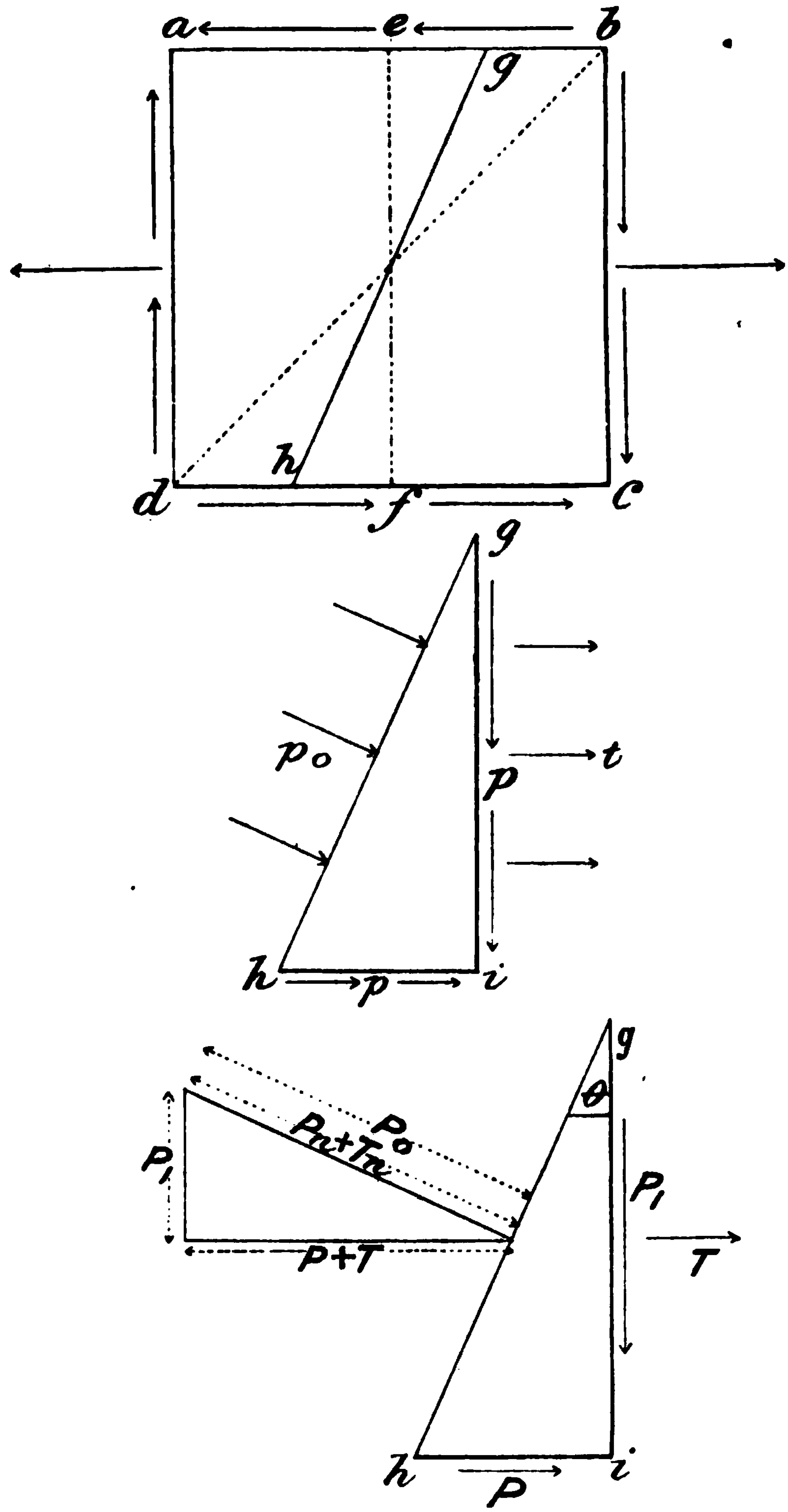


FIG. 298.

to find in terms of the other stresses, be p_0. Then the total direct stress acting normal to the face $gh = p_0 . gh = P_0$.

Now consider the two horizontal forces acting on the element, viz. T and P, and resolve them normally to the face gh as shown, we get T_n and P_n.

$$\text{But } P_n + T_n = \frac{P + T}{\cos\theta} = \frac{p\,.\,\overline{hi} + t\,.\,\overline{gi}}{\cos\theta}$$
$$\text{also } T_n + P_n = P_0 = p_0\,\overline{gh}$$

hence, substituting the above values, we have—

$$\text{hence } t\,.\,\overline{gi} + p\,.\,\overline{hi} = p_0\,\overline{gh}\cos\theta = p_0\,.\,\overline{gi}$$
$$\text{and } t + \frac{p\,.\,\overline{hi}}{\overline{gi}} = p_0$$

Next consider the vertical force acting on the element, viz. P_1, and when resolved normally to the face gh, we get P_0.

$$\text{But } P_1 = P_0 \sin\theta = p_0\,\overline{gh}\sin\theta = p_0\,\overline{hi}$$
$$\text{or } p\,.\,\overline{gi} = p_0\overline{hi}$$
$$\text{and } \frac{p\,.\,\overline{gi}}{p_0} = \overline{hi}$$

Substituting this value of $\overline{hi}$ in the equation above, we have—

$$t + p\frac{p\overline{gi}}{p_0\overline{gi}} = p_0$$
$$\text{or } p_0 t + p^2 = p_0^2$$
$$p_0^2 - p_0 t = p^2$$

Completing the square—

$$p_0^2 - p_0 t + \left(\frac{t}{2}\right)^2 = \frac{t^2}{4} + p^2$$
$$p_0 - \frac{t}{2} = \pm\sqrt{\frac{t^2}{4} + p^2}$$
$$p_0 = \frac{t}{2} + \sqrt{\frac{t^2}{4} + p^2}$$

The minus sign would be retained for finding the stress normal to the other diagonal section ij if the rectangle were completed, viz. $gihj$.

Young's Modulus of Elasticity (E).—We have already stated that experiments show that the strain of an elastic body is proportional to the stress. In some elastic materials the strain is much greater than in others for the same intensity of stress, hence we need some means of concisely expressing the amount of strain that a body undergoes when subjected to a given stress. The usual method of doing this is to state the intensity of stress required to strain the bar by an amount equal to its own length, *assuming the material to remain perfectly elastic.* This stress is known as Young's modulus

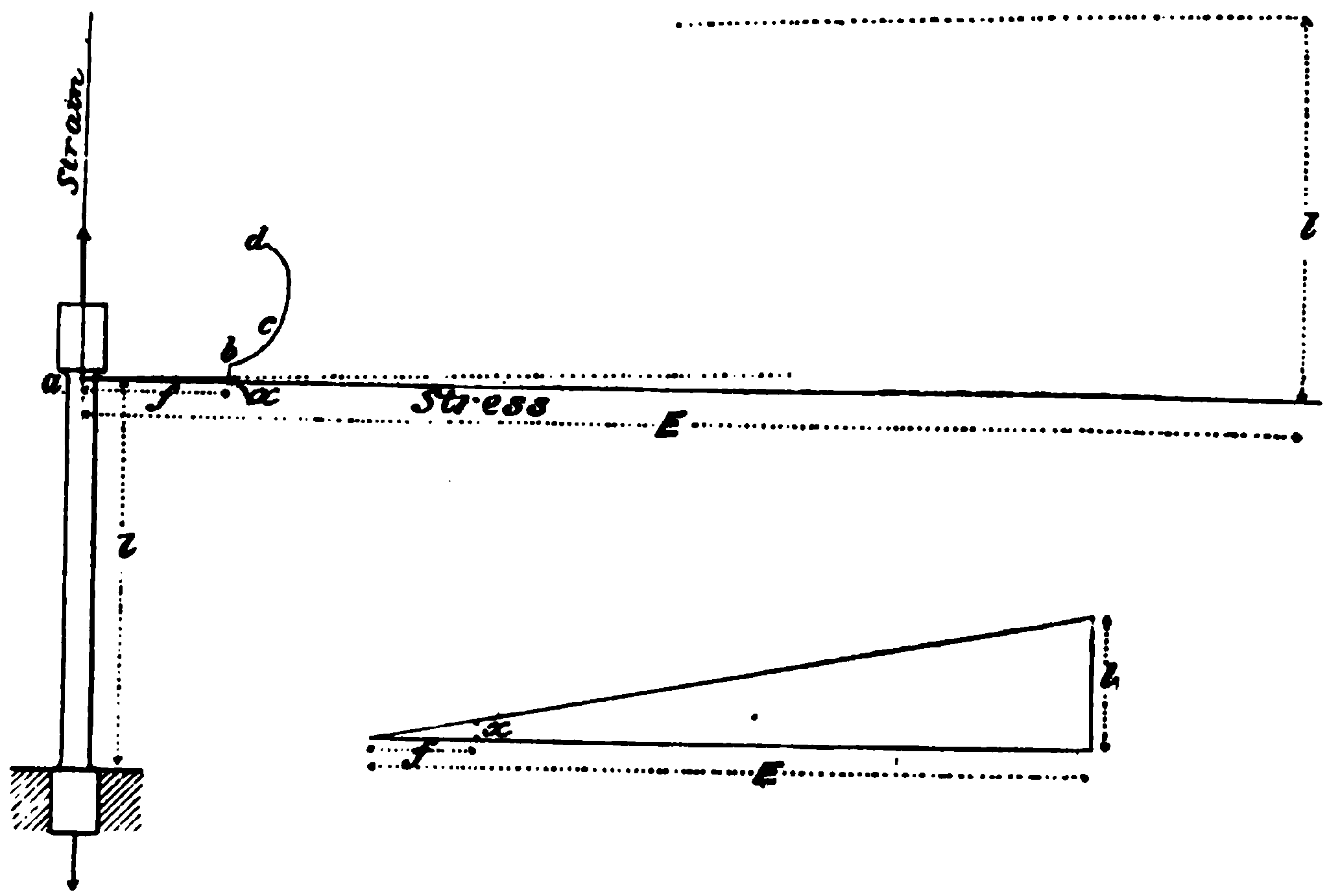


FIG. 299.

of (or measure of) elasticity. We shall give another definition of it shortly.

In the diagram in Fig. 299 we have shown a test-bar of length l between the datum points. The lower end is supposed to be rigidly fixed, and the upper end to be pulled: let a stress-strain diagram be plotted, showing the strain along the vertical and the stress along the horizontal. As the test proceeds we shall get a diagram *abcd* as shown, similar to the diagrams shown on p. 244. Produce the *elastic* line onward as shown (we have had to break it in order to get it on the page) until the *elastic* strain is equal to l; then, if x be the elastic strain at any point along the elastic line of the diagram corresponding to a stress f, we have by similar triangles—

$$\frac{x}{l} = \frac{f}{E}$$

The stress E is termed "Young's modulus of elasticity," and sometimes briefly "The modulus of elasticity." Thus in tension we might have defined the modulus of elasticity as *The stress required to stretch a bar to twice its original length, assuming the material to remain perfectly elastic.* It need hardly be pointed out that no constructive materials used by engineers do remain perfectly elastic when pulled out to twice their original length; in fact, very few materials will stretch much more than the *one-thousandth* of their length and remain elastic. It is of the highest importance that the elastic stretch should not be confused with the stretch beyond the elastic limit. It will be seen in the diagram above that the part *bcd* has *nothing whatever* to do with the modulus of elasticity.

We may write the above expression thus:

$$E = \frac{f}{\frac{x}{l}}$$

Then, if we reckon the strain per unit length as on p. 241, we have $\frac{x}{l}$ = strain, and we may write the above relation thus:

$$\text{Young's modulus of elasticity} = \frac{\text{stress}}{\text{strain}}$$

Thus Young's modulus is often defined as the ratio of the stress to the strain while the material is perfectly elastic, or we may say that it is that stress at which the strain becomes unity, assuming the material to remain perfectly elastic.

The first definition we gave above is, however, by far the clearest and most easily followed.

For compression the diagram must be slightly altered, as in Fig. 300.

In this case the lower part of the specimen is fixed and the upper end pushed down; in other respects the description of the tension figure applies to this diagram, and here, as before, we have—

$$\frac{x}{l} = \frac{f}{E}$$

For most materials the value of E is the same for both

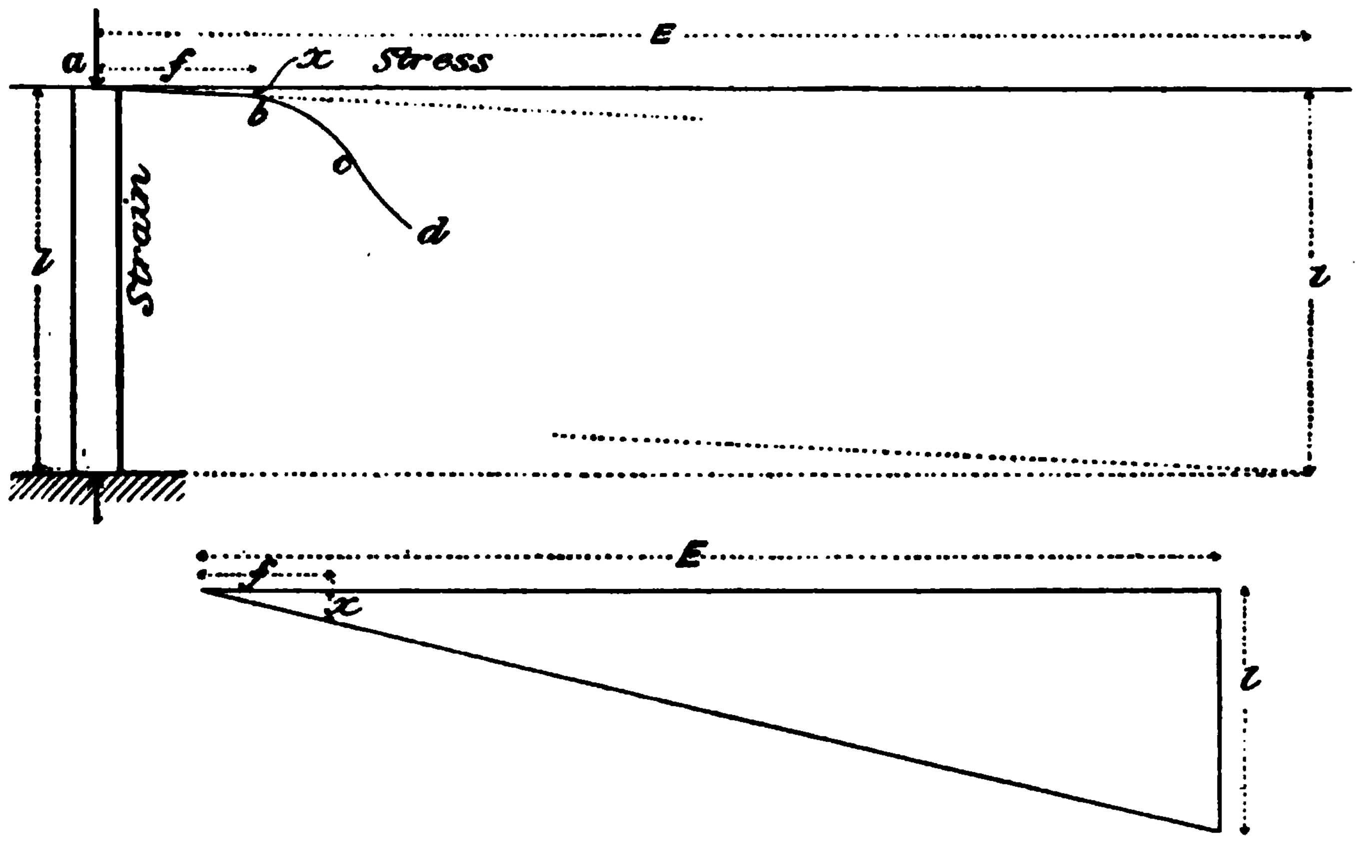


FIG. 300.

tension and compression; the actual values are given in tabular form on p. 289.

The Modulus of Transverse Elasticity, or the Coefficient of Rigidity (G).—The strain or distortion of an element subjected to shear is measured by the slide, x (see p.

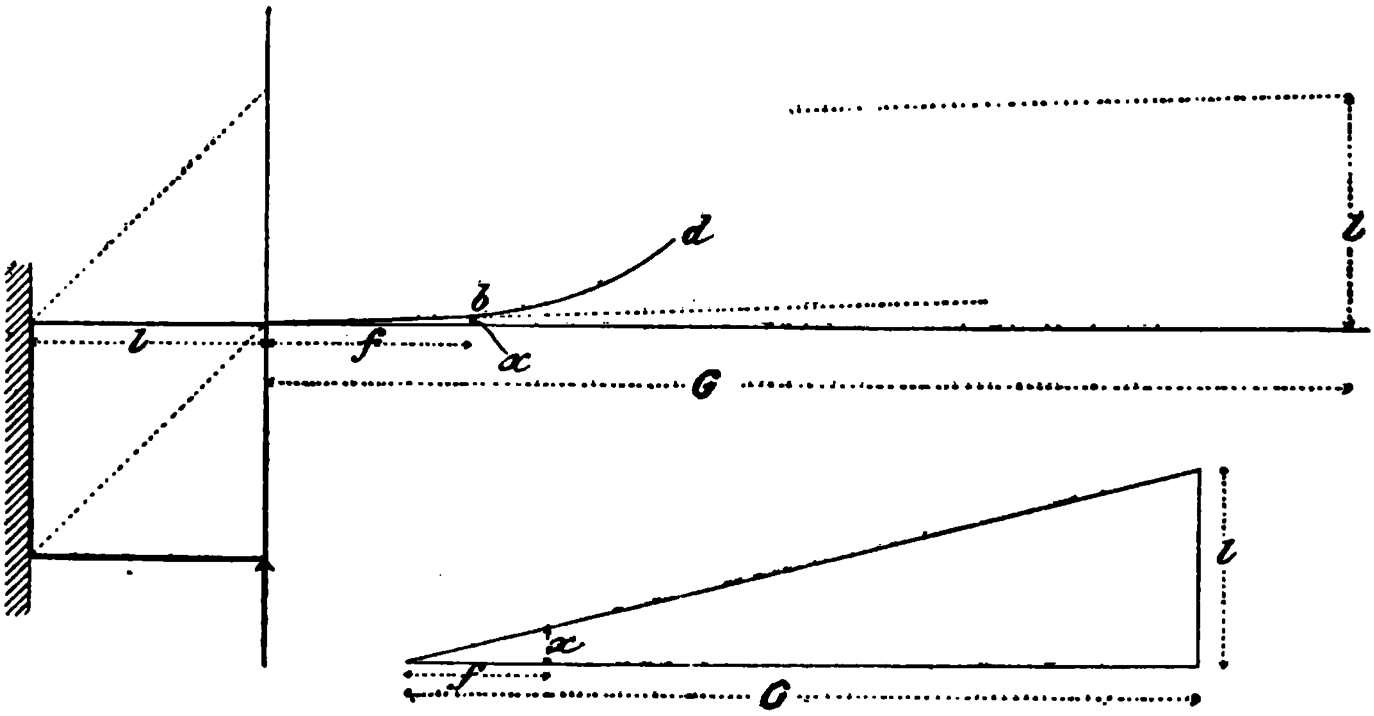


FIG. 301.

241). The shear stress required to make the slide x equal to the length l is termed the modulus of transverse elasticity, or the coefficient of rigidity, G. Assuming, as before, that the material

remains perfectly elastic, we can also represent this graphically by a diagram similar to those given for direct elasticity.

In this case the base of the square element in shear is rigidly fixed, and the outer end sheared, as shown.

From similar triangles, we have—

$$\frac{x}{l} = \frac{f}{G}$$

also—

$$G = \frac{f}{\frac{x}{l}} = \frac{\text{stress}}{\text{strain}}$$

Relation between the Moduli of Direct and Transverse Elasticity.—Let *abcd* be a square element in a perfectly elastic material which is to be subjected to—

(1) Tensile stress equal to the modulus stress; then the length l of the line *ab* will be stretched to $2l$, viz. *ab′*, and the strain reckoned on unit length will be $\frac{2l - l}{l} = 1$.

(2) Shearing stress also equal to the modulus stress; then the length l of the line *ab* will be stretched to $\sqrt{l^2 + l^2} = \sqrt{2}l = 1{\cdot}41l$, and the strain reckoned on unit length will be $\frac{1{\cdot}41l - l}{l} = 0{\cdot}41$.

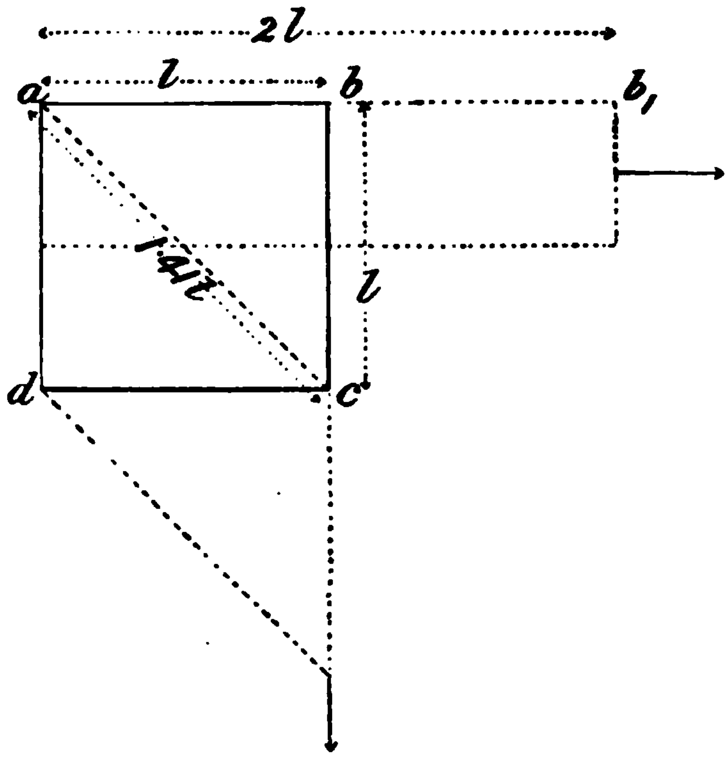


FIG. 302.

Thus, when the modulus stress is reached in shear the strain is 0·41 of the strain when the modulus stress is reached in tension; but the stress is proportional to the strain, therefore the modulus of transverse elasticity is 0·41, or $\frac{2}{5}$ nearly, of the modulus of direct elasticity.

The above proof must be regarded rather as a popular demonstration of this relation than a scientific treatment. The orthodox treatment will be given shortly.

Poisson's Ratio.—When a bar is stretched longitudinally, it contracts laterally; likewise when it is compressed longitudinally, it bulges or spreads out laterally. Then, terming stretches or spreads as positive (+) strains, and compressions or contractions as negative (−) strains, we may say that when the longitudinal strain is positive (+), the lateral strain is negative (−).

Let the lateral strain be $\frac{1}{n}$ of the longitudinal strain. The fraction $\frac{1}{n}$ is generally known as Poisson's ratio, although in reality Poisson's ratio is but a special value of the fraction viz. $\frac{1}{4}$.

Strains resulting from Three Direct Stresses acting at Right Angles.—In the following paragraph it will be convenient to use suffixes to denote the directions in which the forces act and in which the strains take place. Thus any force P which acts, say, normal to the face x will be termed P_x, and the strain per unit length $\frac{X_x}{l_x}$ will be termed S_x, and the stress on the face f_x; then $S_x = \frac{f_x}{E}$ (see p. 263).

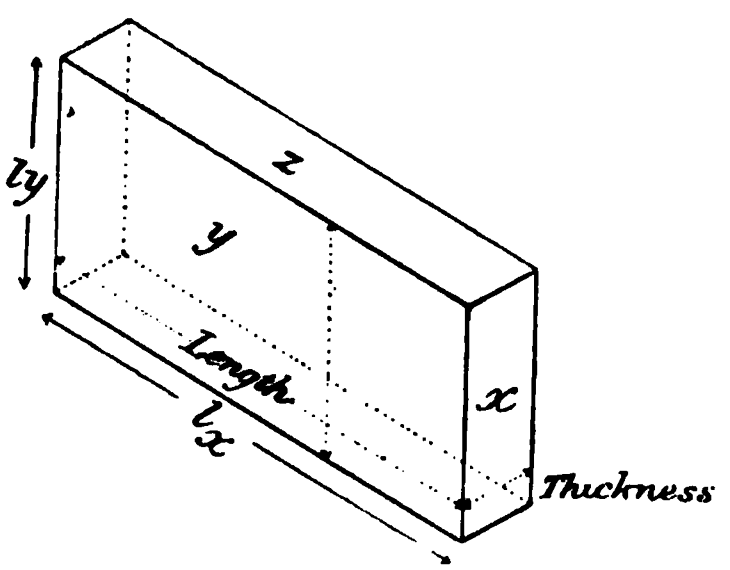


FIG. 303.

Every applied force which produces a stretch or a + strain in its own direction will be termed +, and *vice versâ*.

The strains produced by forces acting in the various directions are shown in tabulated form below.

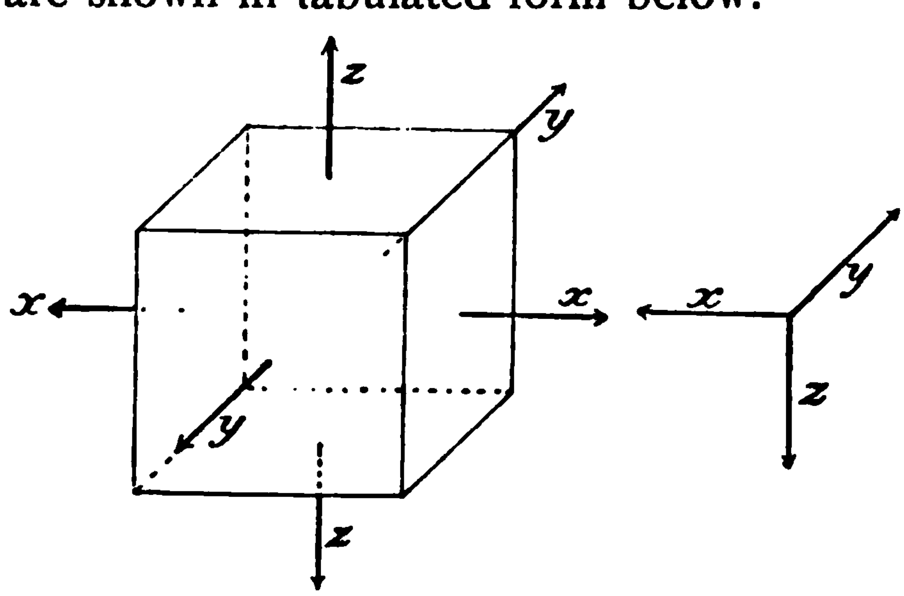


FIG. 304.

Force acting on face of cube.	Strain in direction x. S_x.	Strain in direction y. S_y.	Strain in direction z. S_z.
P_x	$\frac{f_x}{E}$	$-\frac{f_x}{En}$	$-\frac{f_x}{En}$
P_y	$-\frac{f_y}{En}$	$\frac{f_y}{E}$	$-\frac{f_y}{En}$
P_z	$-\frac{f_z}{En}$	$-\frac{f_z}{En}$	$\frac{f_z}{E}$

These equations give us the strains in any direction due to the stresses f_x, f_y, f_z acting alone; if two or more act together, the resulting strain can be found by adding the separate strains, due attention being paid to the signs.

Shear.—We showed above (p. 258) that a shear consists of two equal stresses of opposite sign acting at right angles to one another. The resulting strain can be obtained by adding the strains given in the table above due to the stresses f_1 and f_2, which are of opposite sign and act at right angles to one another.

The strains are—

$$\frac{f_1}{E} + \frac{f_2}{nE} = \frac{f}{E}\left(1 + \frac{1}{n}\right) \text{ in the direction (1)}$$

$$-\frac{f_2}{E} - \frac{f_1}{nE} = -\frac{f}{E}\left(1 + \frac{1}{n}\right) \quad \text{,,} \quad \text{,,} \quad (2)$$

$$-\frac{f_1}{nE} + \frac{f_2}{nE} = 0 \quad \text{,,} \quad \text{,,} \quad (3)$$

Thus the strain in two directions has been increased by $\frac{1}{n}$ due to the superposition of the two stresses, and has been reduced to zero in the third direction.

Let $S = \frac{f}{E}\left(1 + \frac{1}{n}\right)$.

If a square *abcd* had been drawn on the side of the element, it would have become the rhombus *a'b'c'd'* after the strain, the

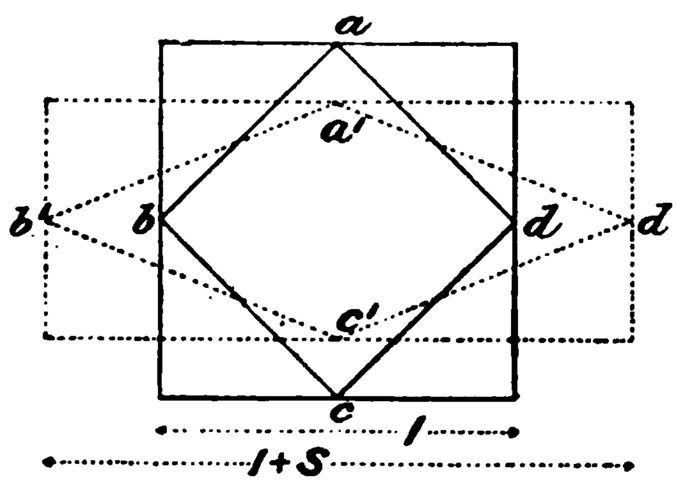


FIG. 305.

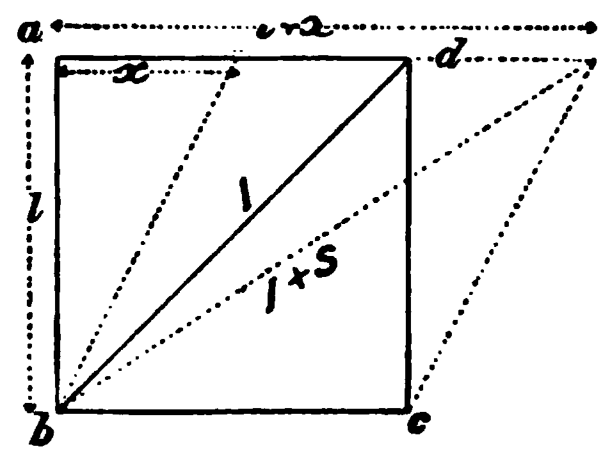


FIG. 306.

long diagonal of the rhombus being to the diagonal of the square as $1 + S$ to 1. The two superposed are shown in Fig. 306. Then we have—

$$l^2 + (l + x)^2 = (1 + S)^2$$
$$\text{or } 2l^2 + 2lx + x^2 = 1 + 2S + S^2$$

But as the diagonal of the square $= 1$, we have—

$$2l^2 = 1$$

And let $\dfrac{x}{l} = S_0$; $x = lS_0$; then by substitution we have—

$$2l^2 + 2l^2S_0 + l^2S_0^{\ 2} = 1 + 2S + S^2$$
$$\text{and } 1 + S_0 + \frac{S_0^{\ 2}}{2} = 1 + 2S + S^2$$

Both S and S_0 are exceedingly small fractions, never more than about $\frac{1}{1000}$, and their squares will be still smaller, and therefore negligible. Hence we may write the above—

$$1 + S_0 = 1 + 2S$$
$$\text{or } S_0 = 2S = \frac{2f}{E}\left(1 + \frac{1}{n}\right)$$

$$\text{But } G = \frac{f}{S_0} = \frac{fE}{2f\left(1 + \frac{1}{n}\right)} = \frac{E}{2}\left(\frac{1}{1 + \frac{1}{n}}\right) = \frac{En}{2(n + 1)}$$

$$\text{hence } n = \frac{2G}{E - 2G}, \text{ and } E = \frac{2G(n + 1)}{n}$$

When $n = 5$, $G = \frac{5}{12}E = 0{\cdot}42E$.
$n = 4$, $G = \frac{2}{5}E = 0{\cdot}40E$.
$n = 3$, $G = \frac{3}{8}E = 0{\cdot}38E$.
$n = 2$, $G = \frac{1}{3}E = 0{\cdot}33E$.

Some values of n will be given shortly.

We have shown above that the maximum strain in an element subject to shear is—

$$S = \frac{f}{E}\left(1 + \frac{1}{n}\right)$$

but the maximum strain in an element subject to a direct stress in tension is—

$$S_t = \frac{f}{E}$$

$$\text{hence } \frac{S}{S_t} = \frac{\frac{f}{E}\left(1 + \frac{1}{n}\right)}{\frac{f}{E}} = \left(1 + \frac{1}{n}\right)$$

$$\text{or } S = S_t\left(1 + \frac{1}{n}\right)$$

	$\frac{S}{S_t}$	$\frac{\text{Safe shear stress}}{\text{safe tensile stress}}$
When $n = 5$	$\frac{6}{5}$	$\frac{5}{6}$
,, $n = 4$	$\frac{5}{4}$	$\frac{4}{5}$
,, $n = 3$	$\frac{4}{3}$	$\frac{3}{4}$
,, $n = 2$	$\frac{3}{2}$	$\frac{2}{3}$

Taking $n = 4$, we see that the same material will take a permanent set, or will pass the elastic limit in shear with $\frac{4}{5}$ of the stress that it will take in tension; or, in other words, the shearing strength of a material is only $\frac{4}{5}$ of the tensile strength. Although this proof only holds while the material is elastic, yet the ratio is approximately correct for the ultimate strength.

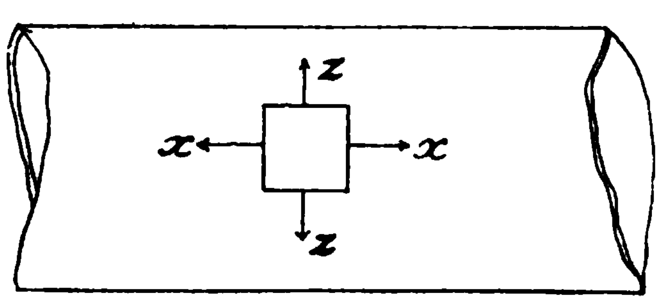


FIG. 307.

Boiler Shell.—On p. 285 we show that $P_z = 2P_x$;

$$f_z = 2f_x\text{ ; } f_x = \frac{f_z}{2}.$$

Strains.

Let $n = 4$.

$$S_x = \frac{f_x}{E} - \frac{f_z}{En} = \frac{f_z}{E}\left(\frac{1}{2} - \frac{1}{n}\right) = \frac{1}{4}\frac{f_z}{E}$$

$$S_y = -\frac{f_x}{En} - \frac{f_z}{En} = -\frac{f_z}{E}\left(\frac{1}{2n} + \frac{1}{n}\right) = -\frac{3}{8}\frac{f_z}{E}$$

$$S_z = -\frac{f_x}{En} + \frac{f_z}{E} = \frac{f_z}{E}\left(1 - \frac{1}{2n}\right) = \frac{7}{8}\frac{f_z}{E}$$

Thus the maximum strain is in the direction S_z.

By the thin-cylinder theory we have the maximum strain $= \frac{f_z}{E}$; thus the real strain is only $\frac{7}{8}$ as great, or a cylindrical boiler shell will stand $\frac{8}{7} = 1·14$ or 14 per cent. more pressure before the elastic limit is reached than is given by ordinary ring theory.

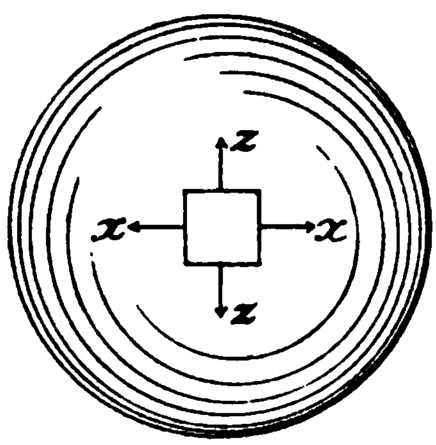


FIG. 308.

Thin Sphere subjected to Internal Fluid Pressure.—In the case of the sphere, we have $P_x = P_z$; $f_x = f_z$.

Strains.—

$$S_x = \frac{f_x}{E} - \frac{f_z}{En} = \frac{f_x}{E}\left(1 - \frac{1}{n}\right) = \frac{3}{4}\frac{f_x}{E}$$

$$S_y = -\frac{f_x}{En} - \frac{f_z}{En} = -\frac{f_x}{E}\left(\frac{1}{n} + \frac{1}{n}\right) = -\frac{1}{2}\frac{f_x}{E}$$

$$S_z = -\frac{f_x}{En} + \frac{f_z}{E} = \frac{f_x}{E}\left(1 - \frac{1}{n}\right) = \frac{3}{4}\frac{f_x}{E}$$

But f_x in this case $= \frac{f_z}{2}$ in the case given above;

$$\therefore S_z \text{ in this case} = \frac{3}{4} \times \frac{f_z}{2E} = \frac{3f_z}{8E}$$

$$\frac{\text{Maximum strain in sphere}}{\text{maximum strain in boiler shell}} = \frac{\frac{3}{8}}{\frac{7}{8}} = \frac{3}{7}$$

Hence, in order that the hemispherical ends of boilers should enlarge to the same extent as the cylindrical shells when under pressure, the plates in the ends should be $\frac{3}{7}$ thickness of the plates in cylindrical portion. If the proportion be not adhered to, bending will be set up at the junction of the ends and the cylindrical part.

Alteration of Volume due to Stress.—If a body were placed in water or other fluid, and were subjected to pressure, its volume would be diminished in proportion to the pressure.

Let V = original volume of body ;
dv = change of volume due to change of pressure ;
dp = „ pressure.

Then $K = \dfrac{dp}{\frac{dv}{V}} = \dfrac{\text{change of pressure}}{\text{change of volume per cubic unit of the body}}$

$$K = \frac{p}{v}$$

K is termed the coefficient of elasticity of volume.

The change of volume is the algebraic sum of all the strains produced. Then, putting $p = f_x = f_y = f_z$ for a fluid pressure, we have, from the table on p. 267, the resulting strains—

$$\frac{p}{E} - \frac{2p}{En} + \frac{p}{E} - \frac{2p}{En} + \frac{p}{E} - \frac{2p}{En} = \frac{3p}{E} - \frac{6p}{En}$$

$$\text{or } v = \frac{3p}{E} - \frac{6p}{En}$$

$$K = \frac{p}{v} = \frac{pEn}{3pn - 6p} = \frac{En}{3n - 6} = \frac{EG}{9G - 3E}$$

$$\text{But } E = \frac{2G(n + 1)}{n} \quad \text{(p. 268)}$$

$$\text{hence } K = \frac{2G(n + 1)}{3n - 6}$$

$$\text{and } n = \frac{2G + 6K}{3K - 2G}$$

The following table gives values of K in tons per square inch also of n :—

Material.	K.	n.
Water ...	140	—
Cast iron ...	6,000	3·8
Wrought iron	8,800	3·6
Steel	11,000	4·0
Brass	6,400	3·1
Copper ...	10,500	2·9
Flint glass ...	2,400	3·9
Indiarubber ...	—	2·0

The n given above has not been calculated by the above

formula, but is the mean of the most reliable published experiments.

Riveted Joints.

Strength of a Perforated Strip.—If a perforated strip of width w be pulled apart in the testing-machine, it will break through the hole, and if the material be only very slightly ductile, the breaking load will be (approximately) $W = f_t(w - d)t$, where f_t is the tensile strength of the metal, and t the thickness of the plate. If the metal be very ductile the breaking load will be higher than this, due to the fact that the tensile strength is always reckoned on the original area of the test bar, and not on the final area at the point of fracture. The difference between the real and the nominal tensile strength, therefore, depends upon the reduction in area. If we could prevent the area from contracting, we should raise the nominal tensile strength. In a perforated bar the minimum area of the section—through the hole—is surrounded by metal not so highly stressed, hence the reduction in area is less and the nominal tensile strength is greater than that of a plain bar. This apparent increase in strength does not occur until the stress is well past the elastic limit, hence we have no need to take it into account in the design of riveted joints.

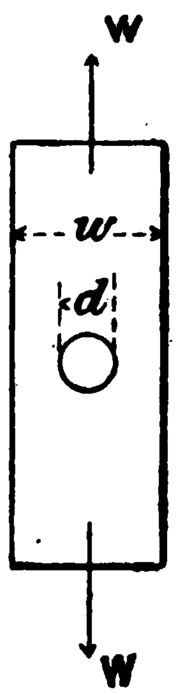


FIG. 309.

Strength of an Elementary Pin Joint.—If a bar, perforated at both ends as shown, were pulled apart through the medium of pins, failure might occur through the tearing of the plates, as shown at aa or bb, or by the shearing of the pins themselves.

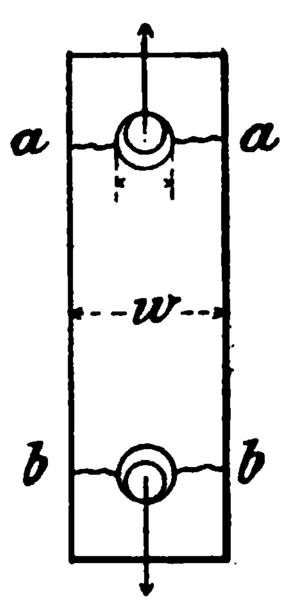


FIG 310.

Let d = the diameter of the pins;
c = the clearance in the holes.

Then the diameter of the holes $= d + c$

Let f_s = the shearing strength of the material in the pins.

For simplicity at the present stage, we will assume the pins to be in single shear. Then, for equal strength of plate and pins, we have—

$$\{w - (d + c)\}tf_t = \frac{\pi d^2}{4}f_s$$

If the holes had been punched instead of drilled, a thin ring of metal all round the hole would have been damaged by the rough treatment of the punch. This damaging action can be very clearly seen by examining the plate under a microscope, and its effect demonstrated by testing two similar strips, in one of which the holes are drilled, and in the other punched, the latter breaking at a lower load than the former. Let the thickness of this damaged ring be $\frac{K}{2}$; then the equivalent diameter of the hole will be $d + c + K$, and for equal strength of plate and pins—

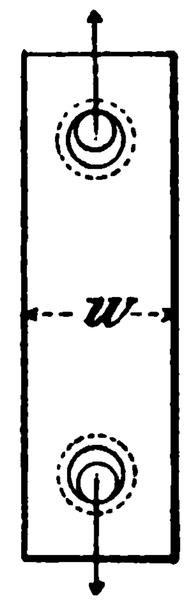


FIG. 311.

$$\{w - (d + c + K)\}tf_t = \frac{\pi d^2}{4}f_s$$

A riveted joint differs from the pin joint in one important respect: the rivets, when closed, completely (or ought to) fill the holes, hence the diameter of the rivet is $d + c$ when closed. In speaking of the diameter of a rivet, we shall, however, always mean the original diameter before closing.

Then, allowing for the increase in the diameter of the rivet, the above expressions become, when the plates are not damaged as in drilling—

$$\{w - (d + c)\}tf_t = \frac{\pi(d + c)^2}{4}f_s$$

When the plates are punched—

$$\{w - (d + c + K)\}tf_t = \frac{\pi(d + c)^2}{4}f_s$$

The value of c, the clearance of a rivet-hole, may be taken at about $\frac{1}{20}$ of the original diameter of the rivet, or $c = 0·05d$.

The diameter of the rivet is rarely less than $\frac{3}{4}$ inch or greater than $1\frac{1}{4}$ inch for boiler work; hence, when convenient, we may write $c = 0·05$ inch.

The equivalent thickness of the ring damaged by punching may be taken at $\frac{1}{10}$ of an inch, or $K = 0·2$ inch. This value

has been obtained by very carefully examining all the most recent published accounts of tests of riveted joints.

For all boiler work we shall take the size of rivet according to Unwin's rule—

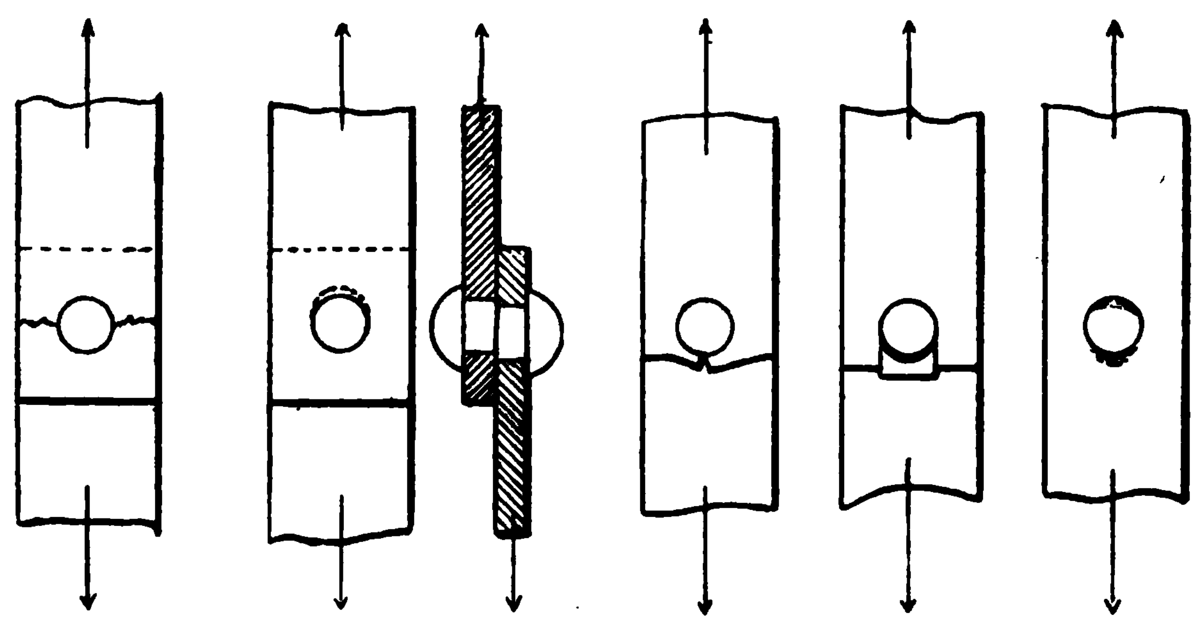
$$d \doteq 1{\cdot}25\sqrt{t}$$
$$\text{and } d^2 = 1{\cdot}56t$$
$$\text{and } (d + c)^2 = 1{\cdot}72t$$

Instead of using f_s, we may write $\frac{4}{5}f_r$ (see p. 269), where f_r is the tensile strength of the rivet material.

The values of f_b, f_r, and f_t may be taken as follows, but if in any special case they differ materially, the actual values should be inserted.

Material.	f_b[1]	f_r	f_t	
Iron	50	24	21	tons per square inch.
Steel	50	28	28	

Ways in which a Riveted Joint may fail.—Riveted joints are designed to be equally strong in tearing the plate



Tearing plate through rivet-holes. Shearing of rivet. Bursting through edge of plate. Shearing through edge of plate. Crushing of rivet.

FIG. 312.

and in shearing the rivet; the design is then checked, to see that the plates and rivets are safe against crushing.

Failure through bursting or shearing of the edge of the plate is easily avoided by allowing sufficient margin between the edge of the rivet-hole and the edge of the plate; usually this is not less than the diameter of the rivet.

[1] See paragraph on Bearing Pressure.

Lap and Single Cover-plate Riveted Joints.—When such joints are pulled, the plates bend, as shown, till the two

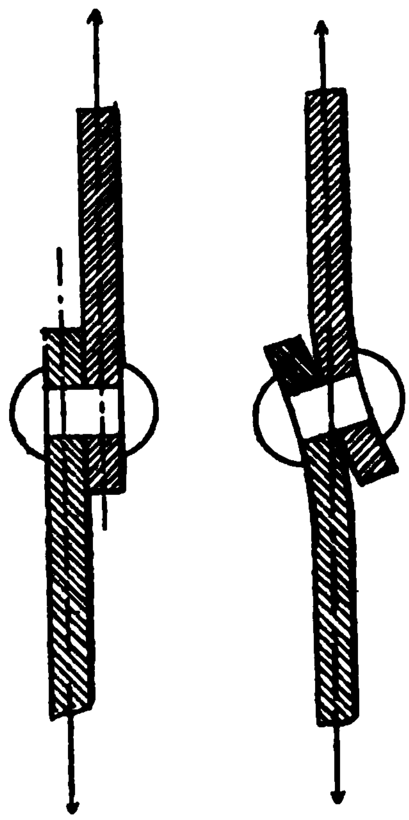


FIG. 313.

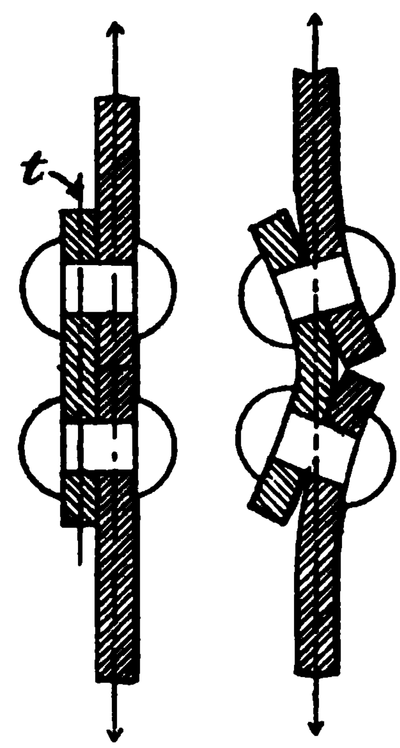


FIG. 314.

lines of pull coincide. This bending action very considerably increases the stress in the material, and consequently weakens the joint. It is not usual to take this bending action into account, although it is as great or greater than the direct stress, and is the cause of the dangerous grooving so often found in lap-riveted boilers.[1]

[1] The bending stress can be approximately arrived at thus: The maximum bending moment on the plates is $\frac{Pt}{2}$ (see p. 357), where P is the total load on a strip of width w. This bending moment decreases as the plates bend. Then, f_t being the stress in the metal between the rivet-holes, the stress in the metal where there are no holes is $f_t\left(\frac{w-d}{w}\right)$; hence—

$$P = wtf_t\left(\frac{w-d}{w}\right)$$

and the bending moment $= \frac{t^2 f_t(w-d)}{2}$

The plate bends in two places along lines where there are no holes; hence—

$$Z = \frac{2wt^2}{6} = \frac{wt^2}{3} \text{ (see Chap. IX.)}$$

and the skin stress due to bending—

$$f = \frac{3}{2} f_t\left(\frac{w-d}{w}\right)$$

Single Row of Rivets.

Punched iron—

$$(w - d - c - K)tf_t = \frac{\pi}{4}(d + c)^2\frac{4f_r}{5}$$

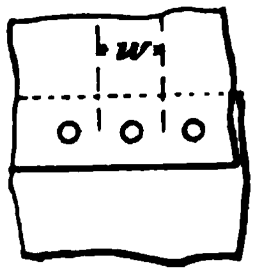


FIG. 315.

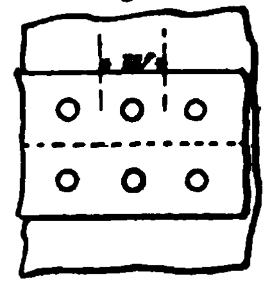


FIG. 316.

Then, substituting 0·05 inch for c on the left-hand side of the equation, and putting in the numerical value of K as given above, also putting $(d + c)^2 = 1{\cdot}72t$, we have on reduction—

$$w - d = 1{\cdot}082\frac{f_r}{f_t} + 0{\cdot}25 \text{ inch}$$

$$w - d = 1{\cdot}49 \text{ inch}$$

Thus the *space between the rivet-holes* $(w - d)$ of all punched iron plates with single lap or cover joints is 1·49 inch, or say $1\frac{1}{2}$ inch for all thicknesses of plate.

By a similar process, we get for—

Punched steel—

$$w - d = 1{\cdot}33 \text{ inch}$$

In the case of drilled plates the constant K disappears, hence $w - d$ is 0·2 inch less than in the case of punched plates; then we have for—

Drilled iron—

$$w - d = 1{\cdot}29 \text{ inch}$$

Drilled steel—

$$w - d = 1{\cdot}13 \text{ inch}$$

Double Row of Rivets.—In this case two rivets have to

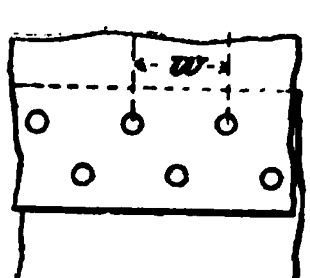


FIG. 317.

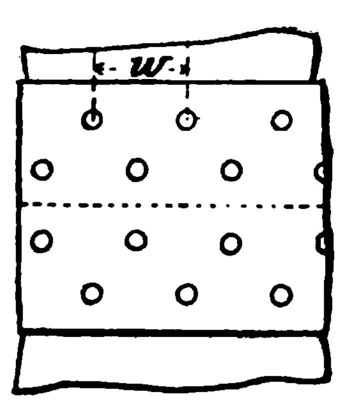


FIG. 318.

be sheared through per unit width of plate w; hence we have—

Punched iron—

$$(w - d - c - \mathrm{K})tf_t = \frac{2\pi}{4}(d + c)^2\frac{4f_r}{5}$$

$$w - d = 2\cdot16\frac{f_r}{f_t} + 0\cdot25 \text{ inch}$$

$$w - d = 2\cdot72 \text{ inches}$$

Punched steel—

$$w - d = 2\cdot41 \text{ inches}$$

Drilled iron—

$$w - d = 2\cdot52 \text{ inches}$$

Drilled steel—

$$w - d = 2\cdot21 \text{ inches}$$

Double Cover-plate Joints.—In this type of joint there is no bending action on the plates. Each rivet is in double shear; therefore, with a single row of rivets the space between the rivet-holes is the same as in the lap joint with two rows of rivets. The joint shown in Fig. 319 has a single row of rivets (*i.e.* in each plate).

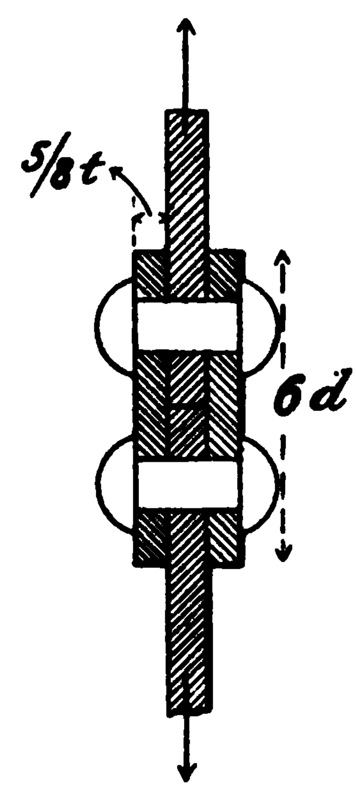


FIG. 319.

Double Row of Rivets.—In this case there are two rivets in double shear, which is equivalent to four rivets in single shear for each unit width of plate w.

Punched iron—

$$(w - d - c - \mathrm{K})tf_t = \frac{4\pi}{4}(d + c)^2\frac{4f_r}{5}$$

$$w - d = 4\cdot328\frac{f_r}{f_t} + 0\cdot25$$

$$w - d = 5\cdot20 \text{ inches}$$

Punched steel—

$$w - d = 4\cdot58 \text{ inches}$$

Drilled iron—

$$w - d = 5\cdot00 \text{ inches}$$

Drilled steel—

$$w - d = 4\cdot38 \text{ inches}$$

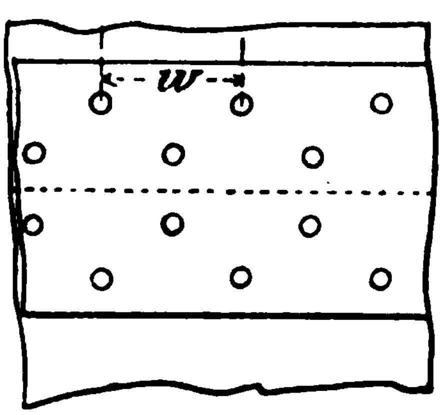


FIG. 320.

Diamond Riveting.—In this case there are five rivets in double shear, which is equivalent to ten rivets in single shear,

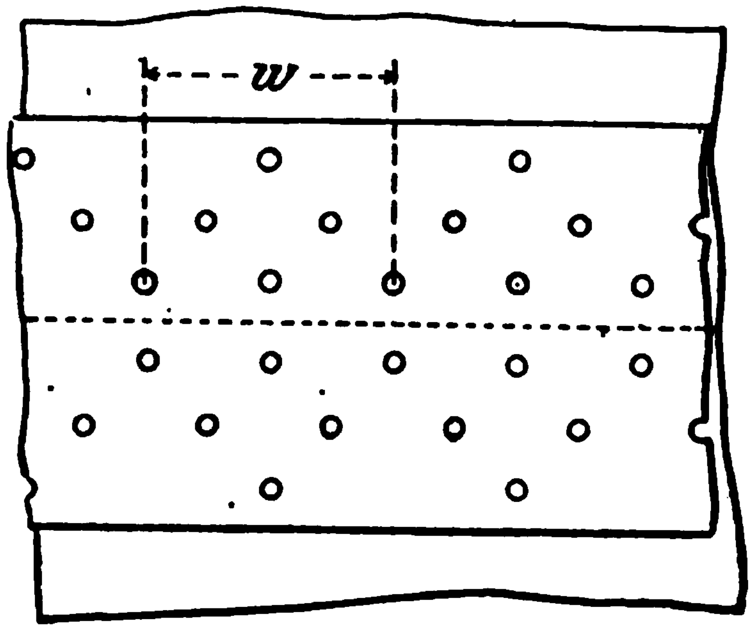


FIG. 321.

in each unit width of plate w; whence we have for drilled steel[1] plates—

$$(w - d - c)tf_t = \frac{10\pi}{4}(d + c)^2\frac{4f_r}{5}$$
$$w - d = 10{\cdot}87 \text{ inches}$$

Combined Lap and Cover-plate Joint.—This joint may fail by—

(1) Tearing through the outer row of rivet-holes.

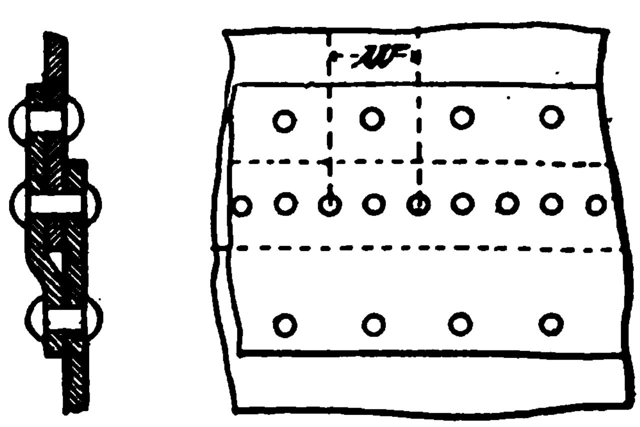


FIG. 322.

(2) Tearing through the inner row of rivet-holes and shearing the outer rivet (single shear).

(3) Shearing three rivets in single shear (one on outer row, and two on inner).

Making (1) = (3), we have for drilled steel—

$$(w - d - c)tf_t = \frac{3\pi}{4}(d + c)^2\frac{4f_r}{5}$$
$$w - d = 3{\cdot}3 \text{ inches}$$

If we make (1) = (3), we get—

$$(w - 2d - 2c)tf_t + \frac{\pi}{4}(d + c)^2\frac{4f_r}{5} = \frac{3\pi}{4}(d + c)^2\frac{4f_r}{5}$$

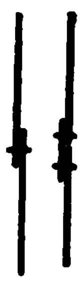
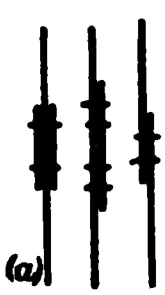
On reduction, this becomes—

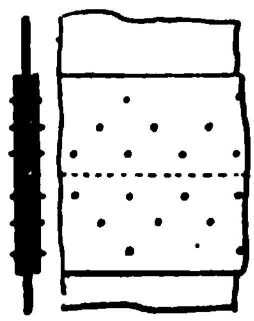
$$w - 2d = 2{\cdot}26 \text{ inches}$$

[1] This joint is rarely used for other than drilled steel plates.

If the value of d be supplied, it will be seen that the two results are practically the same for the usual sizes of rivets; as the former is slightly the simpler method, such joints may be designed by making (1) = (3).

Pitch of Rivets.—The pitch of the rivets, *i.e.* the distance from centre to centre, is simply w; in certain cases, which are very readily seen, the pitch is $\frac{w}{2}$. The pitch for a number of joints is given in the table below. The diameters of the rivets to the nearest $\frac{1}{16}$ of an inch are given in brackets.

Type of joint.	Thickness of plate, t.	Diameter of rivet, $d = 1·25\sqrt{t}$	Iron plates and rivets.		Steel plates and rivets.	
			Punched. $1·49 + d$.	Drilled. $1·29 + d$.	Punched. $1·33 + d$.	Drilled. $1·13 + d$.
A.  FIG. 323.	in.	in.				
	$\frac{3}{8}$	0·76 ($\frac{3}{4}$)	2·25	2·05	2·09	1·89
	$\frac{1}{2}$	0·88 ($\frac{7}{8}$)	2·37	2·17	2·21	2·01
	$\frac{5}{8}$	0·99 (1)	2·48	2·28	2·32	2·12
	$\frac{3}{4}$	1·08 ($1\frac{1}{16}$)	2·57	2·37	2·41	2·21
			$2·72 + d$.	$2·52 + d$.	$2·41 + d$.	$2·21 + d$.
B.  FIG. 324.	$\frac{3}{8}$	0·76 ($\frac{3}{4}$)	(*a*) 2·94 3·48	(*a*) 2·74 3·28	(*a*) 2·46 3·17	(*a*) 2·26 2·97
	$\frac{1}{2}$	0·88 ($\frac{7}{8}$)	(*a*) 3·34 3·60	(*a*) 3·14 3·40	(*a*) 2·79 3·29	(*a*) 2·59 3·09
	$\frac{5}{8}$	0·99 (1)	3·71	3·51	(*a*) 3·10 3·40	(*a*) 2·90 3·20
	$\frac{3}{4}$	1·08 ($1\frac{1}{16}$)	3·80	3·60	(*a*) 3·25 3·49	(*a*) 3·15 3·29
	$\frac{7}{8}$	1·17 ($1\frac{3}{16}$)	3·89	3·69	3·58	3·38
	1	1·25 ($1\frac{1}{4}$)	3·97	3·77	3·66	3·46
			$5·20 + d$.	$5·00 + d$.	$4·58 + d$.	$4·38 + d$.
C.  FIG. 325.	$\frac{1}{2}$	0·88 ($\frac{7}{8}$)	5·28 ~~6·08~~	5·28 ~~5·88~~	4·45 ~~5·46~~	4·25 ~~5·26~~
	$\frac{5}{8}$	0·99 (1)	5·94 ~~6·19~~	5·94 5·99	4·95 ~~5·57~~	4·75 ~~5·37~~
	$\frac{3}{4}$	1·08 ($1\frac{1}{16}$)	6·28	6·08	5·36 ~~5·66~~	5·16 ~~5·46~~
	$\frac{7}{8}$	1·17 ($1\frac{3}{16}$)	6·37	6·17	5·75	5·55
	1	1·25 ($1\frac{1}{4}$)	6·45	6·25	5·83	5·63
	$1\frac{1}{8}$	1·33 ($1\frac{5}{16}$)	6·53	6·33	5·91	5·71

Type of joint.	Thickness of plate, t.	Diameter of rivet. $d = 1·25\sqrt{t}$	Steel plates and rivets (drilled holes). Inner row.	Outer row. $10·87 + d$.
	in.	in.		
D.  FIG. 326.	$\frac{3}{4}$	1·08 ($1\frac{1}{16}$)	5·62	11·23 ~~11·95~~
	$\frac{7}{8}$	1·17 ($1\frac{3}{16}$)	6·02	12·04
	1	1·25 ($1\frac{1}{4}$)	6·06	12·12
	$1\frac{1}{8}$	1·33 ($1\frac{5}{16}$)	6·10	12·20
	$1\frac{1}{4}$	1·40 ($1\frac{3}{8}$)	6·14	12·27
	$1\frac{3}{8}$	1·47 ($1\frac{7}{16}$)	6·17	12·34
				$3·3 + d$.
E.  FIG. 327.	$\frac{1}{2}$	0·88 ($\frac{7}{8}$)	2·09	4·18
	$\frac{5}{8}$	0·99 (1)	2·15	4·29
	$\frac{3}{4}$	1·08 ($1\frac{1}{16}$)	2·19	4·38
	$\frac{7}{8}$	1·17 ($1\frac{3}{16}$)	2·24	4·47
	1	1·25 ($1\frac{1}{4}$)	2·27	4·55
	$1\frac{1}{8}$	1·33 ($1\frac{5}{16}$)	2·32	4·63

It is found that if the pitch of the rivets along the caulked edge of a plate exceeds six times the diameter of the rivet, the plates are liable to pucker up when caulked; hence in the above table all the pitches that exceed this are crossed out with a horizontal line, and the greatest permissible pitch inserted.

Bearing Pressure.—The ultimate bearing pressure on a boiler rivet must on no account exceed 50 tons per square inch, or the rivet and plate will crush. It is better to keep it below 45 tons per square inch. The bearing area of a rivet is dt, or $(d + c)t$. Let f_b = the bearing pressure.

The total load on a group of n rivets in a strip of plate of width w $\Big\} = nf_b(d + c)t$

the load on a strip of plate of width w $\Big\} = (w - d - c)f_t t$, or $(w - d - c - K)f_t t$

then $(w - d - c)f_t t = nf_b(d + c)t$

$$\text{and } f_b = \frac{(w - d - c)f_t}{n(d + c)} \not> 50 \text{ tons per sq. inch}$$

The bearing pressure has been worked out for the joints

given in the table above, and in those instances in which it is excessive they have been crossed out with a diagonal line, and the greatest permissible pitch has been inserted.

Efficiency of Joints.—

$$\left.\begin{array}{l}\text{The efficiency}\\ \text{of a riveted}\\ \text{joint}\end{array}\right\} = \frac{\text{strength of joint}}{\text{strength of plate}}$$

$$= \frac{\text{effective width of metal between rivet-holes}}{\text{pitch of rivets}}$$

$$= \frac{w - d - c}{w}, \text{ or } \frac{w - d - c - \mathrm{K}}{w}$$

The table on the following page gives the efficiency of joints corresponding to the table of pitches given above. All the values are per cent.

Zigzag Riveting.—In zigzag riveting, if the two rows are placed too near together, the plates tear across in zigzag fashion. If the material of the plate were equally strong with and across the grain, then x_1, the zigzag distance between the two holes, should be $\frac{x}{2}$. The plate is, however, weaker along than across the grain, consequently when it tears from one row to the other it partially follows the grain, and therefore tears more readily. The joint is found to be equally strong in both directions when $x_1 = \frac{2x}{3}$.

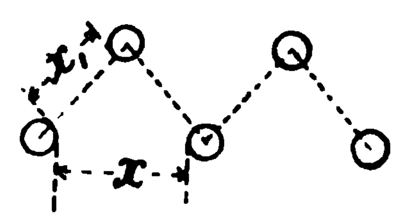


FIG. 328.

Riveted Tie-bar Joints.—When riveting a tie bar, a very high efficiency can be obtained by properly arranging the rivets.

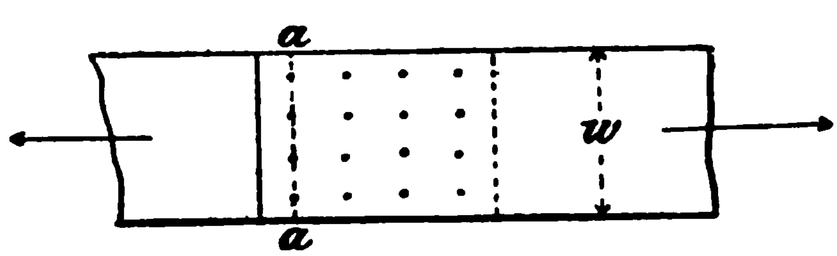


FIG. 329.

The arrangement shown in Fig. 329 is radically wrong for tension joints. The strength of such a joint is, neglecting the clearance in the holes, and damage done by punching—

$(w - 4d)f_t t$ at *aa*

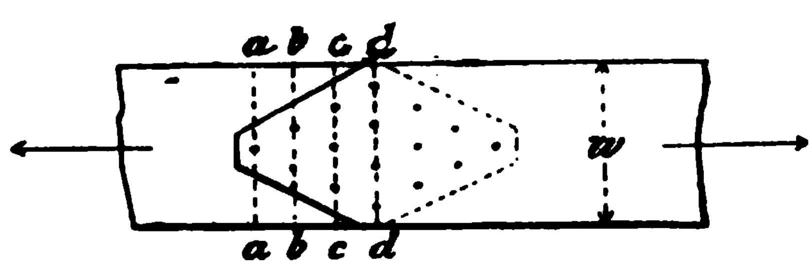
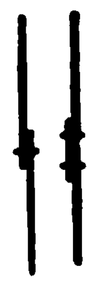
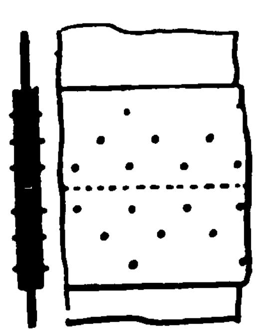


FIG. 330.

Efficiency of Riveted Joints.

Type of joint.	Thickness of plate.	Iron plates and rivets.		Steel plates and rivets.	
		Punched.	Drilled.	Punched.	Drilled.
	in.				
A. 	$\frac{3}{8}$	55	60	52	57
	$\frac{1}{2}$	52	57	49	54
	$\frac{5}{8}$	50	54	47	51
	$\frac{3}{4}$	48	52	45	49
B.  (a)	$\frac{3}{8}$	(a) 66 71	(a) 70 75	(a) 59 68	(a) 64 73
	$\frac{1}{2}$	(a) 66 69	(a) 70 73	(a) 59 66	(a) 64 70
	$\frac{5}{8}$	67	70	(a) 59 64	(a) 64 68
	$\frac{3}{4}$	65	68	(a) 59 62	(a) 64 66
	$\frac{7}{8}$	63	67	60	64
	1	62	66	59	62
C. 	$\frac{1}{2}$	79	82	75	78
	$\frac{5}{8}$	79	82	75	78
	$\frac{3}{4}$	79	81	75	78
	$\frac{7}{8}$	78	80	75	78
	1	77	79	74	77
	$1\frac{1}{8}$	76	78	73	76
D. 	$\frac{3}{4}$				90
	$\frac{7}{8}$				90
	1				89
	$1\frac{1}{8}$				89
	$1\frac{1}{4}$				88
	$1\frac{3}{8}$				88
E. 	$\frac{1}{2}$				78
	$\frac{5}{8}$				76
	$\frac{3}{4}$				74
	$\frac{7}{8}$				73
	1				71
	$1\frac{1}{8}$				70

whereas with the arrangement shown in Fig. 330 the strength is—

$(w - d)f_t t$ at aa (tearing only)

$(w - 2d)f_t t + \frac{\pi}{4} d^2 f_s$ at bb (tearing and shearing one rivet)

$(w - 3d)f_t t + \frac{3\pi}{4} d^2 f_s$ at cc („ „ three rivets)

$(w - 4d)f_t t + \frac{6\pi}{4} d^2 f_s$ at dd („ „ six „)

By assuming some dimensions and working out the strength at each place, the weakest section may be found, which will be far greater than that of the joint shown in Fig. 329. The joint in Fig. 330 will be found to be of approximately equal strength at all the sections, hence for simplicity of calculation it may be taken as being—

$$(w - d)f_t t$$

In the above expressions the constants c and K have been omitted; they are not usually taken into account in such joints.

The working bearing pressure on the rivets should not exceed 8 tons square inch, and where there is much vibration the bearing pressure should not exceed 6 tons square inch.

For bridge, roof, ship plates, and riveted work other than boilers, it is usual to use smaller rivets for the same thickness of plate. Unwin gives—

$$d = 1{\cdot}1\sqrt{t}$$

	in.	in.	in.	in.	in.	in.	in.
Thickness of plate ...	$\frac{3}{8}$	$\frac{1}{2}$	$\frac{5}{8}$	$\frac{3}{4}$	$\frac{7}{8}$	1	$1\frac{1}{8}$
Diameter of rivet ...	$\frac{5}{8}$	$\frac{3}{4}$	$\frac{7}{8}$	$\frac{15}{16}$	1	$1\frac{1}{8}$	$1\frac{3}{16}$

Groups of Rivets.—In the chapter on combined bending and direct stress, it is shown that the stress may be very seriously increased by loading a bar in such a manner that the line of pull or thrust does not coincide with the centre of gravity of the section of the bar. Hence, in order to get the stress evenly distributed over a bar, the centre of gravity of the

group of rivets must lie on the line drawn through the centre of gravity of the cross-section of the bar. And when two bars not in line are riveted, as in the case of the bracing and the boom of a bridge, the centre of gravity of the group must lie

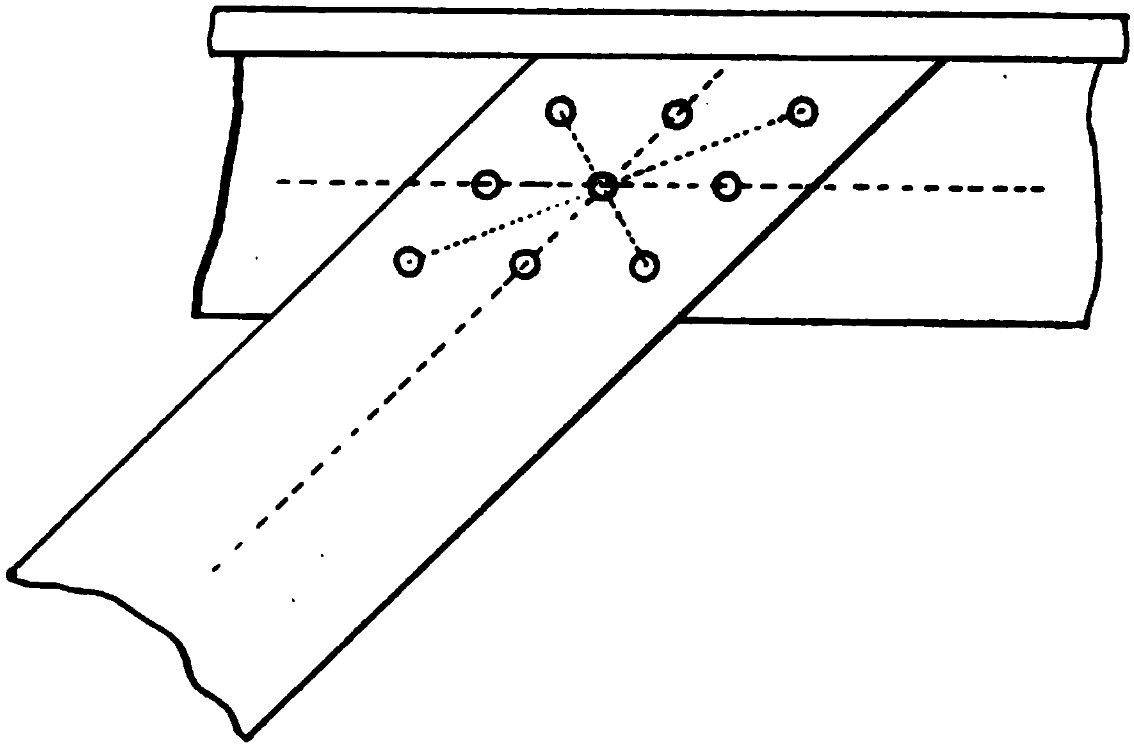


FIG. 331.

on the intersection of the lines drawn through the centres of gravity of the cross-sections of the two bars.

In other words, the rivets must be arranged symmetrically about the two centre lines (Fig. 331).

Strength of Cylinders subjected to Internal Pressure.

Thin Cylinders.—Consider a short cylindrical ring 1 inch

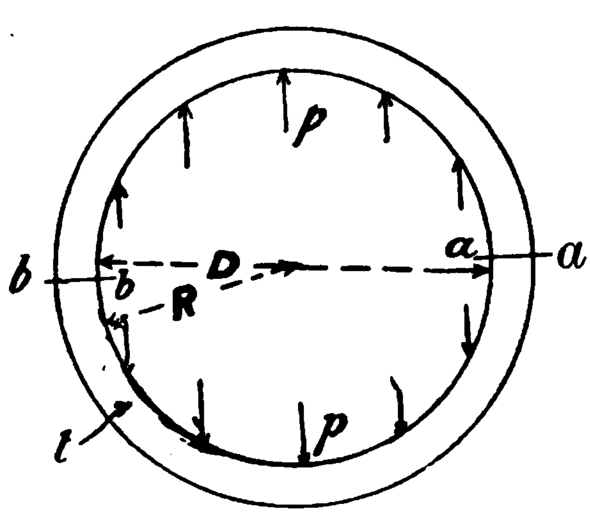


FIG. 332.

in length, subjected to an internal pressure of p lbs. square

inch. Then the total pressure tending to burst the cylinder and tear the plates at *aa* and *bb* is pD, where D is the internal diameter in inches. This bursting pressure has to be resisted by the stress in the ring of metal, which is $2f_t t$, where f_t is the tensile stress in the material.

$$\text{Then } pD = 2f_t t$$
$$\text{or } pR = f_t t$$

When the cylinder is riveted with a joint having an efficiency η, we have—

$$pR = f_t t \eta$$

In addition to the cylinder tending to burst by tearing the plates longitudinally, there is also a tendency to burst circumferentially. The total bursting pressure in this direction is $p\pi R^2$, and the resistance of the metal is $2\pi R t f_t$.

Where f_t is the tensile stress in the material—

$$\text{Then } p\pi R^2 = 2\pi R t f_t$$
$$\text{or } pR = 2tf_t$$

and when riveted—

$$pR = 2tf_t\eta.$$

Thus the stress on a circumferential section is one half as great as on a longitudinal section. On page 269 a method is given for combining these two stresses.

The above relations only hold when the plates are very thin; with thick-sided vessels the stress is greater than the value obtained by the thin cylinder formula.

Thick Cylinders.—There are various theories as to the distribution of the stress in the walls of thick cylinders. If we were dealing with an absolutely perfect material, the theory propounded by Lamé would be in every respect satisfactory; but as the materials we have to deal with are far from being perfect, the results obtained by his formula are not more in accordance with experimental [1] determinations of the distribution

[1] Mr. John Duncan, B.Sc., at the author's suggestion, has experimentally measured the strain on rings at different depths in a thick cylinder, and finds that the distribution of stress is very closely in accordance with Barlow's theory.

of stress than the results obtained by the incomplete but simpler theory of Barlow.

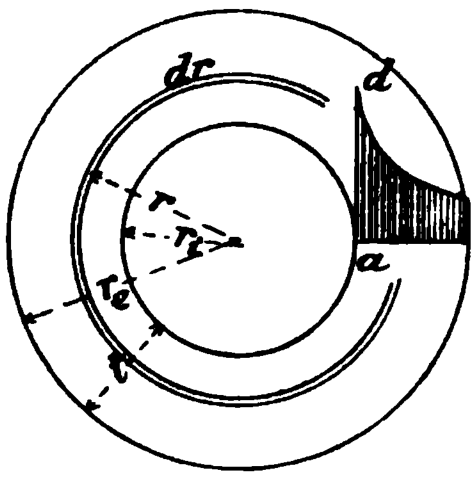


FIG. 333.

For a full discussion of the various theories of thick cylinders the reader is referred to Todhunter and Pearson's "History of Elasticity."

Barlow's Theory.—When the cylinder is exposed to an internal pressure (p), the radii will be increased, due to the stretching of the metal.

When under pressure—

Let r_i be strained to $r_i + n_i r_i = r_i(1 + n_i)$
r_e „ „ $r_e + n_e r_e = r_e(1 + n_e)$

where n is a small fraction indicating the elastic strain of the metal, which never exceeds $\frac{1}{1000}$ for safe working stresses (see p. 243).

The sectional area of the cylinder will be the same (to all intents and purposes) before and after the application of the pressure; hence—

$$\pi(r_e^2 - r_i^2) = \pi\{r_e^2(1 + n_e)^2 - r_i^2(1 + n_i)^2\}$$

which on reduction becomes—

$$r_i^2(n_i^2 + 2n_i) = r_e^2(n_e^2 + 2n_e)$$

n being a very small fraction, its square is still smaller and may be neglected, and the expression may be written—

$$\frac{r_i^2}{r_e^2} = \frac{n_e}{n_i}$$

The material being elastic, the stress f will be proportional to the strain n; hence we may write—

$$\frac{r_i^2}{r_e^2} = \frac{f_e}{f_i}, \text{ or } f_i r_i^2 = f_e r_e^2$$

that is, the stress on any thin ring varies inversely as the square of the radius of the ring.

Consider the stress f in any ring of radius r and thickness dr and of unit width.

$$\left.\begin{array}{l}\text{The total stress on any section of the}\\ \quad\text{elementary ring}\end{array}\right\} = f \, . \, dr$$

$$= \frac{f_i r_i^2}{r^2} dr$$

$$= f_i r_i^2 \, . \, r^{-2} dr$$

$$\left.\begin{array}{l}\text{The total stress on the whole section}\\ \quad\text{of one side of the cylinder}\end{array}\right\} = f_i r_i^2 \int_{r_1}^{r_0} r^{-2} dr$$

$$= f_i r_i^2 \left(\frac{r_e^{1-2} - r_i^{1-2}}{1 - 2} \right)$$

(Substituting the value of $f_i r_i^2$ from above)

$$= \frac{f_i r_i^2 r_e^{-1} - f_i r_i}{-1}$$

$$= \frac{f_e r_e - f_i r_i}{-1}$$

$$= f_i r_i - f_e r_e$$

This total stress on the section of the cylinder is due to the total pressure pr_i; hence—

$$pr_i = f_i r_i - f_e r_e$$

Substituting the value of f_e, we have—

$$pr_i = f_i r_i - \frac{f_i r_i^2 r_e}{r_e^2}$$

Dividing by r_i and reducing, we have—

$$pr_e = f_i r_e - f_i r_i$$
$$pr_e = f_i t$$

For a thin cylinder, we have—

$$pr_i = ft$$

Thus a thick cylinder may be dealt with by the same form of expression as a thin cylinder, taking the pressure to act on the external instead of on the internal radius.

The diagram (Fig. 333 *abcd*) shows the distribution of stress on the section of the cylinder, *ad* representing the stress at the interior, and *bc* at the exterior. The curve *dc* is fr^2 = constant.

Built-up Cylinders.—In order to equalize the stress over the section of a cylinder or a gun, various devices are adopted.

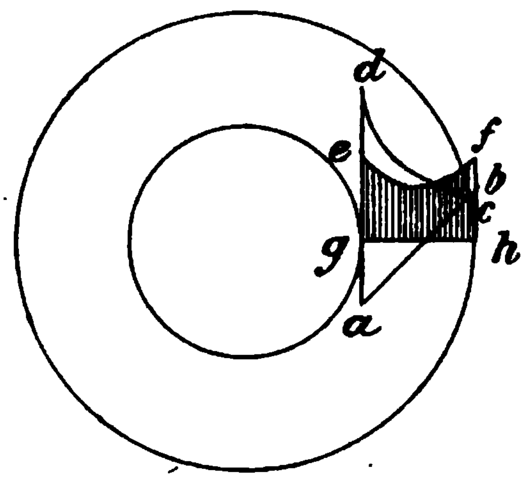


FIG. 334.

In the early days of high pressures, cast-iron guns were cast round chills, so that the metal at the interior was immediately cooled; then when the outside hot metal contracted, it brought the interior metal into compression. Thus the initial stress in a section of the gun was somewhat as shown by the line *ab*, *ag* being compression, and *bh* tension. Then, when subjected to pressure, the curve of stress would have been *dc* as before, but when combined with *ab* the resulting stress on the section is represented by *ef*, thus showing a much more even distribution of stress than before.

This equalizing process is effected in modern guns by either shrinking rings on one another in such a manner that the internal rings are initially in compression and the external rings in tension, or by winding wire round an internal tube to produce the same effect. The exact tension on the wire required to produce the desired effect is regulated by drawing the wire through friction dies mounted on a pivoted arm—in effect, a friction brake.

Strength and Coefficients of Elasticity of Materials in Tons square inch.

Material.	Elastic limit.			Breaking strength.			E. Tension or compression.	G. Shear.	Extension per cent. on 10″.	Reduction in area.
	Tension.	Compression.	Shear.	Tension.	Compression.	Shear.				
Wrought-iron *bars*	12–15	—	10–12	21–24	—	17·19	11,000–13,000	5,000 6,000	10–30	15–40
Plates with grain	13–15	—	—	20–22	—	—	—	—	5–10	7–12
,, across ,,	11–13	—	—	18–20	—	—	—	—	2–6	3–7
Best Yorkshire, with grain ...	13–14	12–14	—	20–23	—	17–19	—	—	15–30	20–50
Best Yorkshire, across grain	13–14	—	—	19–20	—	—	—	—	10–12	12–20
Steel, 0·1 % C. ...	13–14	—	10–11	21–22	—	16–17	13,000 14,000	5,000 6,000	27–30	45–50
,, 0·2 % C.	17–18	16–17	13–14	30–32	—	24–26	,,	,,	20–23	27–32
,, 0·5 % C. ...	20–21	—	16–17	34–35	—	28–29	,,	,,	14–17	17–20
,, 1·0 % C. ...	28–29	22–24	21–23	50–55	—	42–47	14,000	,,	4–5	7–8
Rivet steel ...	15–17	—	12–14	26–28	—	21–22	—	—	20–35	30–50
Steel castings ...	10–11	(not annealed)		20–25	—	—	12,000 to 12,500	—	5–12	6 13
,, ,, ...	15–17	(annealed)		30–40	—	—		—	10–20	15–35
Tool steel ...	35–45	40–50	—	40–70	(unhardened)		14,000 to 15,000	—	1–5	1–5
,, ,, ...	60–80	—	—	60–80	(hardened)			—	*nil*	*nil*
Cast iron	no marked limit			7–11	35·60	8–13	6,000 to 10,000	2,500 to 4,000	practically *nil*	
Copper	2–4	10·13 annealed	—	12–15	20–25	11–12	7,000 to 7,500	—	35–40	50–60
,, ...	10–12	hard drawn		16–20				—	2–5	40–55
Gun-metal ...	3–4	5–6	—	9–16	30–50	8–12	5,000–5,500	—	8–15	10–18
Brass	2–4	—	—	7–10	5–6	—	4,000	2,000	10–12	12–15
Delta, bull metal, etc.—										
Cast	5–8	12–14	—	14–20	60–70	—	5,500 to 6,000	—	8–16	10–22
Rolled ...	15–25	16–22	—	27–34	45–60	—		—	17–34	27–50
Phosphor bronze	7–9	—	—	24–26	—	—	6,000	—	—	—
Muntz metal ...	20–25	(rolled)	—	25–30	—	—	—	—	10–20	30–40
Aluminium ...	2–7	8–10	—	7–10	—	—	2,000 6,000	—	4–15	30–70
Oak	—	—	—	4–6	2–5	with grain 0·2	500-700	—	—	—
Soft woods ...	—	—	—	1–3	1–3	0·07 to 0·1	450-500	—	—	—

CHAPTER IX.

BEAMS

General Theory.—The **T** lever shown in the sketch is hinged at the centre on a pivot or knife-edge, around which the

Fig. 335.

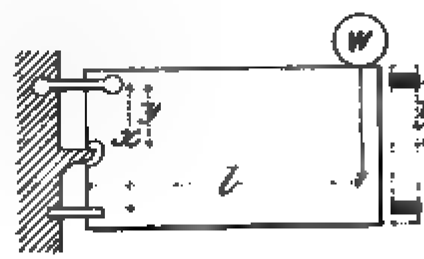


Fig. 336.

lever can turn. The bracket supporting the pivot simply takes the shear. For the lever to be in equilibrium, the two couples acting on it must be equal and opposite, viz. $Wl = px$.

Replace the **T** lever by the model shown in Fig. 336. It is attached to the abutment by two pieces of any convenient material, say indiarubber. The upper one is dovetailed, because it is in tension, and the lower is plain, because it is in compression. Let the sectional area of each block be a; then, as before,

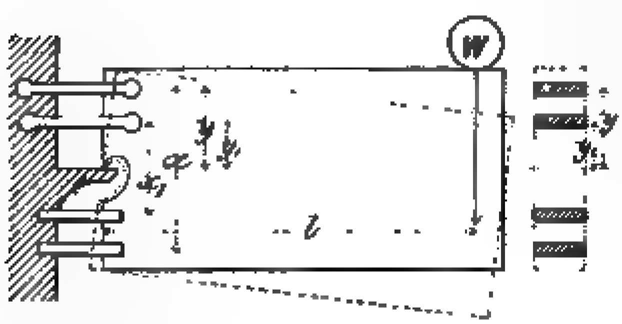


Fig. 337.

we have $Wl = px$. But $p = fa$, where $f =$ the stress in either block in either tension or compression;

$$\text{hence } Wl = fax$$
$$\text{or } = 2fay$$

Or—

$$\left.\begin{array}{l}\text{The moment of}\\ \quad\text{the external}\\ \quad\text{forces}\end{array}\right\} = \left\{\begin{array}{l}\text{the moment of the internal forces, or the}\\ \quad\text{internal moment of resistance of the}\\ \quad\text{beam}\end{array}\right.$$

$$= \text{stress on the area } a \times (\text{moment of the two areas } (a) \text{ about the pivot})$$

Hence the resistance of any section to bending—apart altogether from the strength of the material of the beam—varies directly as the area a and as the distance x, or as the moment ax. Hence the quantity in brackets is termed the "measure of the strength of the section," or the "modulus of the section," and is usually denoted by the letter Z. Hence we have—

$$Wl = M = fZ, \text{ or}$$

$$\left.\begin{array}{l}\text{The bending moment}\\ \quad\text{at any section}\end{array}\right\} = \left\{\begin{array}{c}\text{stress on the}\\ \text{material}\end{array}\right\} \times \left\{\begin{array}{c}\text{modulus of the}\\ \text{section}\end{array}\right.$$

Now take a fresh model with four blocks instead of two. When loaded, the outer end will droop down as shown by the dotted lines, pivoting about the point resting on the bracket. Then the outer blocks will be stretched and compressed or strained more than the inner blocks in the ratio $\frac{x}{x_1}$ or $\frac{y}{y_1}$; *i.e.* the strain is directly proportional to the distance from the point of the pivot.

The enlarged figure shows this more distinctly, where e, e_1 shows the extensions, and c, c_1 show the compressions at the distances y, y_1 from the pivot. From the similar triangles, we have $\frac{e}{e_1} = \frac{y}{y_1}$; also $\frac{c}{c_1} = \frac{y}{y_1}$. But we have previously seen (p. 244) that when a piece of material is strained (*i.e.* stretched or compressed), the stress varies directly as the strain, *provided the elastic limit has not been passed*. Hence, since the strain varies directly as the distance from the pivot, the stress must also vary in the same manner.

Let f = stress in outer blocks, and f_1 = stress in inner blocks; then—

Fig. 338.

$$\frac{f}{f_1} = \frac{y}{y_1} \text{ or } f_1 = \frac{fy_1}{y}$$

Then, taking moments about the pivot as before, we have—

$$Wl = 2fay + 2f_1ay_1$$

Substituting the value of f_1, we have—

$$Wl = 2fay + \frac{2fay_1^2}{y}$$

$$\text{If } y_1 = \frac{y}{2},\ Wl = 2fay + \frac{2fay^2}{4y}$$

$$Wl = 2fay(1 + \tfrac{1}{4})$$

Thus the addition of two inner blocks at one-half the distance of the outer blocks from the pivot has only increased the strength of the beam by $\frac{1}{4}$, or, in other words, the four-block model will only support $1\frac{1}{4}$ times the load (W) that the two-block model will support.

If we had a model with a very large number of blocks, or a beam section supposed to be made up of a large number of layers of area a, a_1, a_2, a_3, etc., and situated at distances y, y_1, y_2, y_3, etc., respectively from the pivot, which we shall now term the *neutral axis*, we should have, as above—

$$Wl = 2fay + \frac{2fa_1y_1^2}{y} + \frac{2fa_2y_2^2}{y} + \frac{2fa_3y_3^2}{y} +, \text{ etc.}$$

$$= \frac{f}{y}2(ay^2 + a_1y_1^2 + a_2y_2^2 + a_3y_3^2 +, \text{ etc.})$$

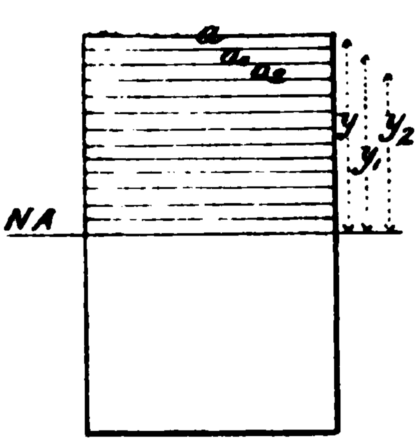


FIG. 339.

The quantity in brackets, viz. each little area (a) multiplied by the square of its distance (y^2) from a given line (N.A.), is termed the second moment or moment of inertia of the upper portion of the beam section; and as the two half-sections are similar, twice the quantity in brackets is the moment of inertia (I) of the whole section of the beam. Thus we have—

$$Wl = \frac{fI}{y}, \text{ or}$$

The bending moment at any section

$$= \frac{\left.\begin{matrix}\text{the stress on the}\\ \text{outermost layer}\end{matrix}\right\} \times \left\{\begin{matrix}\text{second moment, or moment of}\\ \text{inertia of the section}\end{matrix}\right.}{\text{distance of the outermost layer from the neutral axis}}$$

But we have shown above that the stress varies directly as the

distance from the neutral axis; hence the stress on the outermost layer is the maximum stress on any part of the beam section, and we may say—

The bending moment at any section

$$= \frac{\left\{\begin{matrix}\text{the maximum stress}\\ \text{on the section}\end{matrix}\right\} \times \left\{\begin{matrix}\text{second moment, or moment of}\\ \text{inertia of the section}\end{matrix}\right\}}{\text{distance of the outermost layer from the neutral axis}}$$

But we have seen above that $Wl = fZ$, and here we have $Wl = f\frac{I}{y}$;

$$\text{therefore } Z = \frac{I}{y}$$

The quantity $\frac{I}{y}$ is termed the "measure of the strength of the section," or, more briefly, the "modulus of the section;" it is usually designated by the letter Z. Thus we get $Wl = fZ$, or $M = fZ$, or—

$$\left.\begin{matrix}\text{The bending}\\ \text{moment at}\\ \text{any section}\end{matrix}\right\} = \left\{\begin{matrix}\text{the maximum or skin}\\ \text{stress on the section}\end{matrix}\right\} \times \left\{\begin{matrix}\text{the modulus of}\\ \text{the section}\end{matrix}\right.$$

Assumptions of Beam Theory.—To go into the question of all the assumptions made in the beam theory would occupy far too much space. We will briefly consider the most important of them.

First Assumption.—That originally plane sections of a beam remain plane after bending; that is to say, we assume that a solid beam acts in a similar way to our beam model in Fig. 337, in that the strain does increase directly as the distance from the neutral axis. Very delicate experiments by the author show that this assumption is true to within exceedingly narrow limits, *provided the elastic limit of the material is not passed.*

Second Assumption.—That the stress in any layer of a beam varies directly as the distance of that layer from the neutral axis. That the strain does vary in this way we have just seen. Hence the assumption really amounts to assuming that the stress is proportional to the strain. Reference to the elastic curves in Chap. VIII. will show that the elastic line is straight, *i.e.* that the stress does vary as the strain. In most cast materials the line is unquestionably slightly curved, but for low (working) stresses the line is sensibly straight. Hence for

working conditions of beams we are justified in our assumption. After the elastic limit has been passed, this relation entirely ceases. Hence *the beam theory ceases to hold good as soon as the elastic limit has been passed.*

Third Assumption.—That the modulus of elasticity in tension is equal to the modulus of elasticity in compression. Suppose, in the beam model, we had used soft rubber in the tension blocks and hard rubber in the compression blocks, *i.e.* that the modulus of elasticity of the tension blocks was less than the modulus of elasticity of the compression blocks; then the stretch on the upper blocks would be greater than the compression on the lower blocks, with the result that the beam would tend to turn about some point other than the pivot, thus: and the relations given above entirely cease to hold, for the strain and the stress will not vary directly as the distance from the pivot or neutral axis, but directly as the distance from a, which later on we shall see how to calculate.

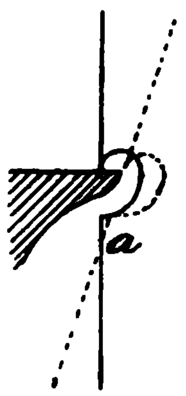


FIG. 340.

For most materials there is no serious error in making this assumption; but in some materials the error is appreciable, but still not sufficient to be of any practical importance.

Neither of the above assumptions are *perfectly* true; but they are so near the truth that for all practical purposes they may be considered to be perfectly true, but *only as long as the elastic limit of the material is not passed.* In other parts of this book (see Appendix), experimental proof will be given of the accuracy of the beam theory.

Graphical Method of finding the Modulus of the Section $(Z = \frac{I}{y})$.—The modulus of the section of a beam might be found by splitting the section up into a great many layers and multiplying the area of each by $\frac{y_0^2}{y}$, as shown above. The process, however, would be very tedious.

But in the graphic method, instead of dealing with each strip separately, we graphically find the *magnitude* of the resultant of all the forces, viz.—

$$fa + f_1a_1 + f_2a_2 +, \text{etc.}$$

acting on each side of the neutral axis, also the *position* or distance apart of these resultants. The product of the two

gives us the moment of the forces on each side of the neutral axis, and the sum of the two moments gives us the total amount of resistance for the beam section, viz. fZ.

Imagine a beam section divided up into a great number of thin layers parallel to the neutral axis, and the stress in each layer varying directly as its distance from it. Then if we construct a figure in which the width, and consequently the area, of each layer is reduced in the ratio of the stress in that layer to the stress in the outermost layer, we shall have the intensity of stress the same in each. Thus, if the original area of the layer be a_1, the reduced area of the layer will be—

$$a_1\left(\frac{f_1}{f}\right) = \text{say } a_1'$$

whence we have $f_1a_1 = fa_1'$

Then the sum of the forces acting over the half section, viz.—

$$fa + f_1a_1 + f_2a_2 +, \text{etc.}$$
$$\text{becomes } fa + fa_1' + fa_2' +, \text{etc.}$$
$$\text{or } f(a + a_1' + a_2' +, \text{etc.}$$

or f(area of the figure on one side of the neutral axis)
or (the whole force acting on one side of the neutral axis)

Then, since the intensity of stress all over the *figure* is the same, the *position* of the resultant will be at the centre of gravity of the figure.

Let A_1 = the area of the figure below the neutral axis;
A_2 = " " above " "
d_1 = the distance of the centre of gravity of the lower figure from the neutral axis;
d_2 = the distance of the centre of gravity of the upper figure from the neutral axis.

Then the moment of all the forces acting on one side of the neutral axis $\Big\} = fA_1 \times d_1$

Then the moment of all the forces acting on *both* sides of the neutral axis $\Big\} = f(A_1d_1 + A_2d_2)$

$= fZ$ (see p. 291)

or $Z = A_1d_1 + A_2d_2$

We shall shortly show that $A_1 = A_2 = A$ (see next paragraph).

$$\text{Then } Z = A(d_1 + d_2)$$
$$Z = AD$$

where D is the distance between the two centres of gravity. In a section which is symmetrical about the neutral axis $d_1 = d_2 = d$, and d $+ d_2 = 2d$ for symmetrical sections.

$$\text{Then } Z = 2Ad$$
$$\text{or } Z = AD$$

The units in which Z is expressed are as follows—

$$Z = \frac{I}{y} = \frac{(\text{length units})^4}{\text{length}} = (\text{length units})^3$$

$$\begin{aligned} \text{or } Z = AD &= \text{area} \times \text{distance} \\ &= (\text{length units})^2 \times \text{length units} \\ &= (\text{length units})^3 \end{aligned}$$

Hence, if a modulus figure be drawn, say, $\frac{1}{n}$ full size, the result obtained must be multiplied by n^3 to get the true value. For example, if a beam section were drawn to a scale of 3 inches = 1 foot, *i.e.* $\frac{1}{4}$ full size, the Z obtained on that scale must be multiplied by $4^3 = 64$.

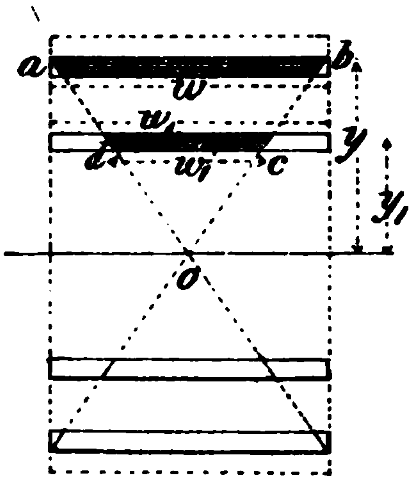


FIG. 341.

We showed above that in order to construct this figure, which we will term a "modulus figure," the width of each strip of the section had to be reduced in the ratio $\frac{f_1}{f}$, which we have previously seen is equal to $\frac{y_1}{y}$. This reduction is easily done thus: Let Fig. 341 represent a section through the indiarubber blocks of the beam model. Join *ao*, *bo* cutting the inner block in *c* and *d*. Then by similar triangles—

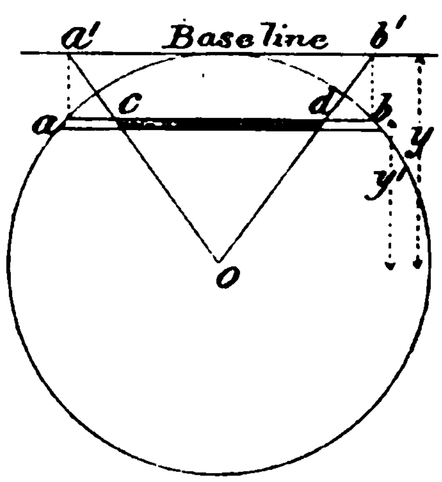


FIG. 342.

$$\frac{cd}{ab} = \frac{y_1}{y} = \frac{f_1}{f}, \text{ or } w_1' = w_1\left(\frac{f_1}{f}\right)$$

In the case of a section in which the strips are not all of the same width, the same construction holds. Project the strip *ab* on to the base-line as shown, viz. *a'b'*. Join *a'o*, *b'o*, cutting the strip in *c* and *d*. By similar triangles we have—

$$\frac{cd}{a'b'} \text{ or } \frac{cd}{ab} = \frac{y_1}{y} = \frac{f_1}{f}, \text{ or } cd = ab\left(\frac{f_1}{f}\right)$$

Several fully worked out sections will be given later on.

By way of illustration, we will work out the strength of the four-block beam model by this method, and see how it agrees with the expression found above on p. 292.

The area A of the modulus figure on one side of the neutral axis $\Big\} = a + a_1'$

The distance d of the c. of g. of the modulus figure from the neutral axis (see p. 58) $\Big\} = \dfrac{ay + a_1'y_1}{a + a_1'}$

But $Wl = fZ = f \times 2Ad$

$$\text{or } Wl = 2f\left(a + a_1' \times \frac{ay + a_1'y_1}{a + a_1}\right) = 2fay + 2fa_1'y_1$$

$$\text{But } a_1' = a\frac{y_1}{y}, \therefore Wl = 2fay + \frac{2fay_1^2}{y}$$

which is the same expression as we had on p. 292.

The graphic method of finding Z should only be used when a convenient mathematical expression cannot be obtained.

Position of Neutral Axis.—We have stated above that the neutral axis in a beam section corresponds with the pivot in the beam model; on the one side of the neutral axis the material is in tension, and on the other side in compression, and at the neutral axis, where the stress changes from tension to compression, there is, of course, no stress (except shear, which we will treat later on). In all calculations, whether graphic or otherwise, the first thing to be determined is the position of the neutral axis with regard to the section.

We have already stated on p. 58 that, if a point be so chosen in a body that the sum of the moments of all the gravitational forces acting on the several particles about the one side of any straight line passing through that point, be equal to the sum of the moments on the other side of the line, that point is termed the centre of gravity of the body; or, if the moments on the one side of the line be termed positive (+), and the moments on the other side of the line be termed negative (−), the sum of the moments will be zero. We are about to show that precisely the same definition may be used for stating the position of the neutral axis; or, in other words, we are about to prove that, accepting the assumptions given above, the neutral axis *invariably* passes through the centre of gravity of the section of a beam.

Let the given section be divided up into a large number of strips as shown—

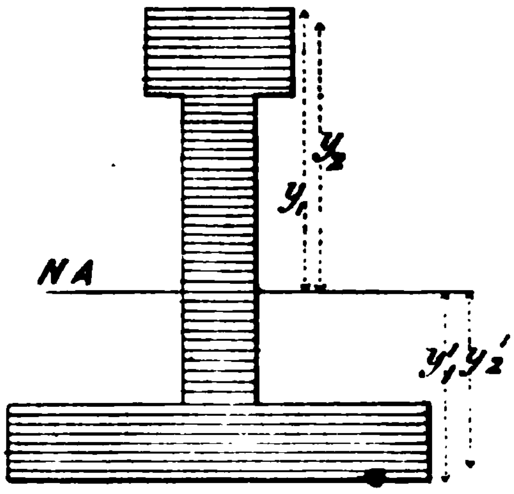


FIG. 343.

Let the areas of the strips above the neutral axis be a_1, a_2, a_3, etc.; ditto below the neutral axis be a_1', a_2', a_3', etc.; and their respective distances from the neutral axis above y_1, y_2, y_3, etc.; ditto below y_1', y_2', y_3', etc.: and the stresses in the several layers above the neutral axis be f_1, f_2, f_3, etc.; ditto below the neutral axis be f_1', f_2', f_3'.

Then, as the stress in each layer varies directly as its distance from the neutral axis, we have—

$$\frac{f_1}{f_2} = \frac{y_1}{y_2}, \text{ or } \frac{f_1}{y_1} = \frac{f_2}{y_2}$$

$$\text{also } \frac{f_1}{y_1} = \frac{f_3}{y_3}, \text{ or } \frac{f_1}{y_1} = \frac{f_2}{y_2} = \frac{f_3}{y_3}, \text{ etc.} = \text{say K (a constant)}$$

then $f = \text{K}y$

The total stress in any layer $= fa = \text{K}ay$

$$\left.\begin{array}{l}\text{The total stress in all the}\\ \quad\text{layers on one side of the}\\ \quad\text{neutral axis}\end{array}\right\} = \text{K}(ay + a_1y_1 + a_2y_2 +, \text{ etc.})$$

$$\left.\begin{array}{l}\text{The total stress in all the}\\ \quad\text{layers on the other side}\\ \quad\text{of the neutral axis}\end{array}\right\} = \text{K}(a'y' + a'_1y'_1 + a'_2y'_2 +, \text{ etc.})$$

But as the tensions and compressions form a couple, the total amount of tension on the one side of the neutral axis must be equal to the total amount of compression on the other side; hence—

$$\text{K}(ay + a_1y_1 + a_2y_2 +, \text{etc.}) = \text{K}(a'y' + a_1'y_1' + a_2'y_2' +, \text{etc.})$$
$$\text{or } ay + a_1y_1 + a_2y_2 +, \text{etc.} = a'y' + a_1'y_1' + a_2'y_2' +, \text{etc.}$$

or, expressed in words, the sum of the moments of all the elemental areas on the one side of the neutral axis are equal to the sum of the moments on the other side of the neutral axis; but, referring to the statement above, it will be seen that this is precisely the definition of a line which passes through the centre of gravity of the section. Hence, *the neutral axis passes through the centre of gravity of the section.*

It should be noticed that not one word has been said in the above proof about the *material* of which the beam is made; all

that is taken for granted in the above proof is that the modulus of elasticity in tension is equal to the modulus of elasticity in compression. The position of the neutral axis has *nothing whatever to do with the relative strengths of the material in tension and compression*, as is so frequently stated in text-books, and by correspondents in engineering journals.

Unsymmetrical Sections.—In a symmetrical section, the centre of gravity is equidistant from the skin in tension and compression; hence the maximum stress on the material in tension is equal to the maximum stress in compression. Now, some materials, notably cast iron, are from five to six times as strong in compression as in tension; hence, if we use a symmetrical section in cast iron, the material fails on the tension side at from $\frac{1}{5}$ to $\frac{1}{6}$ the load that would be required to make it fail in compression. In order to make the beam equally strong in tension and compression, we make the section of cast-iron beams of such a *form* that the neutral axis is about five or six times[1] as far from the compression flange as from the tension flange, so that the stress in compression shall be five or six times as great as the stress in tension. It should be particularly noted that the reason why the neutral axis is nearer the one flange than the other is entirely due to the *form* of the section, and not to the material; the neutral axis would be in precisely the same place if the material were of wrought iron, or lead, or stone, or timber (provided assumption 3, p. 294 is true). We have shown above on p. 292 that $M = f\frac{I}{y}$, and that $Z = \frac{I}{y}$, where y is the distance of the skin from the neutral axis. In a symmetrical section there is no difficulty in finding the value of y—it is simply *the half depth of the section*; but in the unsymmetrical section y may have two values: the distance of the tension skin from the neutral axis, or the distance of the compression skin from the neutral axis. Which, then, are we to choose? If the maximum *tensile* stress f_t is required, the y_t must be taken as the distance of the tension skin from the neutral axis; and likewise when the maximum compressive stress f_c is required, the y_c must be measured from the *compression skin*. Thus we have either—

$$M = f_t\frac{I}{y_t} \text{ or } f_c\frac{I}{y_c}$$

[1] We shall show later on that such a great difference as 5 or 6 is undesirable for practical reasons.

and as $\frac{f_t}{y_t} = \frac{f_c}{y_c}$, we get precisely the same value for the bending moment whichever we take. We also have two values of Z, viz. $\frac{I}{y_t} = Z_t$ and $\frac{I}{y_c} = Z_c$;

$$\text{and } M = f_tZ_t \text{ or } f_cZ_c$$

We shall invariably take f_tZ_t when dealing with cast-iron sections, mainly because such sections are always designed in such a manner that they fail in tension.

The construction of the modulus figures for such sections is a simple matter.

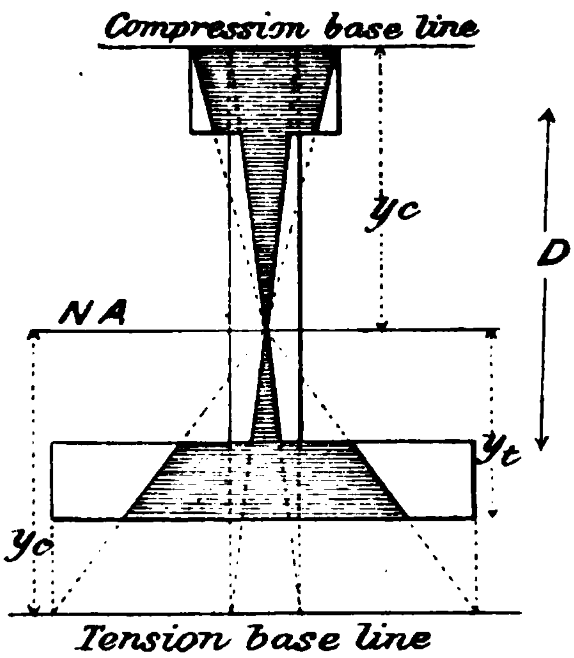


FIG. 344.

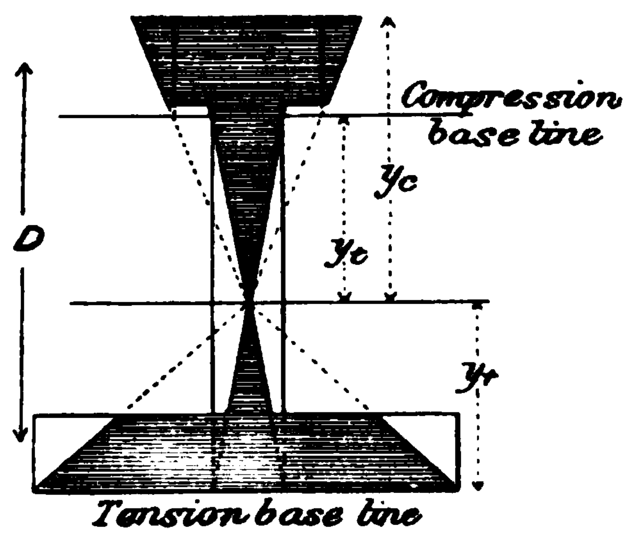


FIG. 345.

Construction for Z_c (Fig. 344).—Find the centre of gravity of the section, and through it draw the neutral axis parallel with the flanges. Draw a compression base-line touching the outside of the compression flange; set off the tension base-line parallel to the neutral axis, at the same distance from it as the compression base-line, viz. y_c; project the parts of the section down to each base-line, and join up to the central point which gives the shaded figure as shown. Find the centre of gravity of each figure (cut out in cardboard and balance). Let D = distance between them; then Z_c = shaded area above *or* below the neutral axis $\times$ D.

Construction for Z_t (Fig. 345).—Proceed as above, only the tension base-line is made to touch the outside of the tension flange, and the compression base-line cuts the figure; the parts of the section above the compression base-line have been projected down on to it, and the modulus figure beyond it passes

outside the section; at the base-lines the figure is of the same width as the section. The centre of gravity of the two figures is found as before, also the Z.

The reason for setting the base-lines in this manner will be evident when it is remembered that the stress varies directly as the distance from the neutral axis; hence, the stress on the tension flange f_t is to the stress on the compression flange f_c as y_t is to y_c.

N.B.—The tension base-line touches the tension flange when the figure is being drawn for the tension modulus figure Z_t, and likewise for the compression.

As the tensions and compressions form a couple, the total amount of tension is equal to the total amount of compression, therefore the area of the figure above the neutral axis must be equal to the area of the figure below the neutral axis, whether the section be symmetrical or otherwise; but the moment of the tension is not equal to the moment of the compression about the neutral axis in unsymmetrical sections. The accuracy of the drawing of modulus figures should be tested by measuring both areas; if they only differ slightly (say not more than 5 per cent.), the mean of the two may be taken; but if the error be greater than this, the figure should be drawn again.

If, in any given instance, the Z_c has been found, and the Z_t is required or *vice versâ*, there is no need to construct the two figures, for—

$$Z_t = Z_c \times \frac{y_c}{y_t}, \text{ or } Z_c = Z_t \times \frac{y_t}{y_c}$$

hence the one can always be obtained from the other.

Most Economical Sections for Cast-iron Beams.—Experiments by Hodgkinson and others show that it is undesirable to adopt so great a difference as 5 or 6 to 1 between the compressive and tensile stresses. This is mainly due to the fact that, if sections were made with such a great difference, the tension flange would have to be very thick or very wide compared with the compression flange; if a very thick flange were used as the casting cooled, the thin compression flange and web would cool first, and the thick flange afterwards, and set up serious initial cooling stresses in the metal.

The author, when testing large cast-iron girders with very unequal flanges, has seen them break with their own weight before any external load was applied, due to this cause. Very wide flanges are undesirable, because they bend transversely when loaded, as in Fig. 346.

Experiments seem to show that the most economical section for cast iron is obtained when the proportions are roughly those given in the figures in Fig. 346.

"Massing up" Beam Sections.—Thin hollow beam

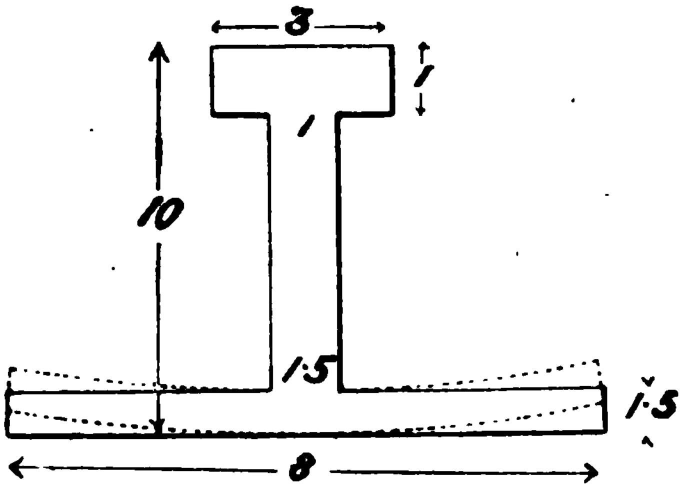


Fig. 346.

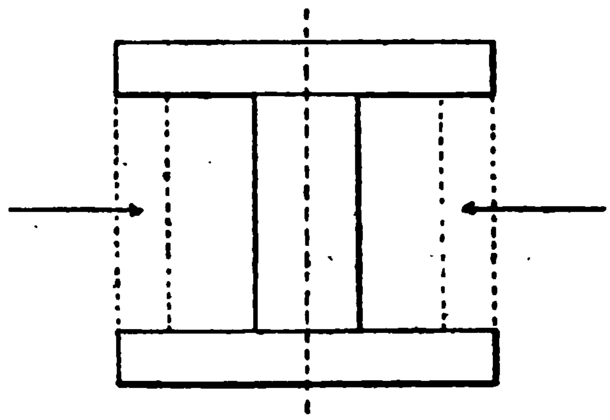
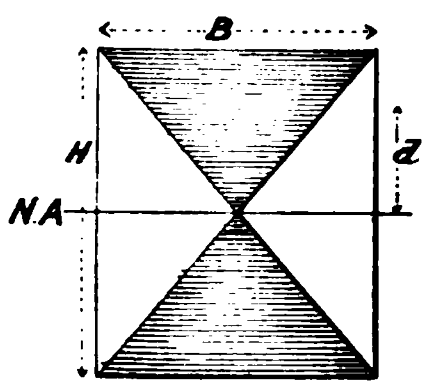


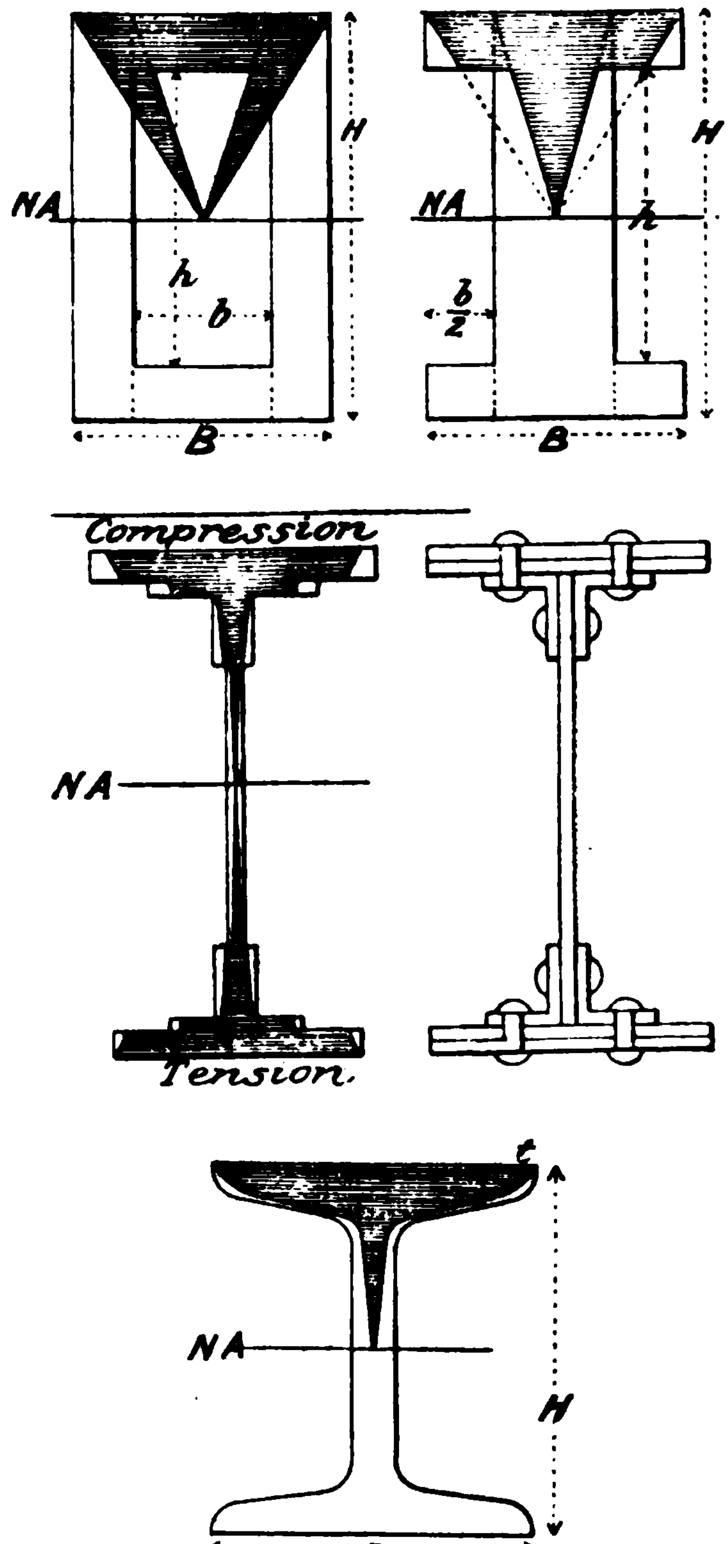
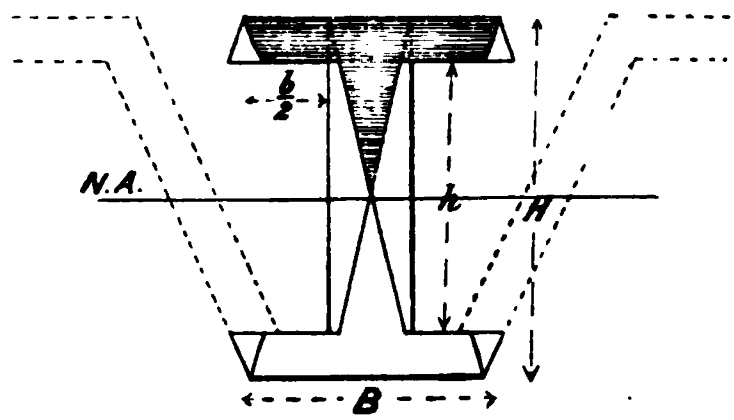
Fig. 347.

sections are usually more convenient to deal with graphically if they are "massed up" about a centre line to form an equivalent solid section. "Massing up" consists of sliding in the sides of the section parallel to the neutral axis until they meet as shown in Fig. 347.

The dotted lines show the original position of the sides, and the full lines the sides after sliding in. The "massing up" process in no way affects the Z, as the distance of each section from the neutral axis remains unaltered; it is done merely for convenience in drawing the modulus figure. In the table of sections several instances are given.

Section.	Examples of Modulus Figures.
Rectangular.	 Fig. 348.
Square.	B = H = S (the side of the square)

Modulus of the section Z.	REMARKS.
$\frac{BH^2}{6}$	The moment of inertia (see p. 80) $= \frac{BH^3}{12}$ $y = \frac{H}{2}$ $Z = \frac{\frac{BH^3}{12}}{\frac{H}{2}} = \frac{BH^2}{6}$ Also by graphic method—
(Square) $\frac{S^3}{6}$	The area $A = \frac{BH}{4}$ $d = \frac{2}{3} \times \frac{H}{2} = \frac{H}{3}$ $Z = 2Ad = 2 \times \frac{BH}{4} \times \frac{H}{3} = \frac{BH^2}{6}$

Section.	Examples of Modulus Figures.
Hollow rectangles and girder sections.	 FIG. 349.
One corrugation of a trough flooring.	 FIG. 350.

Modulus of the section Z.

$\dfrac{BH^3 - bh^3}{6H}$

Moment of inertia for outer rectangle $= \dfrac{BH^3}{12}$

" " inner " $= \dfrac{bh^3}{12}$

" " hollow " $= \dfrac{BH^3 - bh^3}{12}$

$$y = \frac{H}{2}$$

$$Z = \frac{\dfrac{BH^3 - bh^3}{12}}{\dfrac{H}{2}} = \frac{BH^3 - bh^3}{6H}$$

Approximately, when the web is thin, as in a rolled joist—

BtH

where t is the mean thickness of the flange.

This might have been obtained direct from the Z for the solid section, thus:

Z for outer section $= \dfrac{BH^2}{6}$

Z " inner " $= \dfrac{bh^2}{6} \times \dfrac{h}{H} = \dfrac{bh^3}{6H}$

Z " hollow " $= \dfrac{BH^2}{6} - \dfrac{bh^3}{6H} = \dfrac{BH^3 - bh^3}{6H}$

The Z for the inner section was multiplied by the ratio $\dfrac{h}{H}$, because the stress on the interior of the flange is less than the stress on the exterior in the ratio of their distances from the neutral axis.

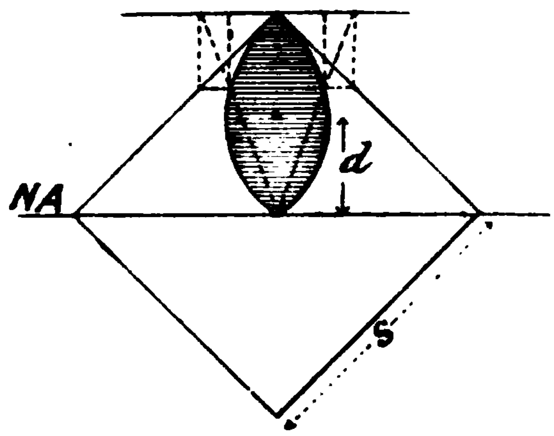
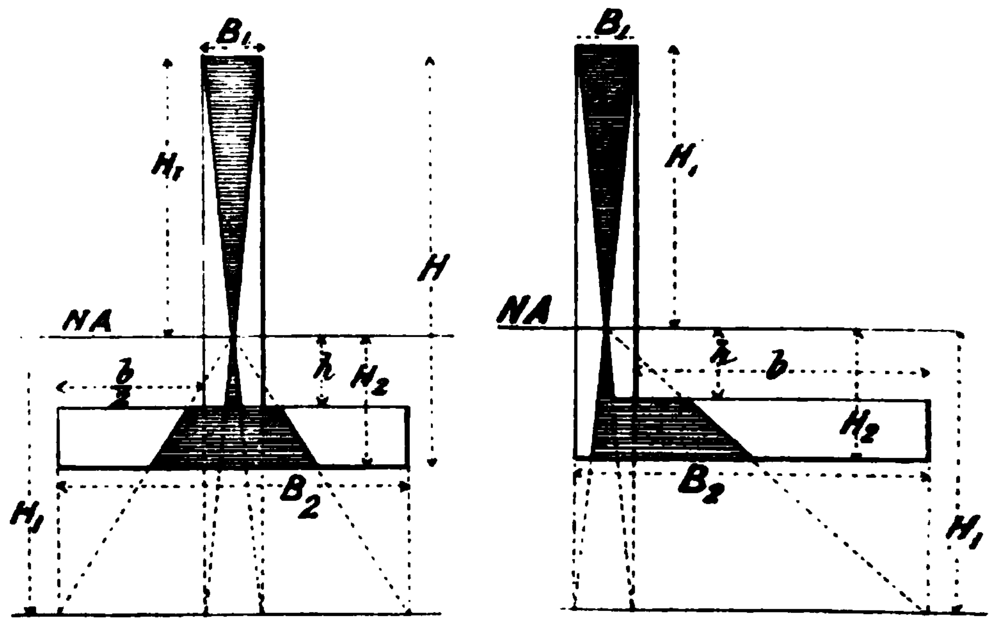
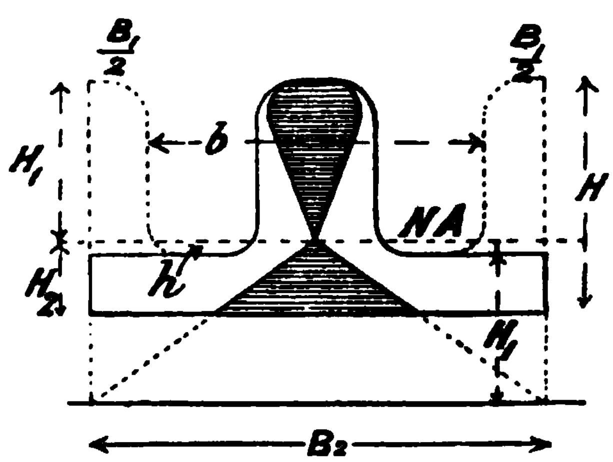
The approximate methods neglect the strength of the web, and assume the stress evenly distributed over the two flanges.

For rolled joists BtH is rather nearer the truth than BtH_0, where H_0 is measured to the middle of the flanges, and is more readily obtained. For almost all practical purposes the approximate method is sufficiently accurate.

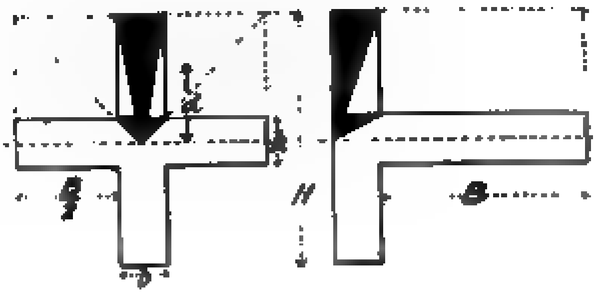
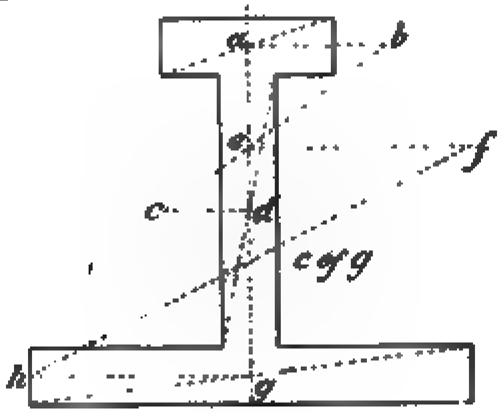
N.B.—The safe loads given in makers' lists for their rolled joists are usually too high. The author has tested some hundreds in the testing-machine on both long and short spans, and has rarely found that the strength was more than 75 per cent. of that stated in the list.

In corrugated floorings or built-up sections, if there are rivets in the tension flanges, the area of the rivet-holes should be deducted from the Bt. Thus, if there are n rivets of diameter d in any one cross-section, the Z will be—

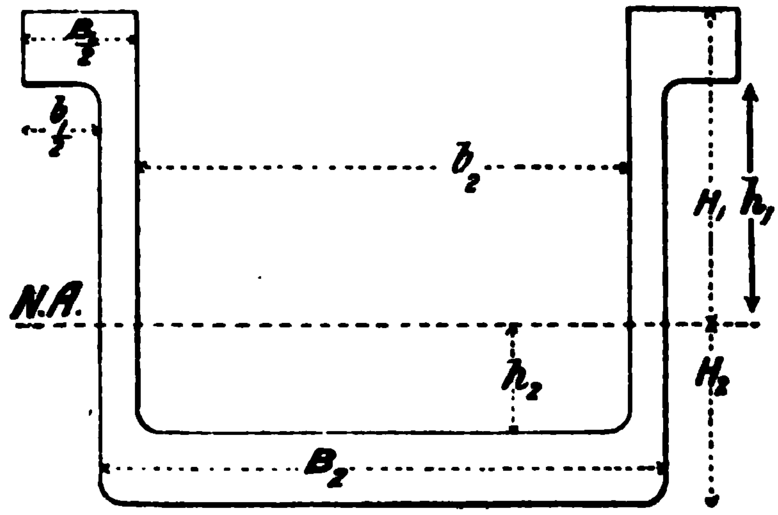
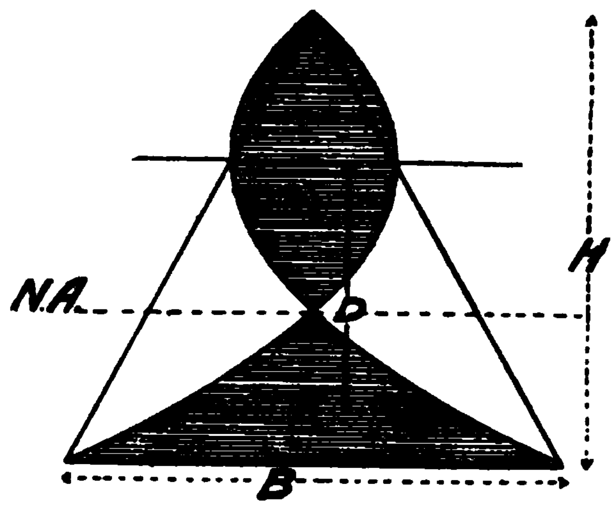
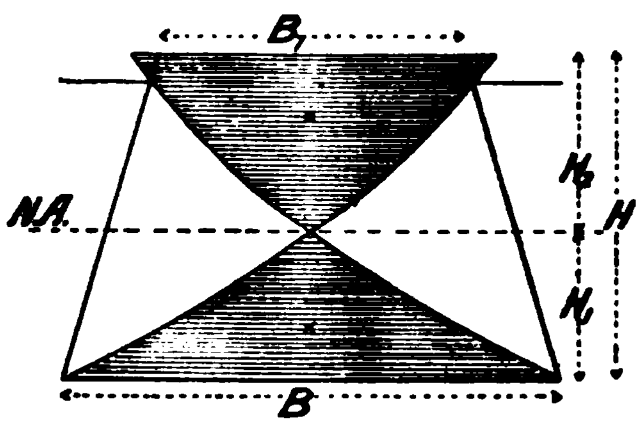
$$(B - nd)tH \text{ (approx.)}$$

Section.	Examples of Modulus Figures.
Square on edge.	 FIG. 351.
T's and angles and ⊔ sections.	 FIG. 352.  This figure becomes a T or L when massed up about a vertical line. FIG. 353.

Modulus of the section Z.	
$0{\cdot}118S^3$	The moment of inertia (see p. 88) $= \dfrac{S^4}{12}$ $y = \dfrac{S}{\sqrt{2}}$ $Z = \dfrac{\frac{S^4}{12}}{\frac{S}{\sqrt{2}}} = \dfrac{\sqrt{2}S^3}{12} = 0{\cdot}118S^3$
$\dfrac{B_1H_1^3+B_2H_2^3-bh^3}{3H_1}$ $H_1 = 0{\cdot}7H$ approx. $H_2 = 0{\cdot}3H$,,	The moment of inertia for the part above the N.A. $= \dfrac{B_1H_1^3}{3}$ Ditto below $= \dfrac{B_2H_2^3 - bh^3}{3}$ Ditto for whole section $= \dfrac{B_1H_1^3 + B_2H_2^3 - bh^3}{3}$ Z for stress at top $= \dfrac{B_1H_1^3 + B_2H_2^3 - bh^3}{3H_1}$ If the position of the centre of gravity be calculated for the form of section usually used, it will be found to be approximately 0·3H from the bottom. Rolled sections, of course, have not square corners as shown in the above sketches, but the error involved is not material if a mean thickness be taken.

Section.	Examples of Modulus Figures.
⊤ section on edge and cruciform sections.	 Fig. 354.
Unequal flanged sections.	 Fig. 355. Fig. 356.

Modulus of the section Z.	
$\dfrac{bH^3 + Bh^3}{6H}$ Approx. Z $\dfrac{bH^2}{6}$	Moment of inertia for vertical part $= \dfrac{bH^3}{12}$,, ,, horizontal ,, $= \dfrac{Bh^3}{12}$,, ,, whole section $= \dfrac{bH^3+Bh^3}{12}$ Z for ,, $= \dfrac{bH^3+Bh^3}{6H}$ Or this result may be obtained direct from the moduli of the two parts of the section. Thus: Z for vertical part $= \dfrac{bH^2}{6}$,, horizontal ,, $= \dfrac{Bh^3}{6} \times \dfrac{h}{H}$ (see p. 305) ,, whole section $= \dfrac{bH^2}{6} + \dfrac{Bh^3}{6H} = \dfrac{bH^3+Bh^3}{6H}$ It should be observed in the figure how very little the horizontal part of the section adds to the strength. The approximate Z neglects this part. The strength of the T section when bent thus is very much weaker than when bent as shown in the previous figure.
$\dfrac{B_1H_1^3+B_2H_2^3-b_1h_1^3-b_2h_2^3}{3H_2}$	Moment of inertia for upper part $= \dfrac{B_1H_1^3}{3} - \dfrac{b_1h_1^3}{3}$ Ditto lower part $= \dfrac{B_2H_2^3}{3} - \dfrac{b_2h_2^3}{3}$ Moment of inertia for whole section $= \dfrac{B_1H_1^3+B_2H_2^3-b_1h_1^3-b_2h_2^3}{3} = I$ $Z = \dfrac{I}{H_2}$ for the stress on the tension flange

Section.	Examples of Modulus Figures.
	 FIG. 357. This figure becomes the same as the last when massed about a centre line.
Triangle.	 FIG. 358.
Trapezium.	 FIG. 359.

Modules of the section Z.

The construction given in Fig. 355 is a convenient method of finding the centre of gravity of such sections.

Where ab = area of web, cd = area of top flange, ef = area of bottom flange, $gh = ab + cd$. The method is fully described on p. 63.

For stress at base—

$$\frac{BH^2}{12}$$

For stress at apex—

$$\frac{BH^2}{24}$$

The moment of inertia for a triangle about its c. of g. $\Big\} = \frac{BH^3}{36}$ (see p. 82)

$$y \text{ for stress at base} = \frac{H}{3}, \text{ at apex} = \frac{2H}{3}$$

$$Z \text{ for stress at base} = \frac{\frac{BH^3}{36}}{\frac{H}{3}} = \frac{BH^2}{12}$$

$$\text{,,} \qquad \text{apex} = \frac{BH^2}{24}$$

For stress at wide side—

$$\frac{BH^2}{12}\left(\frac{n^2+4n+1}{2n+1}\right)$$

For stress at narrow side—

$$\frac{BH^2}{12}\left(\frac{n^2+4n+1}{n+2}\right)$$

Approximate value for Z—

Moment of inertia $\Big\} = \frac{BH^3}{36}\left(\frac{n^2+4n+1}{n+1}\right)$ (see p. 86)

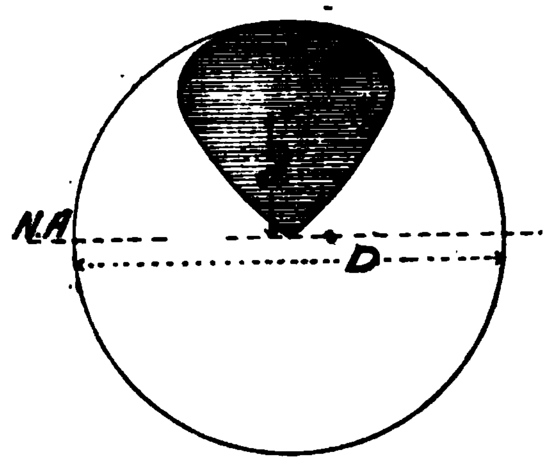
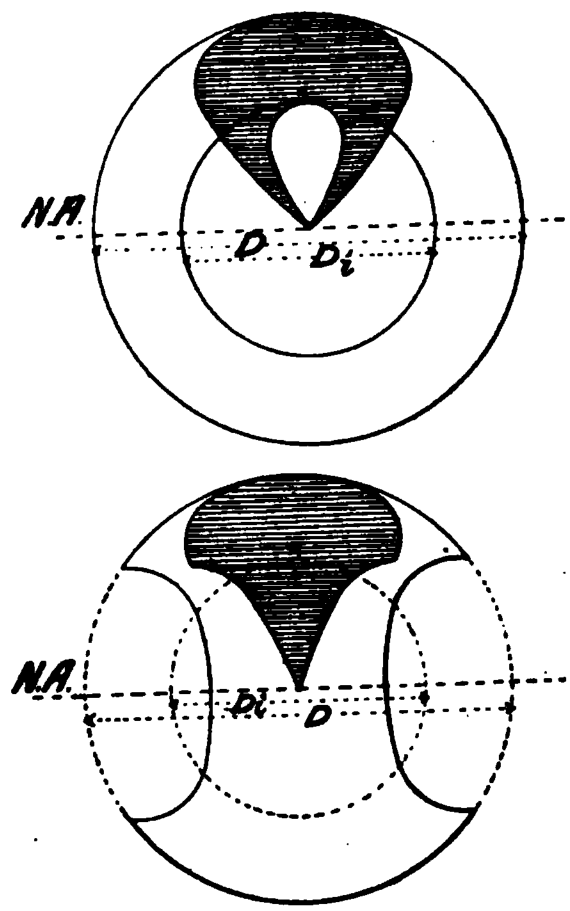
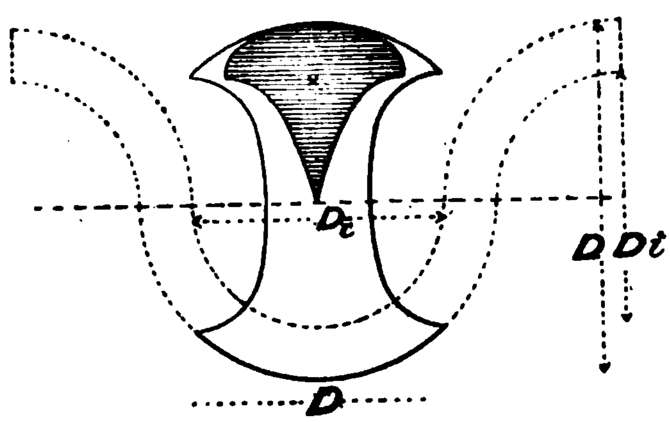
$$y \text{ for stress at wide side} = H_1 = \frac{H}{3}\left(\frac{2n+1}{n+1}\right)$$

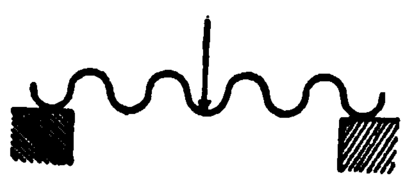
$$Z \text{ for stress at wide side} = \frac{\frac{BH^3}{36}\left(\frac{n^2+4n+1}{n+1}\right)}{\frac{H}{3}\left(\frac{2n+1}{n+1}\right)}$$

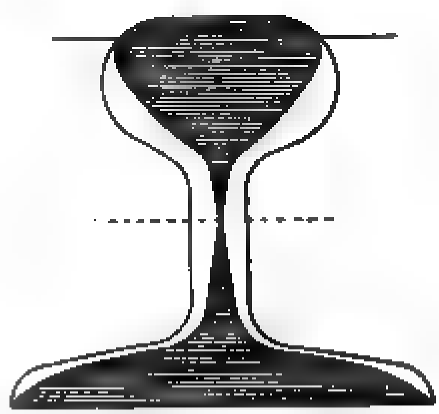
$$= \frac{BH^2}{12}\left(\frac{n^2+4n+1}{2n+1}\right)$$

$$y \text{ for stress at narrow side} = H_2 = \frac{H}{3}\left(\frac{n+2}{n+1}\right)$$

and dividing as above, we get the value given in the

Section.	Examples of Modulus Figures.
Circle.	 FIG. 360.
Hollow circle and corrugated section.	 FIG. 361.  FIG. 362.

Modules of the section Z.	
$\frac{\pi D^3}{32}$ or $\frac{D^3}{10{\cdot}2}$	The moment of inertia of a circle about a diameter (see p. 88) $\Big\} = \frac{\pi D^4}{64}$ $y = \frac{D}{2}$ $Z = \frac{\frac{\pi D^4}{64}}{\frac{D}{2}} = \frac{\pi D^3}{32}$
$\frac{\pi(D^4 - D_i^4)}{32D}$	The moment of inertia of a hollow circle about a diameter (see p. 88) $\Big\} = \frac{\pi(D^4 - D_i^4)}{64}$ $y = \frac{D}{2}$ $Z = \frac{\frac{\pi(D^4 - D_i^4)}{64}}{\frac{D}{2}}$ $Z = \frac{\pi(D^4 - D_i^4)}{32D}$ This may be obtained direct from the Z thus: Z for outer circle $= \frac{\pi D^3}{32}$,, inner ,, $= \frac{\pi D_i^3}{32} \times \frac{D_i}{D}$ (see p. 305) ,, hollow ,, $= \frac{\pi(D^4 - D_i^4)}{32D}$ For corrugated sections in which the corrugations are not perfectly circular, the error involved is very slight if the diameters D and D_t are measured vertically. The expression given is for one corrugation. It need hardly be pointed out that the corrugations must not be placed as in Fig. 362*a*.  FIG. 362*a*. The strength, then, is simply that of a rectangular section of height H = thickness of plate.

Irregular sections.	Examples of Modulus Figures.
Bull-headed rail.	Fig. 363.
Flat-bottomed rail.	 Fig. 364.
Tram rail.	Fig. 365.

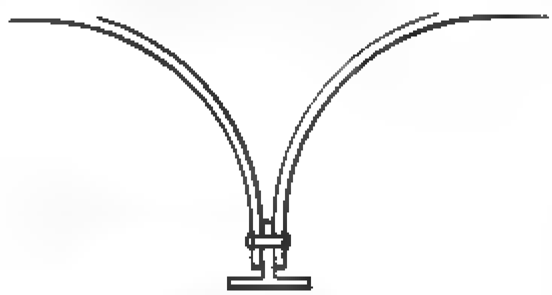
Irregular sections.

Examples of Modulus Figures.

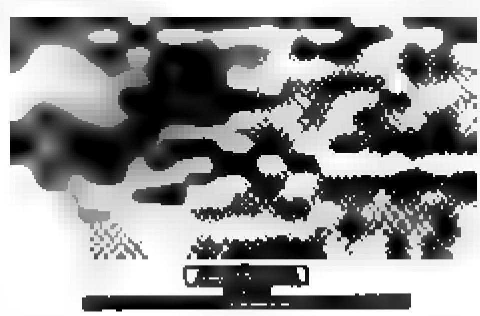
Bulb section.

Fig. 366.

Hobsons's patent flooring.



One section.



Four sections massed up.

Fig. 367.

Fireproof flooring.

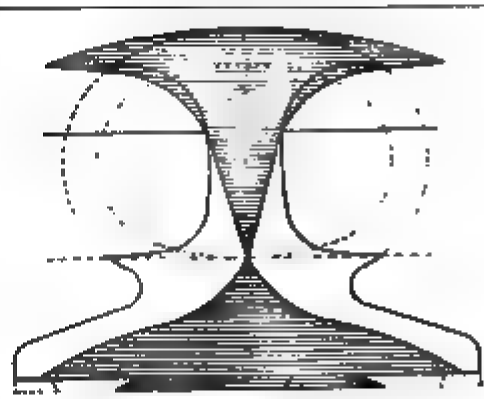
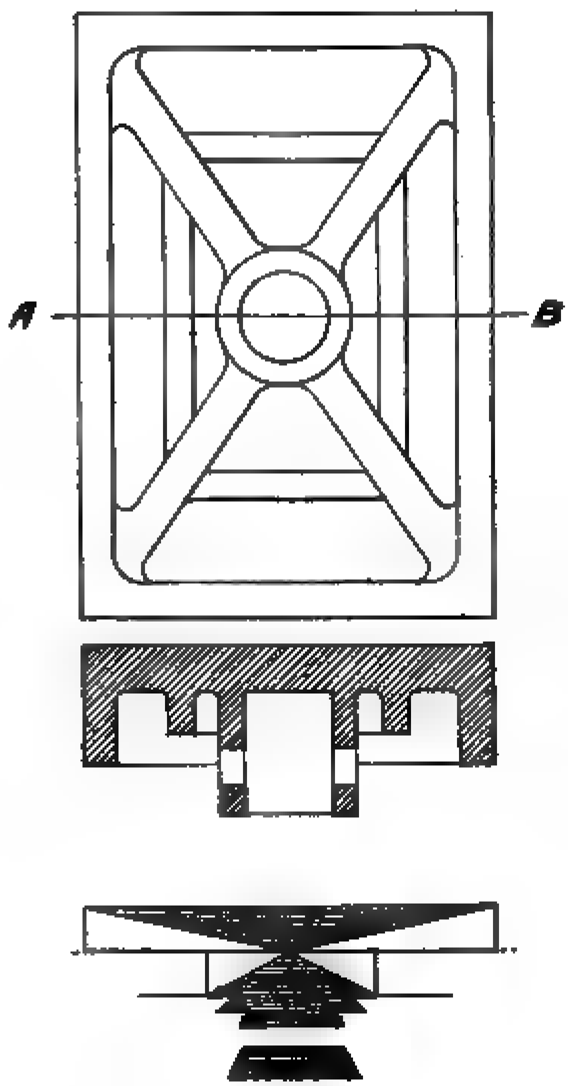


Fig. 368.

Irregular sections.	Examples of Modulus Figures.
Fireproof flooring.	Fig. 369.
Table of hydraulic press.	 Fig. 370.

Shear on Beam Sections.—In the Fig. 371 the rectangular element *abcd* on the unstrained beam becomes *d′b′c′d′* when the beam is bent, and the element has undergone a shear. The total shear force on any vertical section = W, and, assuming for the present that the shear stress is evenly distributed over the whole section, the mean shear stress $= \frac{W}{A}$, where A = the area of the section; or we may write it $\frac{W}{B.H}$. But we have shown (p. 275) that the shear stress along any two parallel sides of a rectangular element is equal to the shear stress along the other two parallel sides, hence the shear stress along *cd* is also equal to $\frac{W}{A}$.

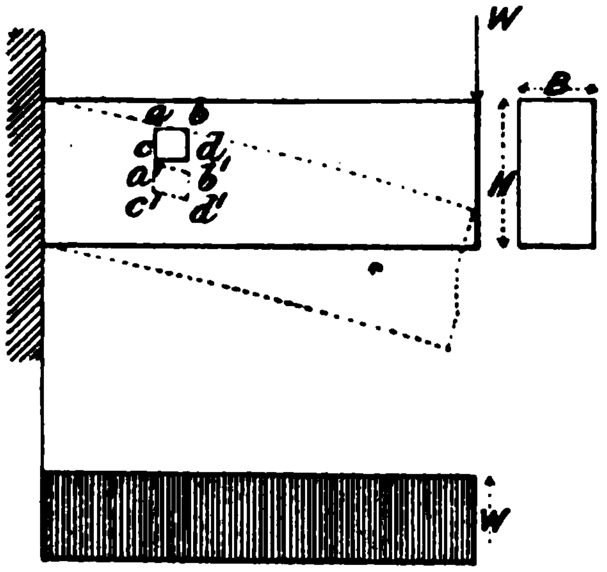


Fig. 371.

The shear on vertical planes tends to make the various parts of the beam slide downwards as shown in Fig. 372, *a*, but the shear on the horizontal planes tends to make the parts of the beam slide as in Fig. 372, *b*. This action may be illustrated by bending some thin strips of wood, when it will be found that they slide over one another in the manner shown. Solid timber beams often fail in this manner when tested.

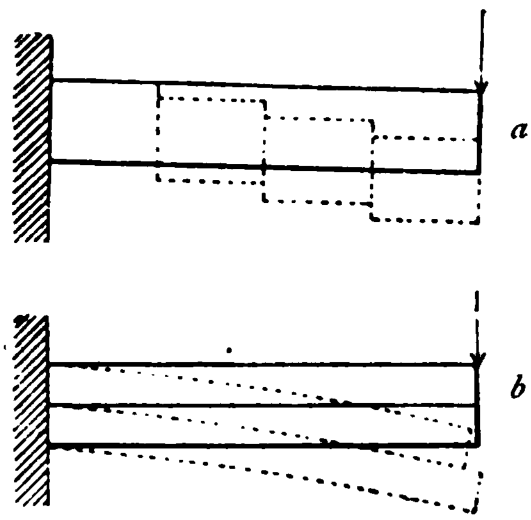


Fig. 372.

In the paragraph above we assumed that the shear stress was evenly distributed over the section; this, however, is far from being the case, for the shearing force at any part of a beam section is the algebraic sum of the shearing forces acting on either side of that part of the section (see p. 331). We will now work out one or two cases to show the distribution of the shear on a

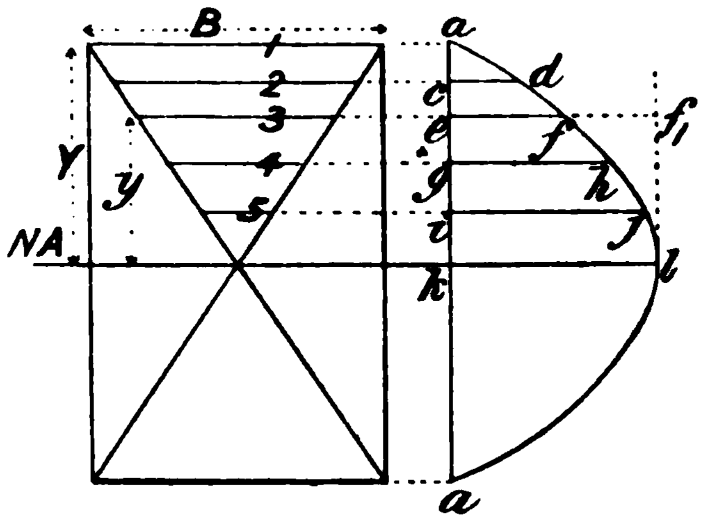


Fig. 373.

section by a graphical method, and afterwards find a mathematical expression for the same.

In Fig, 373 the distribution of stress is shown by the width of the modulus figure. Divide the figure up as shown into strips, and construct a figure at the side on the base-line *aa*, the ordinates of which represent the shear at that part of the section, *i.e.* the sum of the forces acting to either side of it, thus—

The shear at 1 is zero
" " 2 is the area of the strip between 1 and 2 = *cd* on a given scale.
" " 3 is the area of the strip between 1 and 3 = *ef* on a given scale.
" " 4 is the area of the strip between 1 and 4 = *gh* on a given scale.
" " 5 is the area of the strip between 1 and 5 = *ij* on a given scale.
" " 6 is the area of the strip between 1 and 6 = *kl* on a given scale.

Let the width of the modulus figure at any point distant y from the neutral axis $= b$; then—

$$\text{the shear at } y = \frac{\text{BY}}{2} - \frac{by}{2}$$

$$\text{But } \frac{b}{y} = \frac{\text{B}}{\text{Y}}, \text{ and } b = \frac{\text{B}y}{\text{Y}}$$

$$\text{the shear at } y = \frac{\text{BY}}{2} - \frac{\text{B}y^2}{2\text{Y}} = kl - \frac{\text{B}}{2\text{Y}}(y^2)$$

$$\text{in the figure } kl = \frac{\text{BY}}{2} \qquad ff' = \frac{\text{B}}{2\text{Y}}(y^2)$$

Thus the shear curve is a parabola, as the ordinates ff', etc., vary as y^2; hence the maximum ordinate $kl = \frac{3}{2}$ (mean ordinate) (see p. 30), or the maximum shear on the section is $\frac{3}{2}$ of the mean shear.

In Figs. 374, 375 similar curves are constructed for a circular and for an **I** section.

It will be observed that in the **I** section nearly all the shear is taken by the web; hence it is usual, in designing plate girders of this section, to assume that the whole of the shear is taken by the web. The shear at any section (Fig. 376) $ab = \frac{\text{W}}{2}$, and the

intensity of shear on the above assumption $= \dfrac{W}{2A_w}$, where $A_w =$ the sectional area of the web, or the intensity of shear $= \dfrac{W}{2ht}$. But the intensity of shear stress on aa' = the intensity of shear

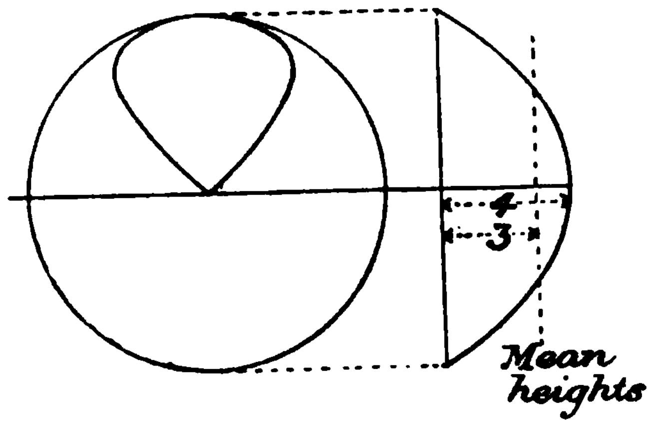


FIG. 374.

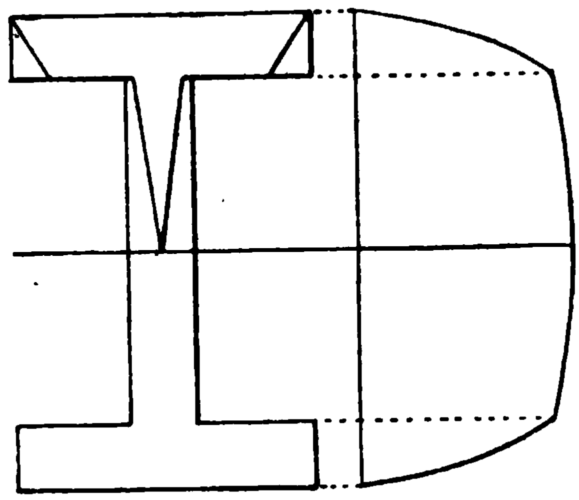


FIG. 375.

stress on ab, hence the intensity of the shear stress between the web and flange is also $= \dfrac{W}{2ht}$. We shall make use of this when working out the requisite spacing for the rivets in the angles between the flanges and the web of a plate girder.

In all the above cases it should be noticed that the shear

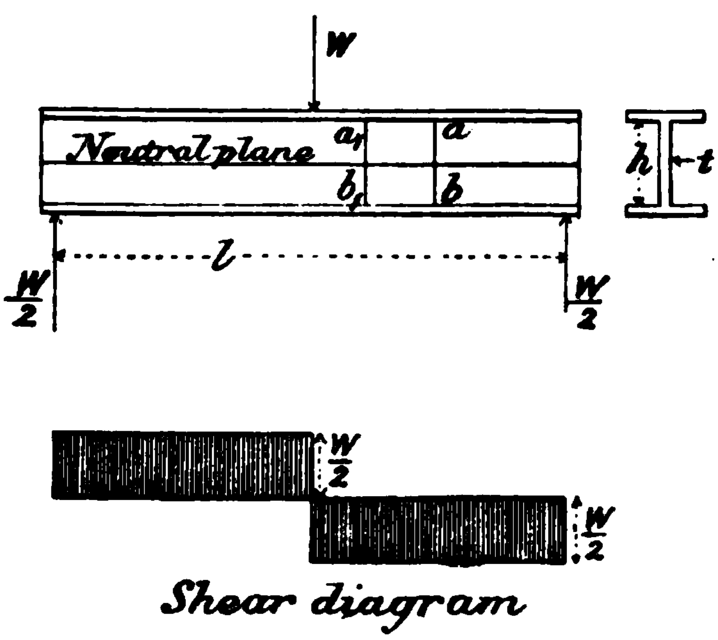


FIG. 376.

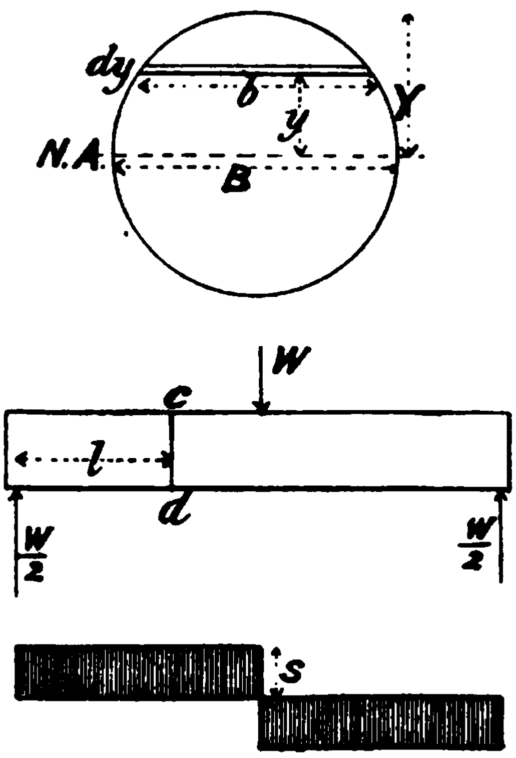


FIG. 377.

stress is a maximum at the neutral plane, and the total shear there is equal to the total direct tension or compression acting above or below it.

We will now get out an expression for the shear at any part of a beam section.

We have shown a circular section, but the argument will be seen to apply equally to any section.

Let b = breadth of the section at a distance y from the neutral axis;
F = stress on the skin of the beam distant Y from the neutral axis;
f = stress at the plane b distant y from the neutral axis;
M = bending moment on the section cd;
I = moment of inertia of the section;
S = shear force on section cd.

$$\left.\begin{array}{r}\text{The area of the strip distant } y\\ \text{from the neutral axis}\end{array}\right\} = b \,.\, dy$$

$$\text{the total force acting on the strip} = f \,.\, b \,.\, dy$$

$$\text{But } \frac{f}{F} = \frac{y}{Y}, \text{ or } f = \frac{Fy}{Y}$$

Substituting the value of f in the above, we have—

$$\frac{F}{Y} b \,.\, y \,.\, dy$$

$$\text{But } M = \frac{FI}{Y}, \text{ or } \frac{F}{Y} = \frac{M}{I} \text{ (see p. 292)}$$

Substituting the value of $\frac{F}{Y}$ in the above, we have—

$$\text{the total force acting on the strip} = \frac{M}{I} b \,.\, y \,.\, dy$$

$$\text{But } M = Sl \text{ (see p. 333)}$$

Substituting in the equation above, we have—

$$\frac{Sl}{I} b \,.\, y \,.\, dy$$

Dividing by the area of the plane, viz. $b \,.\, l$, we get—

$$\text{The intensity of shearing stress on that plane} = \frac{Sl}{Ibl}\int_y^Y b \,.\, y \,.\, dy$$

$$\left.\begin{array}{r}\text{the maximum value at the neutral axis,}\\ \text{when } b = B, \text{ and } y = 0\end{array}\right\} = \frac{S}{IB}\int_0^Y b \,.\, y \,.\, dy$$

$$\text{the mean intensity of shear stress} = \frac{S}{A}$$

where A = the area of the section.

$$\left.\begin{array}{l}\text{The ratio of the maximum}\\ \text{intensity to the mean}\end{array}\right\} = \frac{\dfrac{S}{IB}\displaystyle\int_0^Y b\,.\,y\,.\,dy}{\dfrac{S}{A}}$$

$$= \frac{A}{IB}\int_0^Y b\,.\,y\,.\,dy$$

Deflection due to Shear.—The shearing action of a load on a beam causes the deflection to be greater than it otherwise would be. In the figure the beam is supposed to be subject to shear only, which deflects it an amount = x.

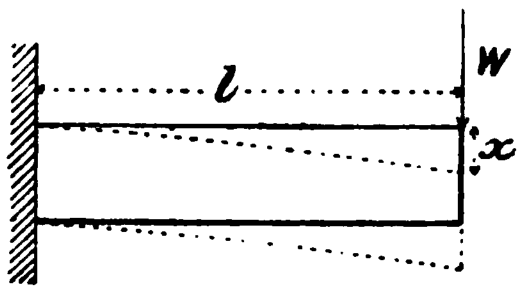


FIG. 378.

Then (p. 265)—

$$\frac{x}{l} = \frac{f}{G}$$

where f = the shear stress on the section ;
G = the coefficient of rigidity.

But $f = \frac{KW}{A}$, where K is the ratio of the maximum to the mean shear stress (see last paragraph); A = sectional area of beam. Hence—

$$x = \frac{KWl}{AG}$$

For a rectangular section, K = 1·5, A = BH

$$\text{and } x = \frac{1\cdot 5Wl}{BHG}$$

The deflection due to bending is (see p. 361)—

$$\delta = \frac{Wl^3}{3EI} = \frac{Wl^3}{3 \times \frac{5}{2}G \times \frac{BH^3}{12}} \text{ (see pages 268 and 80)}$$

$$\delta = \frac{8Wl^3}{5GBH^3}$$

and the ratio—

$$\frac{\text{deflection due to shear}}{\text{deflection due to bending}} = \frac{x}{\delta} = 0{\cdot}94\frac{\text{H}^2}{l^2}$$

$\frac{\text{H}}{l}$ varies from $\frac{1}{10}$ to $\frac{1}{20}$. Hence the above ratio = from about $\frac{1}{100}$ to $\frac{1}{400}$; thus it is quite a negligible quantity for practical purposes.

For plate girders and web sections it is of much greater importance, and frequently has to be taken into account.

Discrepancies between Experiment and Theory.—Far too much is usually made of the slight discrepancies between experiments and the theory of beams; it has mainly arisen through an improper application of the beam formula, and to the use of very imperfect appliances for measuring the elastic deflection of beams. The author has made a special study of this subject, and finds that the more delicate the apparatus he uses, the more nearly do experiments and theory agree.

The discrepancies may be dealt with under three heads—

(1) Discrepancies below the elastic limit
(2) " at " "
(3) " after " "

(1) The discrepancies below the elastic limit are partly due to the fact that the modulus of elasticity (E_c) in compression is not always the same as in tension (E_t).

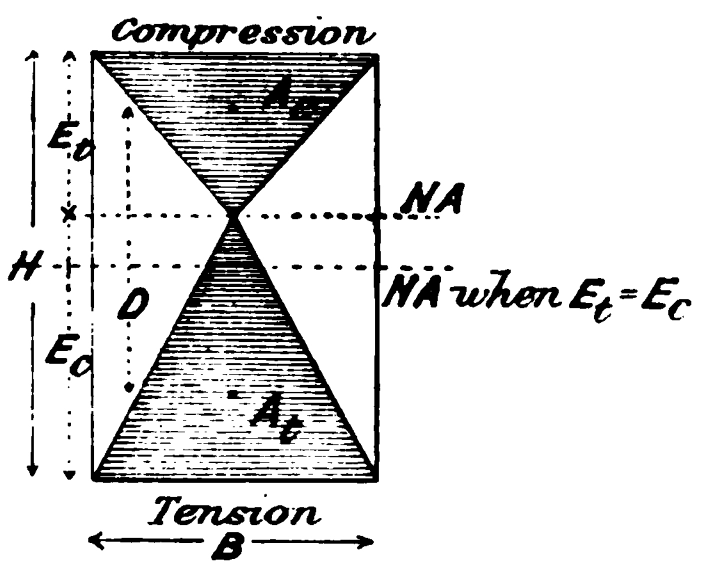


Fig. 379.

For example, suppose a piece of material to be tested by pure tension for E_t, and the same piece of material to be afterwards tested as a beam for E_b (modulus of elasticity from a bending test); then, by the usual beam theory, the two results should be identical, but in the case of cast materials it will probably be found that the bending test will give the higher result; for if, as is often the case, the E_c is greater than the E_t, the compression area A_c of the modulus figure will be smaller than the tension area A_t, for the tensions and compressions form a

couple, and $A_c E_c = A_t E_t$. The modulus of the section will now be $A_t \times D$, or $\frac{BH}{4} \times \frac{E_c}{\frac{E_c + E_t}{2}} \times D$; thus the Z is increased in the ratio $\frac{2E_c}{E_c + E_t}$. If the E_c be 10 per cent. greater than E_t, the E_b found from the beam will be about 5 per cent. greater than the E_t found by pure traction.

This difference in the elasticity will certainly account for considerable discrepancies, and will nearly always tend to make the E_b greater than E_t. There is also another discrepancy which has a similar tendency, viz. that some materials do not *perfectly* obey Hooke's law; the strain increases slightly more rapidly than the stress (see Appendix). This tends to increase the size of the modulus figures, as shown exaggerated in dotted lines, and thereby to increase the value of Z, and this again tends to increase its strength and stiffness, and consequently make the E_b greater than E_t.

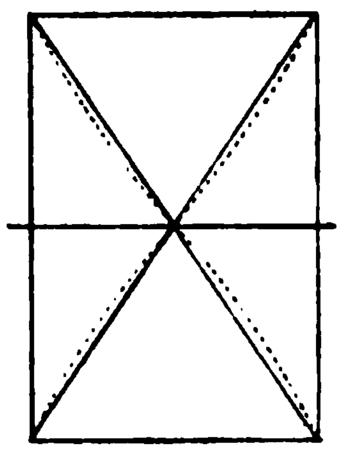


FIG. 380.

On the other hand, the deflection due to the shear (see p. 321) is usually neglected in calculating the value E_b, and consequently tends to make the deflection greater than calculated, and reduces the value of E_b. And, again, experimenters often measure the deflection of beams thus, between the bottom of the beam and the supports as shown. The supports slightly dig into the beam when loaded, and moreover spring slightly, both of which tend to make the deflection greater than it should be, and consequently reduce the value of E_b.

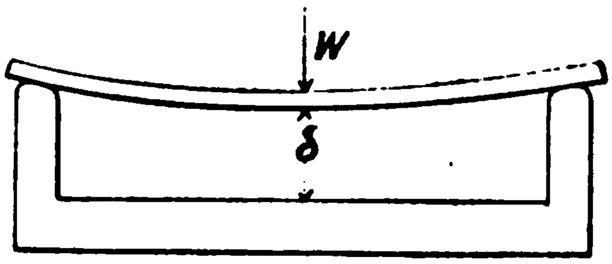


FIG. 381.

The discrepancies, however, between theory and experiment in the case of beams which are not loaded beyond the elastic limit are very, very small, far smaller than the errors usually made in estimating the loads on beams.

(2) The discrepancies at the elastic limit are more imaginary than real. A beam is usually assumed to pass the elastic limit when the rate of increase of the deflection per unit increase of load increases rapidly under a gradually increasing load, *i.e.* when the tangent of slope of the stress-strain diagram increases rapidly, or when a marked permanent set is produced, but the

load at which this occurs is, however, far beyond the true elastic limit of the material. In the case of a tension bar the stress is evenly distributed over any cross-section, hence the whole section of the bar passes the elastic limit at the same instant; but in the case of a beam the stress is not evenly distributed, consequently only a very thin skin of the metal passes the elastic limit at first, while the rest of the section remains elastic, hence there cannot possibly be a sudden increase in the strain (deflection) such as is experienced in tension. When the load is removed, the elastic portion of the section restores the beam to very nearly its original form, and thus prevents any marked permanent set. Further, in the case of a tension specimen, the sudden stretch at the elastic limit occurs over the whole length of the bar, but in a beam only over a very small part of the length, viz. just where the bending moment is a maximum, hence the load at which the sudden stretch occurs is much less definitely marked in a beam than in a tension bar.

In the case of a beam of, say, mild steel, the distribution of stress in a section just after passing the elastic limit is approximately that shown in the shaded modulus figure of Fig. 382, whereas if the material had remained perfectly elastic, it would have been that indicated by the triangles *aob*. By the methods described in the next chapter, the deflection after the elastic limit can be calculated, and thereby it can be readily shown that the rate of increase in the deflection for stresses far above the elastic limit is very gradual, hence it is practically impossible to detect the true elastic limit of a piece of material from an ordinary bending test. Results of tests will be found in the Appendix.

Fig. 382.

Fig. 383.

(3) The discrepancies after the elastic limit have occurred. The word "discrepancy" should not be used in this connection at all, for if there is one principle above all others that is laid down in the beam theory, it is that the material is taken to be perfectly elastic, *i.e.* that it has not passed the elastic limit, and yet one is constantly hearing of the

"error in the beam theory," because it does not hold under conditions in which the theory expressly states that it will not hold. But, for the sake of those who wish to account for the apparent error, they can do it approximately in the following way. The beam theory assumes the stress to be proportional to the distance from the neutral axis, or to vary as shown by the line *ab*; under such conditions we get the usual modulus figure. When, however, the beam is carried beyond the elastic limit, the distribution of stress in the section is as shown by the line *adb*, hence the width of the modulus figure must be increased in the ratio of the widths of the two curves as shown, and the Z thus corrected is the shaded area × D as before.[1] This in many instances will entirely account for the so-called error. Similar figures corrected in this manner are shown below, from which it will be seen that the difference is much greater in the circle than in the rolled joist, and, for obvious reasons,

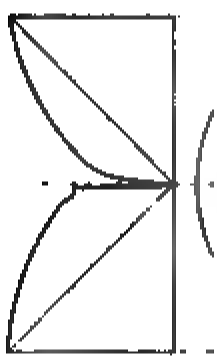


FIG. 384.

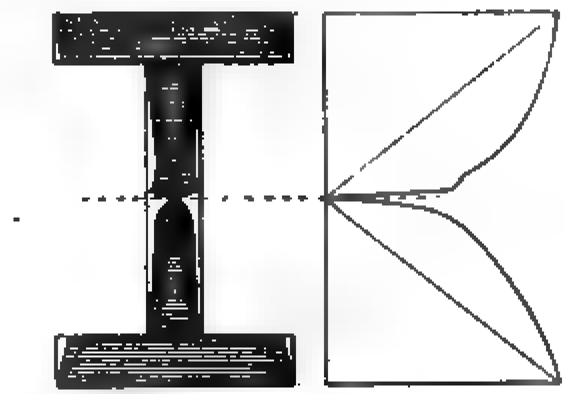


FIG. 385.

it will be seen that the difference is greatest in those sections in which much material is concentrated about the neutral axis.

But before leaving this subject the author would warn readers against such reasoning as this. The actual breaking strength of a beam is very much higher than the breaking strength calculated by the beam formula, hence much greater stresses may be allowed on beams than in the same material in tension and compression. Such reasoning is utterly misleading, for the apparent error only occurs after the elastic limit has been passed.

[1] It should also be remembered that when one speaks of the tensile strength of a piece of material, one always refers to the *nominal* tensile strength, not to the *real*, the difference, of course, is due to the reduction of the section as the test proceeds. Now, no such reduction in the Z occurs in the beam, hence we must multiply this corrected Z by the ratio of the real to the nominal tensile stress at the maximum load.

CHAPTER X.

BENDING MOMENTS AND SHEAR FORCES.

Bending Moments.—*When two*[1] *equal and opposite couples are applied at opposite ends of a bar in such a manner as to tend to rotate it in opposite directions, the bar is said to be subjected to a bending moment.*

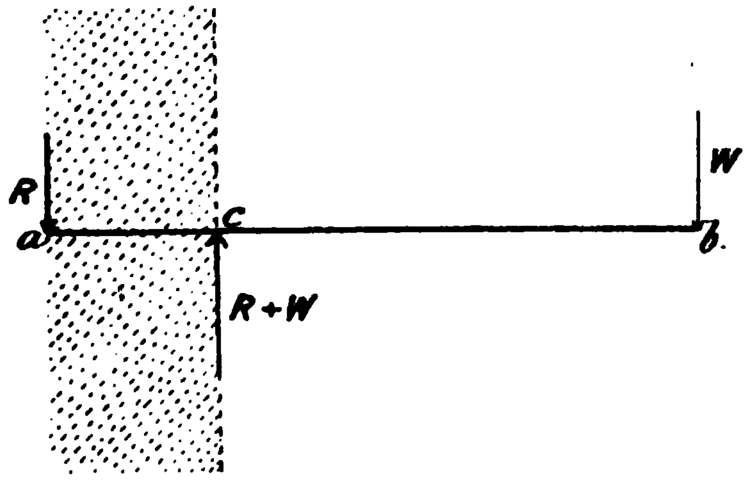


FIG. 386.

Thus, in Fig. 386, the bar ab is subjected to the two equal and opposite couples R . ac and W . bc, which tend to make the two parts of the bar rotate in opposite directions round the point C; or, in other words, they tend to *bend* the bar, hence the term "bending moment." Likewise in Fig. 387 the couples are R_1ac and R_2bc, which have the same effect as the couples in Fig. 386. The bar in Fig. 386 is termed a "cantilever." The couple R . ac is due to the resistance of the wall into which it is built.

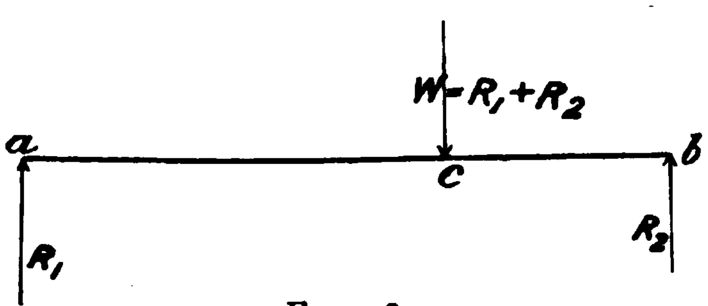


FIG. 387.

The bar in Fig. 387 is termed a "beam."

When a cantilever or beam is subjected to a bending moment which tends to bend it

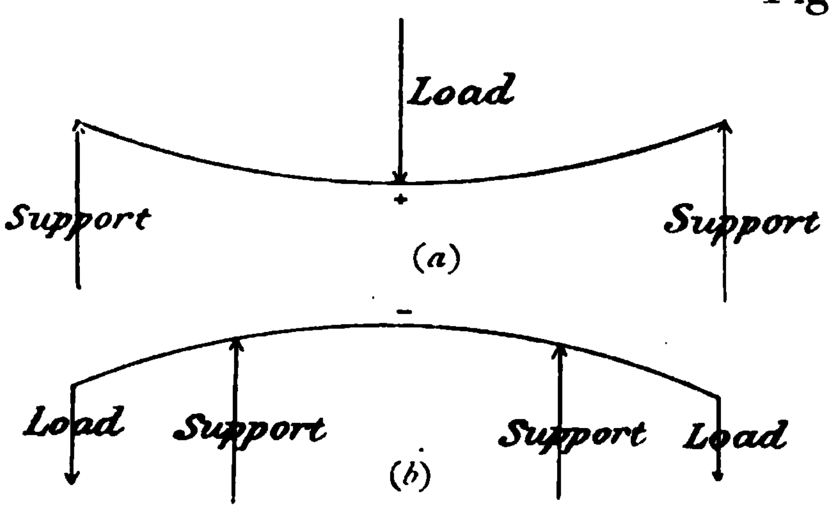


FIG. 388

[1] If there be more than two couples, they can always be reduced to two.

concave upwards, as in Fig. 388 *a*, the bending moment will be termed positive (+), and when it tends to bend it the reverse way, as in Fig. 388 *b*, it will be termed negative (−).

Bending-moment Diagrams.—In order to show the variation of the bending moments at various parts of a beam, we frequently make use of bending-moment diagrams. The bending moment at the point *c* in Fig. 389 is W . *bc*; set down from *c* the ordinate *cc'* = W . *bc* on some given scale. The bending moment at *d* = W . *bd*; set down from *d*, the ordinate *dd'* = W . *bd* on the same scale; and so on for any number of points: then, as the bending moment at any point increases directly as the distance of that point from W, the points *b*, *d'*, *c'*, etc., will lie on a straight line. Join up these points as shown, then the depth of the diagram below any point in the beam represents on the given scale the bending moment at that point. This diagram is termed a "bending-moment diagram."

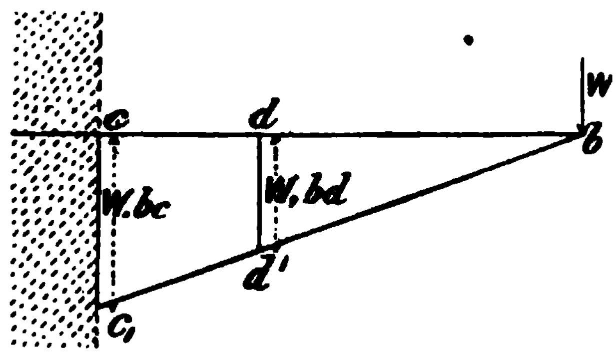


FIG. 389.

In precisely the same manner the diagram in Fig. 390 is obtained. The ordinate *dd'* represents on a given scale the bending moment R_1ad, likewise *cc'* the bending moment R_1ac or R_2bc, also *ee'* the bending moment R_2be.

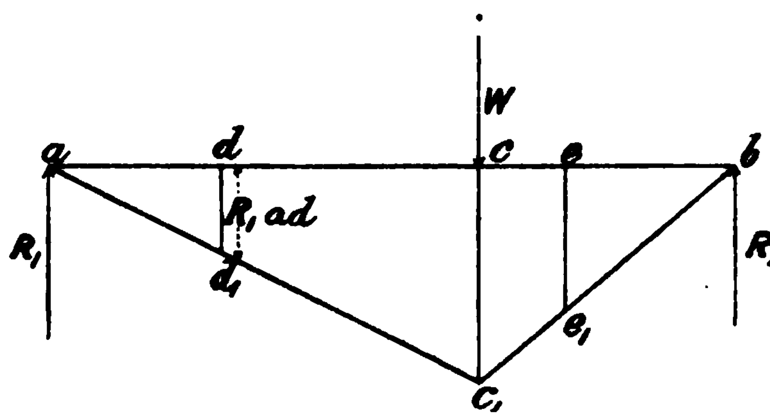


FIG. 390.

The reactions R_1 and R_2 are easily found by the principle of moments thus. Taking moments about the point *b*, we have—

$$R_1ab = Wbc \qquad R_1 = \frac{W \cdot bc}{ab} \qquad R_2 = W - R_1$$

In the cantilever in Fig. 389, let W = 800 lbs., *bc* = 6·75 feet, *bd* = 4·5 feet.

$$\begin{aligned}\text{The bending moment at } c &= W \cdot bc \\ &= 800 \text{ (lbs.)} \times 6{\cdot}75 \text{ (feet)} \\ &= 5400 \text{ (lbs.-feet)}\end{aligned}$$

Let 1 inch on the bending-moment diagram = 12,000 (lbs.-feet), or 12,000 (lbs.-feet) per inch, or $\dfrac{12000 \text{ lbs.-feet}}{1 \text{ (inch)}}$

Then the ordinate $cc' = \dfrac{5400 \text{ (lbs.-feet)}}{\dfrac{12000 \text{ (lbs.-feet)}}{1 \text{ (inch)}}} = 0{\cdot}45$ (inch)

Measuring the ordinate dd', we find it to be 0·3 inch.

Then $0{\cdot}3 \text{ (inch)} \times \dfrac{12000 \text{ (lbs.-feet)}}{1 \text{ (inch)}} = 3600$ (lbs.-feet) bending moment at d

In this instance the bending moment could have been obtained as readily by direct calculation; but in the great majority of cases, the calculation of the bending moment is long and tedious, and can be very readily found from a diagram.

In the beam (Fig. 390), let W = 1200 lbs., $ac = 5$ feet, $bc = 3$ feet, $ad = 2$ feet.

$$R_1 = \frac{W \,.\, bc}{ab} = \frac{1200 \text{ (lbs.)} \times 3 \text{ (feet)}}{8 \text{ (feet)}} = 450 \text{ lbs.}$$

the bending moment at $c = 450 \text{ (lbs.)} \times 5 \text{ (feet)}$
$= 2250$ (lbs.-feet)

Let 1 inch on the bending-moment diagram = 4000 lbs.-feet), or 4000 (lbs.-feet) per inch, or $\dfrac{4000 \text{ (lbs.-feet)}}{1 \text{ (inch)}}$.

Then the ordinate $cc' = \dfrac{2250 \text{ (lbs.-feet)}}{\dfrac{4000 \text{ (lbs.-feet)}}{1 \text{ (inch)}}} = 0{\cdot}56$ (inch)

Measuring the ordinate dd', we find it to be 0·225 (inch).

Then $\dfrac{0{\cdot}225 \text{ (inch)} \times 4000 \text{ (lbs.-feet)}}{1 \text{ (inch)}} = \left\{ \begin{array}{l} 900 \text{ (lbs.-feet) bending} \\ \text{moment at } d \end{array} \right.$

General Case of Bending Moments.—*The bending moment at any section of a beam is the algebraic sum of all the moments of the external forces about the section acting either to the left or to the right of the section.*

Thus the bending moment at the section f in Fig. 391 is, taking moments to the left of f—

$$R_1 af - W_1 cf - W_2 df$$

or, taking moments to the right of f—

$$R_2 bf - W_3 ef$$

That the same result is obtained in both cases is easily shown by taking a numerical example.

Let $W_1 = 30$ lbs., $W_2 = 50$ lbs., $W_3 = 40$ lbs.; $ac = 2$ feet, $cd = 2{\cdot}5$ feet, $df = 1{\cdot}8$ feet, $fe = 2{\cdot}2$ feet, $eb = 3$ feet.

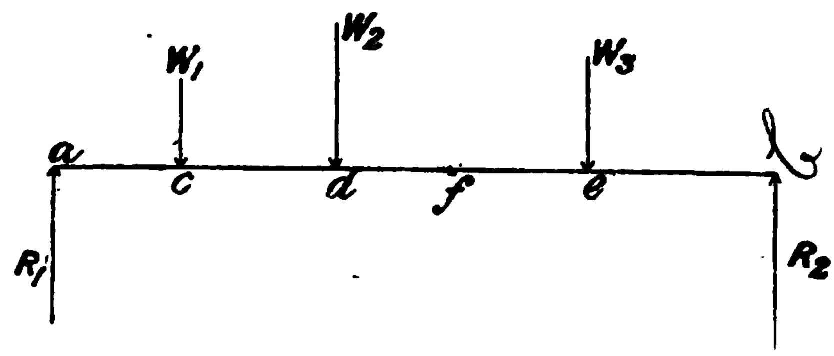


FIG. 391.

We must first calculate the values of R_1 and R_2. Taking moments about b, we have—

$$R_1 ab = W_1 cb + W_2 db + W_3 eb$$

$$R_1 = \frac{W_1 cb + W_2 db + W_3 eb}{ab}$$

$$R_1 = \frac{30(\text{lbs.}) \times 9{\cdot}5(\text{feet}) + 50(\text{lbs.}) \times 7(\text{feet}) + 40(\text{lbs.}) \times 3(\text{feet})}{11{\cdot}5\ (\text{feet})}$$

$$= \frac{755\ (\text{lbs.-feet})}{11{\cdot}5\ (\text{feet})} = 65{\cdot}65 \text{ lbs.}$$

$$R_2 = W_1 + W_2 + W_3 - R_1$$

$$R_2 = 30\,(\text{lbs.}) + 50\,(\text{lbs.}) + 40\,(\text{lbs.}) - 65{\cdot}65\,(\text{lbs.}) = 54{\cdot}35\,(\text{lbs.})$$

The bending moment at f, taking moments to the left of f,

$$= R_1 af - W_1 cf - W_2 df$$

$$= 65{\cdot}65\,(\text{lbs.}) \times 6{\cdot}3\,(\text{feet}) - 30\,(\text{lbs.}) \times 4{\cdot}3\,(\text{feet}) - 50\,(\text{lbs.}) \times 1{\cdot}8\,(\text{feet}) = 194{\cdot}6\,(\text{lbs.-feet})$$

The bending moment at f, taking moments to the right of f,

$$= R_2 bf - W_3 ef$$

$$= 54{\cdot}35\,(\text{lbs.}) \times 5{\cdot}2\,(\text{feet}) - 40\,(\text{lbs.}) \times 2{\cdot}2\,(\text{feet})$$

$$= 194{\cdot}6\,(\text{lbs.-feet})$$

Thus the bending moment at f is the same whether we take moments to the right or to the left of the point f. The calculation of it by both ways gives an excellent check on the accuracy of the working, but generally we shall choose that side of the section that involves the least amount of calculation. Thus, in the case above, we should have taken moments to the right of the section, for that only involves the calculation of two moments, whereas if we had taken it to the left it would have involved three moments.

The above method becomes very tedious when dealing with many loads. For such cases we shall adopt graphical methods.

Shearing Forces.—When couples are applied to a beam in the way described above, the beam is not only subjected to a bending moment, but also to a shearing action. In a long beam or cantilever, the bending is by far the most important, but a short stumpy beam or cantilever will nearly always fail by shear.

Let the cantilever in Fig. 392 be loaded until it fails. It

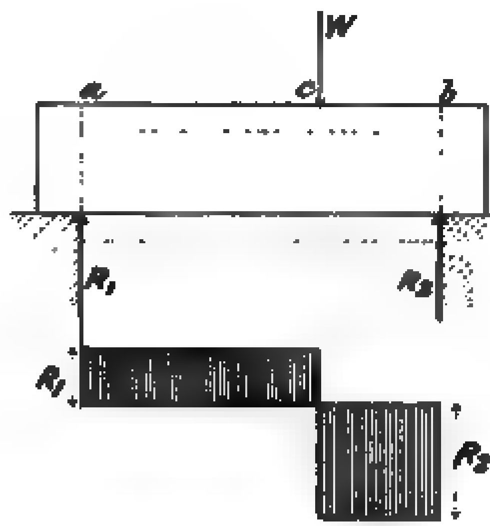


Fig. 392.

Fig. 393.

will bend down slightly at the outer end, but that we may neglect for the present. The failure will be due to the outer part shearing or sliding off bodily from the built-in part of the cantilever, as shown in dotted lines.

The shear on all vertical sections, such as ab or $a'b'$, is of the same value, and equal to W.

In the case of the beam in Fig. 393, the middle part will shear or slide down relatively to the two ends, as shown in dotted lines. The shear on all vertical sections between b and c is of the same value, and equal to R_2, and on all vertical sections between a and c is equal to R_1.

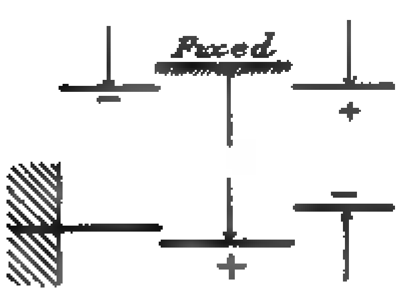


Fig. 394.

We have spoken above of positive and negative bending moments. We shall also find it convenient to speak of positive and negative shears.

When the sheared part slides in a $\left\{\begin{matrix}\text{clockwise} \\ \text{contra-clockwise}\end{matrix}\right\}$ direction relatively to the fixed part, we term it a $\left\{\begin{matrix}\text{positive } (+) \text{ shear} \\ \text{negative } (-) \text{ shear}\end{matrix}\right\}$

Shear Diagrams.—In order to show clearly the amount of shear at various sections of a beam, we frequently make use of shear diagrams. In cases in which the shear is partly positive and partly negative, we shall invariably place the *positive* part of the shear diagram *above* the base-line, and the *negative* part *below* the base-line. Attention to this point will save endless trouble.

In Fig. 392, the shear is positive and constant at all vertical sections, and equal to W. This is very simply represented graphically by constructing a diagram immediately under the beam or cantilever of the same length, and whose depth is equal to W on some given scale, then the depth of this diagram at every point represents on the same scale the shear at that point. Usually the shear diagram will not be of uniform depth. The construction for various cases will be shortly considered. It will be found that its use greatly facilitates all calculations of the shear in girders, beams, etc.

In Fig. 393, the shear at all sections between *a* and *c* is constant and equal to R_1. It is also positive (+), because the slide takes place in a clockwise direction; and, again, the shear at all sections between *b* and *c* is constant and equal to R_2, but it is of negative (−) sign, because the slide takes place in a contra-clockwise direction; hence the shear diagram between *a* and *c* will be above the base-line, and that between *b* and *c* below the line, as shown in the diagram. The shear changes sign immediately under the load, and the resultant shear at that section is $R_1 - R_2$.

General Case of Shear.—*The shear at any section of a beam or cantilever is the algebraic sum of all the forces acting to the right or to the left of that section.*

One example will serve to make this clear.

In Fig. 395 three forces are shown acting on the cantilever fixed at *d*, two acting downwards, and one acting upwards.

The shear at any section between *a* and $d = +W$ due to W

" " " *b* " $d = -W_1$ " W_1

" " " *c* " $d = +W$ " W_2

Construct the diagrams separately for each shear as shown, then combine by superposing the − diagram on the + diagram. The unshaded portion shows where the − shear neutralizes the

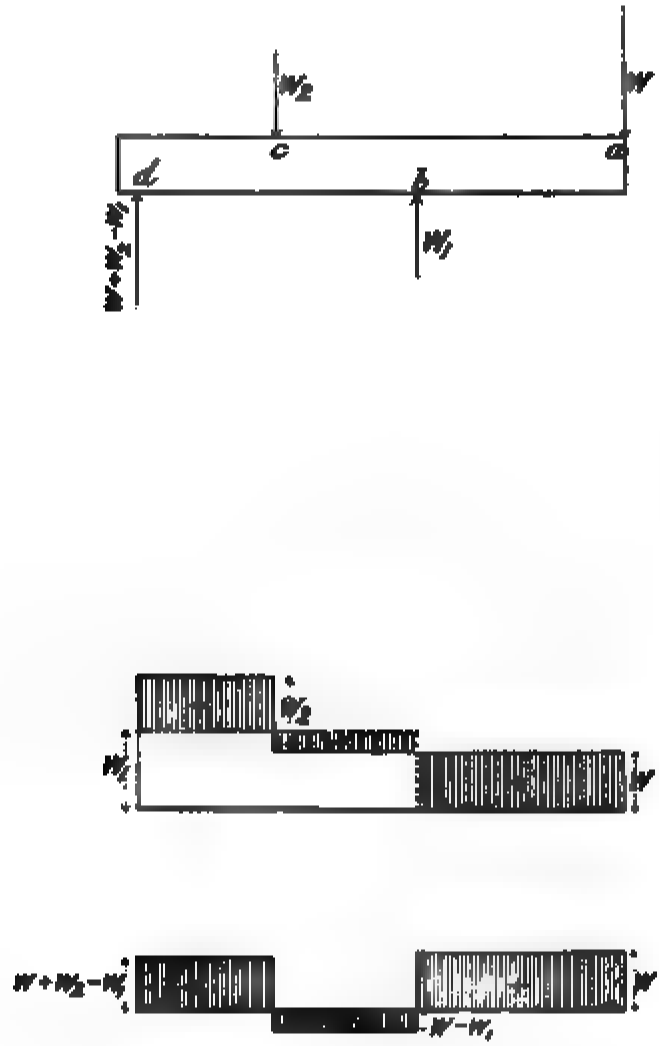


Fig. 395.

+ shear; then bringing the + portions above the base-line and the − below, we get the final figure.

Resultant shear at any section—		
Between	To the right.	To the left.
a and *b*	W	$-W_1 + W_2 - (W + W_2 - W_1 = -W$
b and *c*	$W - W_1$	$W_2 - (W + W_2 - W_1) = -(W - W_1)$
c and *d*	$W_2 - W_1 + W$ or $W + W_2 - W_1$	$-(W + W_2 - W_1)$

In the table above are given the algebraic sum of the forces to the right and to the left of various sections. On comparing them with the results obtained from the diagram, they will be found to be identical. In the case of the shear between the sections *b* and *c*, the diagram shows the shear as negative. The table, in reality, does the same, because W′ in this case is greater than W. It should be noticed that when the shear is taken to the left of a section, the sign of the shear is just the reverse of what it is when taken to the right of the section.

Connection between Bending-moment and Shear Diagrams.—In the construction of shear diagrams, we make their depth at any section equal, on some given scale, to the shear at that section, *i.e.* to the algebraic sum of the forces to the right or left of that section, and the length of the diagrams equal to the distance from that section.

Let any given beam be loaded thus: Loads W_1, W_2, W_3, $-W_4$, $-W_5$ at distances l_1, l_2, l_3, l_4, l_5 respectively from any given section *a*, as shown in Fig. 396.

The bending moment at *a* is $=W_2l_2+W_3l_3-W_4l_4$ or $W_1l_1-W_5l_5$.

But W_2l_2 is the area of the shear diagram between W_2 and the section *a*, likewise W_3l_3 is the area between W_3 and *a*, also $-W_4l_4$ is the area between $-W_4$ and *a*. The positive areas are partly neutralized by the negative areas. The parts not neutralized are shown shaded.

The shaded area $= W_2l_2 + W_3l_3 - W_4l_4$, but we have shown above that this quantity is equal to the bending moment at *a*. In the same manner, it can be shown that the shaded area of the shear diagram to the left of the section *a* is equal to

$W_1 l_1 - W_5 l_5$, *i.e.* to the bending moment at *a*. Hence we get this relation—

The bending moment at any section of a beam is equal to the area of the shear diagram up to that point measured on the length-load scale adopted.

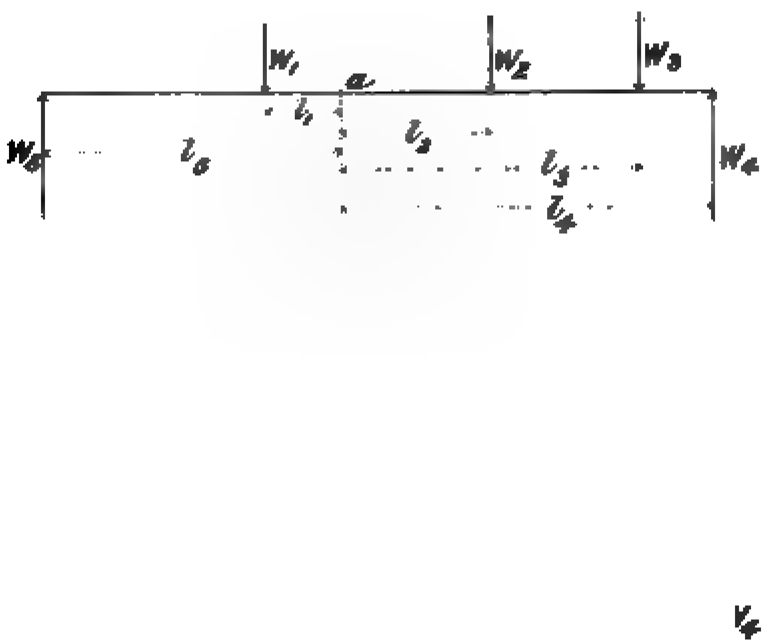


FIG. 396.

Due attention must, of course, be paid to positive and negative areas in the shear diagram.

To make this quite clear, we will work out a numerical example.

In the figure, let	$W_1 = 50$ lbs.	$l_1 = 1$ foot
	$W_2 = 80$ lbs.	$l_2 = 2$ feet
	$W_3 = 70$ lbs.	$l_3 = 4$ feet
By moments we find	$W_4 = 132 \cdot 2$ lbs.	$l_4 = 5$ feet
	$W_5 = 67 \cdot 8$ lbs.	$l_5 = 4$ feet

The figure is drawn to the following scales—

Length 1 inch = 4 feet
load 1 inch = 160 lbs.

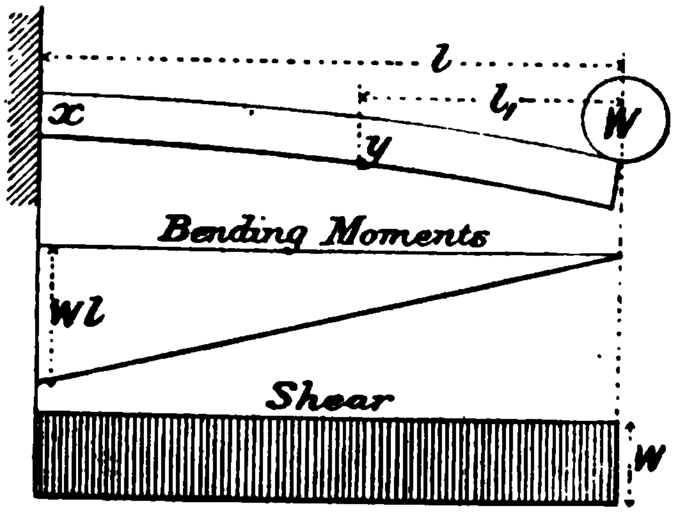
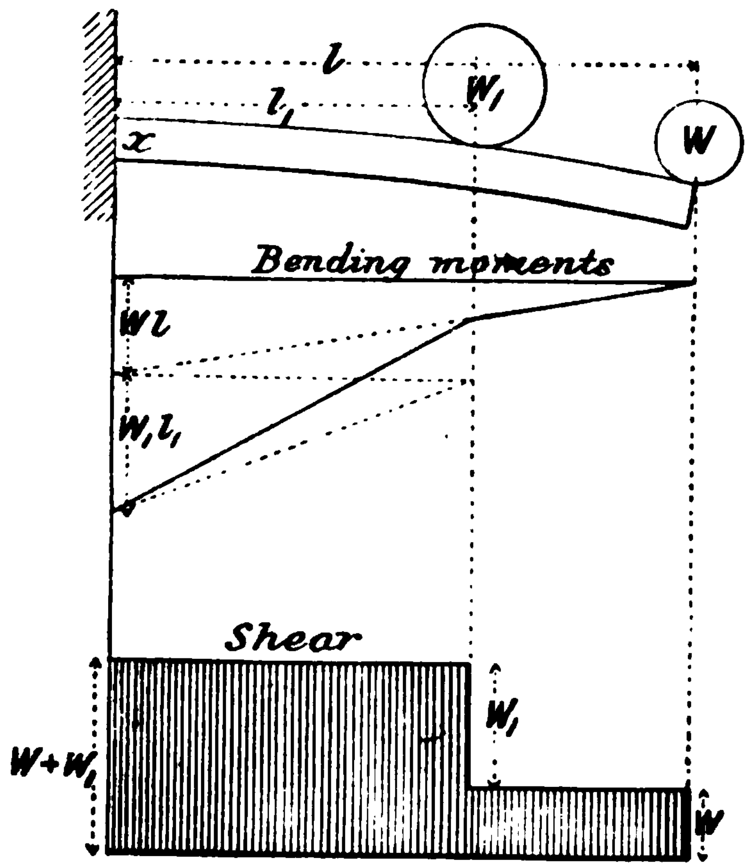
hence 1 square inch on the shear diagram = 4 (feet) × 160 (lbs.)
= 640 (lbs.-feet)

The area of the negative part of the shear diagram below

the base-line is $-0{\cdot}401$ sq. inch, and the positive part above the base-line is $0{\cdot}056$ sq. inch, thus the area of the shear diagram up to the section *a* is $-0{\cdot}401 + 0{\cdot}056 = 0{\cdot}345$ sq. inch, but 1 sq. inch on the shear diagram = 640 (lbs. feet) bending moment; thus the bending moment at the section a = $0{\cdot}345 \times 640 = 221$ (lbs. feet). The area of the shear diagram to the left of a $= 0{\cdot}345$ sq. inch, *i.e.* the same as the area to the right of the section. As a check on the above, we will calculate the bending moment at *a* by the direct method, thus—

$$
\begin{aligned}
\text{The bending moment at } a &= W_2 l_2 + W_3 l_3 - W_4 l_4 \\
&= 80 \text{ (lbs.)} \times 2 \text{ (feet)} + 70 \text{ (lbs.)} \times 4 \text{ (feet)} - 132{\cdot}2 \text{ (lbs.)} \times 5 \text{ (feet)} \\
&= 221 \text{ (lbs.-feet)}
\end{aligned}
$$

which is the same result as we obtained above from the shear diagram.

Cantilever with single load at free end.	 FIG. 397.
Cantilever with two loads.	 FIG. 398.

Bending moment M in lbs.-inches.	Depth of bending-moment diagram in inches. Scale of W, m lbs. = 1 inch. Scale of l, $\frac{1}{n}$ full size.	REMARKS.
$M_x = Wl$ $= W\text{ (lbs.)} \times l\text{ (in.)}$ $M_y = Wl_1$ $M = d\,.\,mn$ at any section where d = depth of bending-moment diagram in inches	$\frac{Wl}{mn}$ $\frac{Wl_1}{mn}$	The only moment acting to the right of x is Wl, which is therefore the bending moment at x. Likewise at y. The complete statement of the units for the depth of the bending moment diagram is as follows:— $m\text{(lbs.)} = 1$ inch on diagram, or $\frac{m\text{(lbs.)}}{1\text{(inch)}}$ $n\text{(in.)} = 1$,, ,, $\frac{W\text{ (lbs.)}}{\frac{m\text{ (lbs.)}}{1\text{ inch}}} \times \frac{l_1\text{(inches)}}{n\text{ (inches)}} = \frac{Wl}{mn}$ (inches)
$M_x = Wl + W_1l_1$ $M = d\,.\,mn$	$\frac{Wl + W_1l_1}{mn}$	This is a simple case of combining two such bending-moment diagrams as we had above. The lower one is tilted up from the diagram shown in dotted lines.

Cantilever with an evenly distributed load of w lbs. per *inch* run.

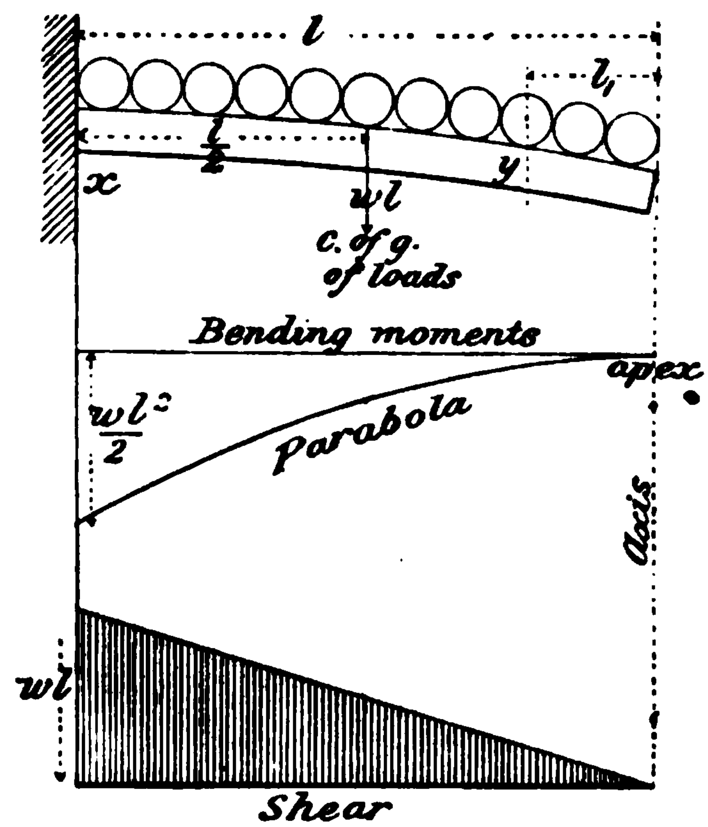
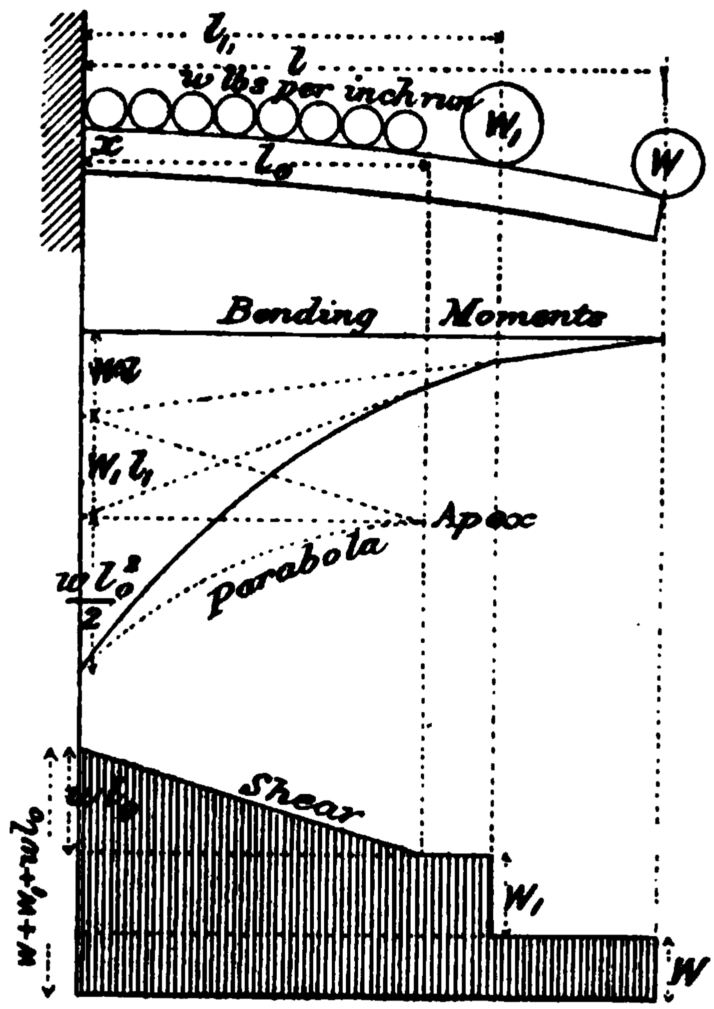
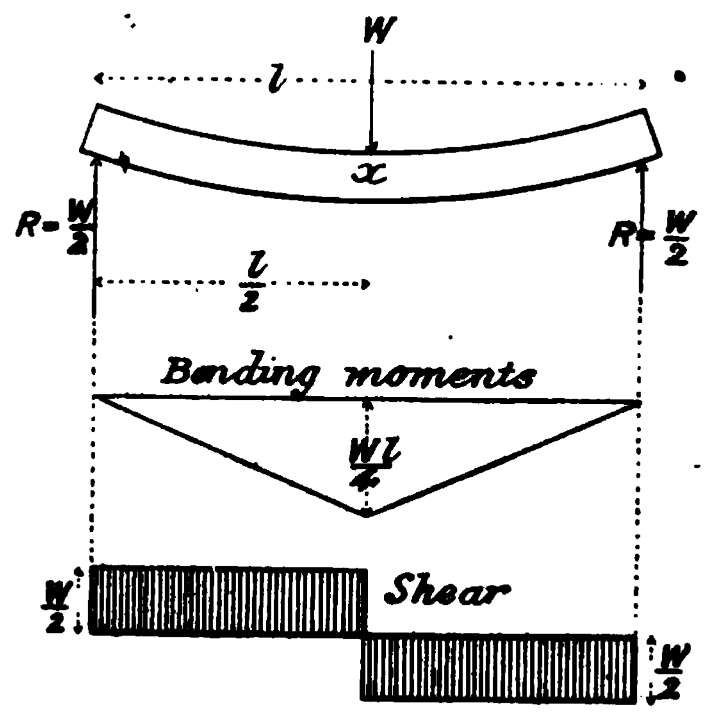


FIG. 399.

Bending moment M in lbs.-inches.	Depth of bending-moment diagram in inches. Scale of W, m lbs. = 1 inch. Scale of l, $\frac{1}{n}$ full size.	REMARKS.
$M_x = \frac{wl^2}{2}$ $\frac{w\,(\text{lbs.})}{\text{inches}} \times \frac{l^2\,(\text{inches})^2}{2\,(\text{constant})}$ $= wl^2 \frac{(\text{lbs. inches})}{\text{constant}}$ Let $W = wl$ $M_x = \frac{Wl}{2}$ $M = d\,.\,mn$	$\frac{Wl}{2mn}$	In statics any system of forces may always be replaced by their resultant, which in this case is situated at the centre of gravity of the loads ; and as the distribution of the loading is uniform, the resultant acts at a distance $\frac{l}{2}$ from x. The total load on the beam is wl, or W ; hence the bending moment at $x = wl \times \frac{l}{2} = \frac{wl^2}{2}$. At any other section, $y = wl_1 \times \frac{l_1}{2} = \frac{wl_1^2}{2}$. Thus the bending moment at any section varies as the square of the distance of the section from the free end of the beam, therefore the bending moment diagram is a parabola. As the beam is fully covered with loads, the sum of the forces to the right of any section varies directly as the length of the beam to the right of the section ; therefore the shearing force at any section varies directly as the distance of that section from the free end of the beam, and the depth of the shearing-force diagram varies in like manner, and is therefore triangular, with the apex at the free end as shown, and the depth at any point distant l_0 from the free end is wl_0, *i.e.* the sum of the loads to the right of l_0. and the area of the shear diagram up to that point is $\frac{wl_0 \times l_0}{2} = \frac{wl_0^2}{2}$, *i.e.* the bending moment at that point.

Cantilever irregularly loaded.	 FIG. 400.
Beam supported at both ends, with a central load.	 FIG. 401.

Bending moment M in lbs.-inches.	Depth of bending-moment diagram in inches. Scale of W, m lbs. = 1 inch. Scale of l. $\frac{1}{n}$ full size.	REMARKS.
$M_x = Wl + W_1 l_1 + \frac{w l_0^2}{2}$ $M = d \cdot mn$	$\frac{Wl + W_1 l_1 + \frac{w l_0^2}{2}}{mn}$	This is simply a case of the combination of the diagrams in Figs. 398 and 399. However complex the loading may be, this method can always be adopted, although the graphic method to be described later on is generally more convenient for many loads.
$M_x = \frac{Wl}{4}$ $M = d \cdot mn$	$\frac{Wl}{4mn}$	Each support or abutment takes one-half the weight $= \frac{W}{2}$. The only moment to the right or left of the section x is $\frac{W}{2} \times \frac{l}{2} = \frac{Wl}{4}$. At any other section the bending moment varies directly as the distance from the abutment; hence the diagram is triangular in form as shown. The only force to the right or left of x is $\frac{W}{2}$; hence the shear diagram is of constant depth as shown, only positive on one side of the section x, and negative on the other side.

Beam supported at both ends, with two symmetrically placed loads.

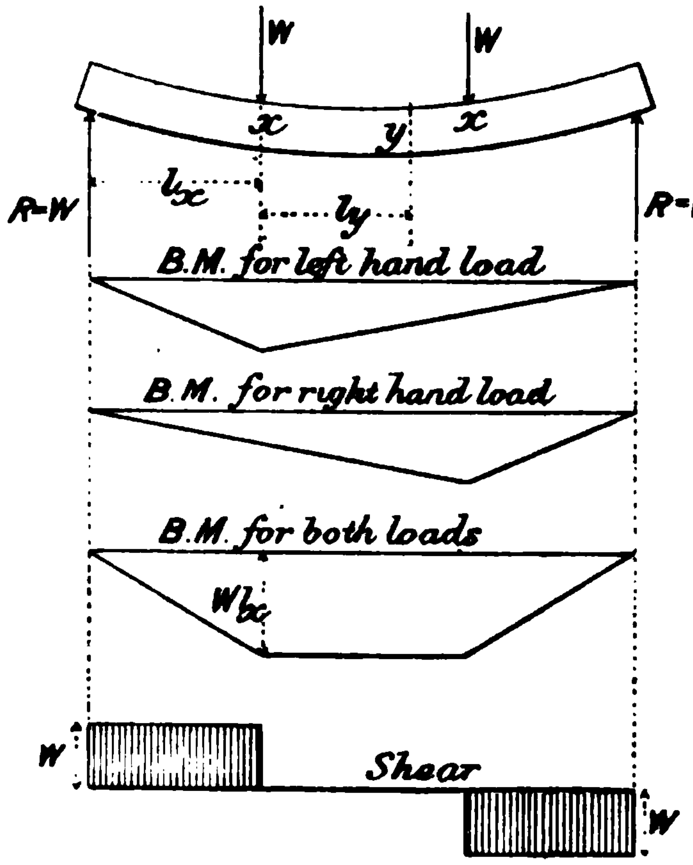


FIG. 402.

Bending moment M in lbs.-inches.	Depth of bending-moment diagram in inches. Scale of W, m lbs. = 1 inch. Scale of l, $\frac{1}{n}$ full size.	REMARKS.
$M_x = Wl_x$ $M_y = Wl_x$ $M = d \cdot mn$	$\frac{Wl_x}{mn}$	The beam being symmetrically loaded, each abutment takes one weight $= W = R$. The only moment to the right of the right-hand section x is $W \cdot l_x$; likewise with the left-hand section. At any other section y between the loads, and distant ly from one of them, we have, taking moments to the left of y, $R(l_x + ly) - W \cdot l_y = R \cdot lx + R \cdot ly - R \cdot l_y = R \cdot l_x$ or $W \cdot l_x$, *i.e.* the bending moment is constant between the two loads. The sum of the forces to the right or left of $y = W - R = 0$, and to the right of the right-hand section the sum of the forces $= R = W$ at every section.

Beam supported at both ends, load evenly distributed, *w* lbs. per *inch* run.

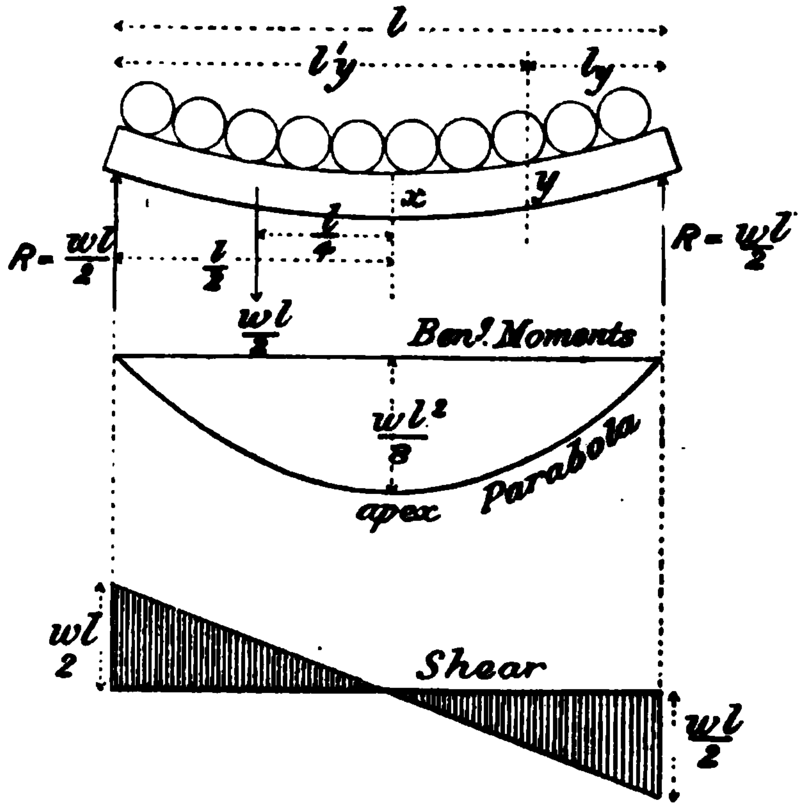


FIG. 403.

Bending moment M in lbs.-inches.	Depth of bending-moment diagram in inches. Scale of W, m lbs. = 1 inch. Scale of l, $\frac{1}{n}$ full size.	REMARKS.
$M_x = \frac{wl^2}{8}$ Let $W = wl$ $M_x = \frac{Wl}{8}$ N.B.—Be very careful to reduce the distributed load to pounds per *inch* run if the dimensions of the beam are in *inches*.	$\frac{Wl}{8mn}$	As in the case of the uniformly loaded cantilever, we must replace the system of forces by their resultant. The load being symmetrically placed, the abutments each take one-half the load $= \frac{wl}{2}$ Then, taking moments to the left of x, we have— $\frac{wl}{2} \times \frac{l}{2} - \frac{wl}{2} \times \frac{l}{4} = \frac{wl^2}{8}$ The $\frac{wl}{2}$ shown midway between x and the abutment is the resultant of the loads on half the beam, acting at the centre of gravity of the load, viz. $\frac{l}{4}$ from x, or the abutment. The bending moment at any other section y, distant l_y from the abutment, is: taking moments to the right of y— $\frac{wl}{2} \times l_y - wl_y \times \frac{l_y}{2} = \frac{wl_y}{2}(l - l_y)$ $= \frac{w}{2}(l_y \times l_y')$ where $l_y' = l - l_y$. Thus the bending moment at any section is proportional to the product of the segments into which the section divides the beam. Hence the bending-moment diagram is a parabola, with its axis vertical and under the middle of the beam as shown. The forces acting to the right of the section $x = \frac{wl}{2} - \frac{wl}{2} = 0$; *i.e.* the shear at the middle section is zero. At the section $y = wl_y - \frac{wl}{2} = w\left(l_y - \frac{l}{2}\right)$. Hence the shear varies inversely as the distance from the abutment, and at the abutment, where $l_y = 0$, it is $= \frac{l}{2}$.

Beam supported at two points equidistant from the ends, and a load of w lbs. per inch run evenly distributed.

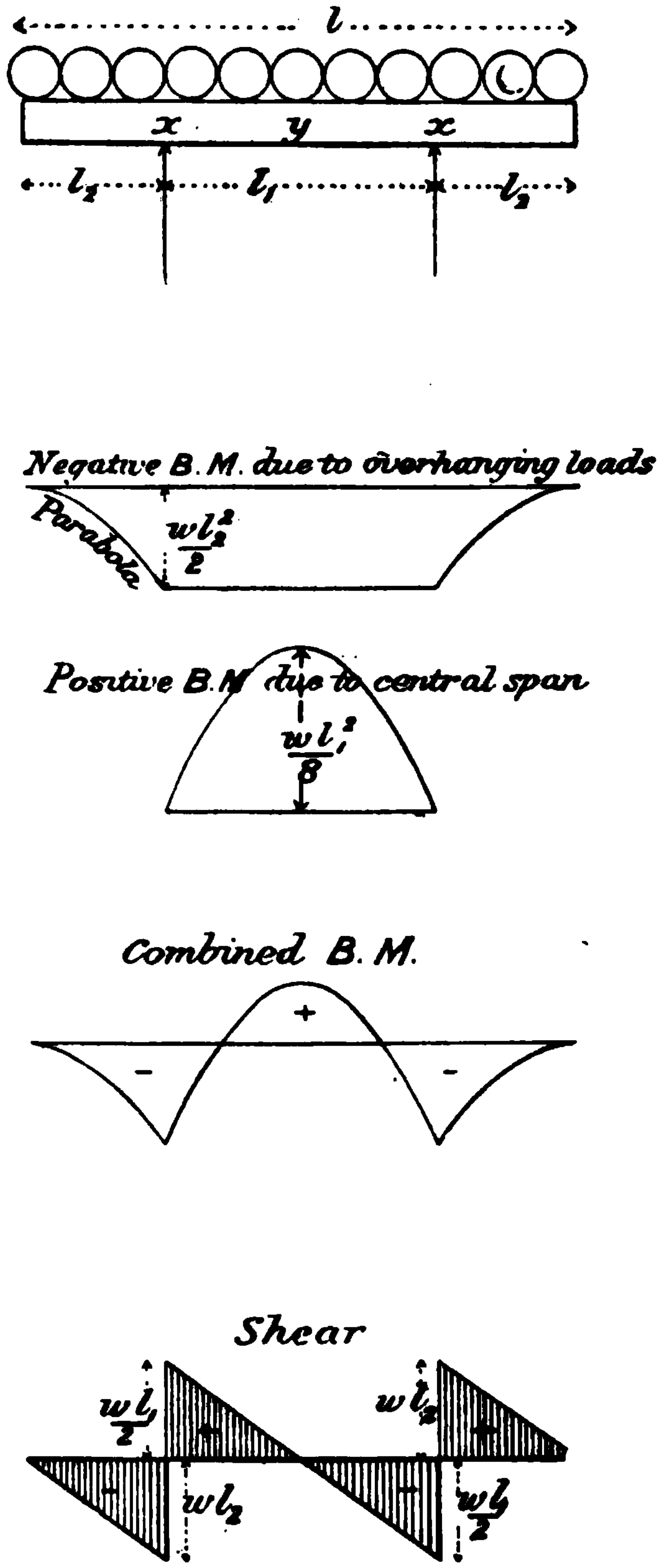


FIG. 404.

Bending moment M in lbs.-inches.	Depth of bending-moment diagram in inches. Scale of W, m lbs. = 1 inch. Scale of l, $\frac{1}{n}$ full size.	REMARKS.
$M_x = -\frac{wl_2^2}{2}$ $M_y = \frac{wl_1^2}{8} - \frac{wl_2^2}{2}$ $= \frac{w}{2}\left(\frac{l_1^2}{4} - l_2^2\right)$	$\frac{M_x}{mn}$ $\frac{M_y}{mn}$	The bending-moment diagram for the loads on the overhanging ends is a combination of Figs. 399 and 402, and the diagram for the load on the central span is simply Fig. 403. Here we see the importance of signs for bending moments. The beam will be subject to the smallest bending moment when $M_x = M_y$; or when $\frac{wl_2^2}{2} = \frac{wl_1^2}{8} - \frac{wl_2^2}{2}$ $wl_2^2 = \frac{wl_1^2}{8} \quad \frac{l_1^2}{l_2^2} = 8$ $l_1 = 2{\cdot}83 l_2$ But $l_1 + 2l_2 = l$ substituting the value of l_1 above $\Big\} = 2{\cdot}83 l_2 + 2l_2 = l$ or $l = 4{\cdot}83 l_2$ or say $l_2 = \frac{1}{5}l$ for the conditions of maximum strength of the beam. The shear diagram will be seen to be a combination of Figs. 399 and 403.

Beam supported at each end and irregularly loaded.

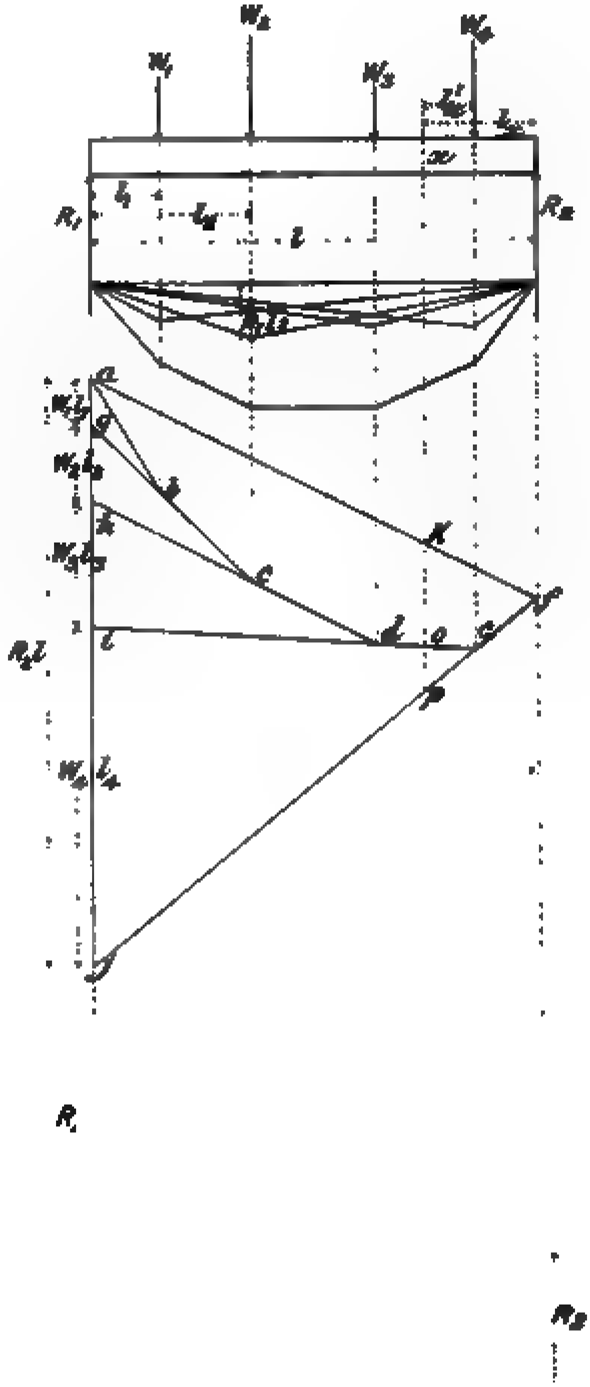


Fig. 405.

Bending moment M in lbs.-inches.	Depth of bending-moment diagram in inches. Scale of W, m lbs. = 1 inch. Scale of l, $\frac{1}{n}$ full size.	REMARKS.
$M = d.mn$	$\frac{W_1 l_1}{mn}$ or $\frac{W_2 l_2}{mn}$ etc.	The method shown in the upper figure is simply that of drawing in the triangular bending-moment diagram for each load treated separately, as in Fig. 401, then adding the ordinates of each to form the final diagram by stepping off with a pair of dividers. In the lower diagram, the heights *ag*, *gh*, etc., are set off on the vertical drawn through the abutment = $W_1 l_1$, $W_2 l_2$, etc., as shown. The sum of these, of course, = $R_2 l$. From the starting-point *a* draw a sloping line *ab*, cutting the vertical through W_1 in the point *b*. Join *gb* and produce to *c*, join *hc* and produce to *d*, and so on, till the point *f* is reached; join *fa*, which completes the bending-moment diagram, the depth of which in inches multiplied by *mn* gives the bending moment. The proof of the construction is as follows: The bending moment at any point *x* is $R_2 l_x - W_4 l'_x$. On the bending-moment diagram $\frac{Kp}{aj} = \frac{l_x}{l}$; or $Kp = \frac{aj \times l_x}{l} = \frac{R_2 l \times l_x}{l} = R_2 l_x$ $Op = \frac{ij \times l'_x}{l_4} = \frac{W_4 l_4 \times l'_x}{l_4} = W_4 l'_x$ and the depth of the bending-moment diagram $= KO = Kp - Op = R_2 l_x - W_4 l'_x$ It will be observed that this construction does not involve the calculation of R_1 and R_2. For the shear diagram R_2 can be obtained thus: Measure off *aj* in inches; then $\frac{aj \times mn}{l} = R_2$, where l is the actual length of the beam in inches.

Beam supported at the middle and irregularly loaded.	Fig. 406

Bending moment M in lbs.-inches.	REMARKS.
	Make the height of the load lines on the beam proportional to the loads, viz. $\frac{W_1}{m}$, $\frac{W_2}{m}$, etc., inches. Drop perpendiculars through each as shown. On a vertical fb set off $fc = \frac{W_1}{m}$, $ed = \frac{W_2}{m}$, etc. Choose any convenient point O distant Oh from the vertical. Oh is termed the "polar distance." Join fO, eO, etc. From any point j on the line passing through R_1 draw a line jm parallel to fO; from m draw mK parallel to eO, and so on, till the line through R_2 is reached in g. Join gj, and draw Oa on the vector polygon parallel to this last line; then the reaction $R_1 = fa$, and $R_2 = ba$. Then the vertical depth of the bending-moment diagram at any given section is proportional to the bending moment at that section.
	Proof.—The two triangles jpm and Oaf are similar, for jm is parallel to fO, and jp to aO, and pm to fa; also jq is drawn at right angles to the base mp. Hence—
$M = m \, . \, n(D \times Oh)$	$$\frac{\text{height of } \triangle jpm}{\text{height of } \triangle Oaf} = \frac{\text{base of } \triangle jpm}{\text{base of } \triangle Oaf}$$ $$\text{or } \frac{jq}{mp} = \frac{Oh}{af}$$ $$\therefore \frac{l_1}{D_x} = \frac{Oh}{R_1}$$
	For $jq = l_1$ and $af = R_1$; and let $mp = D_x$, *i.e.* the depth of the bending-moment diagram at the section x, or $R_1 l_1 = D_x Oh = M_x =$ the bending moment at x.
	By similar reasoning we have— $$R_1 l_2 = rt \times Oh$$ $$\text{also } W_1(l_2 - l_1) = r\text{K} \times Oh$$ $$\text{the bending-moment at } y = M_y = R_1 l_2 - W_1(l_2 - l_1) = Oh(rt - r\text{K}) = Oh(\text{K}t) = Oh \, . \, D_y$$ where D_y = the depth of the bending-moment diagram at the section y.
	Thus the bending moment at any section is equal to the depth of the diagram at that section multiplied by the polar distance, both taken to the proper scales, which we will now determine. The diagram is drawn so that—
	1 inch on the load scale = m lbs. 1 ,, ,, length ,, = n inches.
	Hence the measurements taken from the diagram in inches must be multiplied by mn.
	The bending moment expressed in *lbs.-inches* at any section $\Big\} = M = m \, . \, n \, . \, (D \, . \, Oh)$

Beam supported at two points with overhanging ends and irregularly loaded.

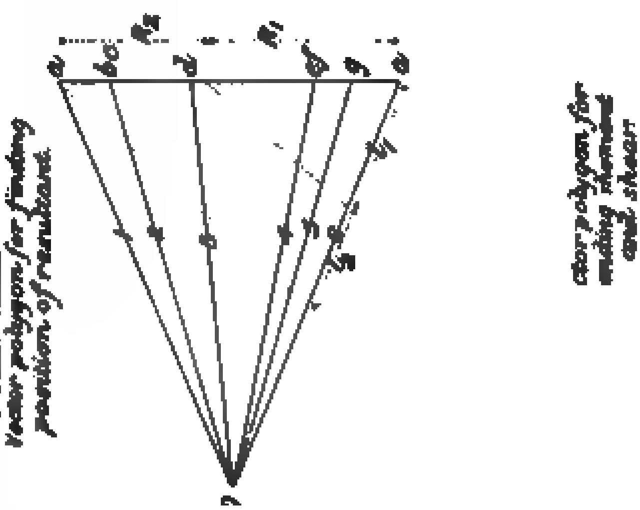


Fig. 407.

Bending moment M in lbs.-inches.

where D is the depth of the diagram in inches at that section, and O*h* is the polar distance in inches.

In Chap. IV. we showed that the resultant of such a system of parallel forces as we have on the beam passes through the meet of the first and last links of the link polygon, viz. through *u*, where *jm* cuts *gh*. Then, as the whole system of loads may be replaced by the resultant, we have $R_1 l_3 = R_2 l_4$. But we have shown above that the triangles *juw* and O*af* are similar; hence $\frac{jv}{Oh} = \frac{uw}{af}$. But $jv = l_3$, therefore $af \times l_3 = Oh \times uw = R_1 l_3$, or $af = R_1$. Similarly it may be shown that $ab = R_2$.

$M = m \, . \, n(D \times Oh)$

Remarks.

The method of constructing this diagram is precisely the same as in the last case. The only points that need be mentioned are—

(1) The two reactions must be determined by calculation, or, better, by finding the resultant of all the loads by the method given in Chap. IV., which is shown on the upper part of the diagram, upside down for convenience. The magnitude of the resultant is, of course, the sum of the loads. When the position has been found, it is a simple matter to find how much is carried by each support, thus: Set off the lengths l_1 and l_2 along the line 6; join *a* on line 1 to the end point of l_1; from the end of l_2 draw a line parallel: this divides the vertical in the ratio R_2 to R_1, because—

$$\frac{R_1}{l_2} = \frac{R_2 + R_1}{l_2 + l_1}$$
$$R_1 l_1 = R_2 l_2$$

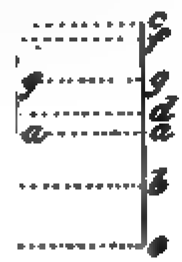


Fig. 407*a*.

(2) The reactions must be arranged as shown on the upper part of the beam, and the loads set off on the vertical line, as in Fig. 407*a*, which simply amounts to setting *down* all loads and setting *up* the reactions.

Care must be taken, in drawing the link polygon, to get the lines in the right spaces. Thus the line *b*O on the vector polygon is parallel to the link in the space *b* in the link polygon; likewise *c*O in the space *c*. The space *a* extends all along the under part of the beam.

The construction of the shear diagram will be evident when it is remembered that the shear at any section is the algebraic sum of the forces to the right or to the left of the section.

This method can be used for all cases of continuous girders resting on three or more supports, but the reactions are not so easily obtained as in the last case. The method of finding them is fully dealt with in Chap. XI.

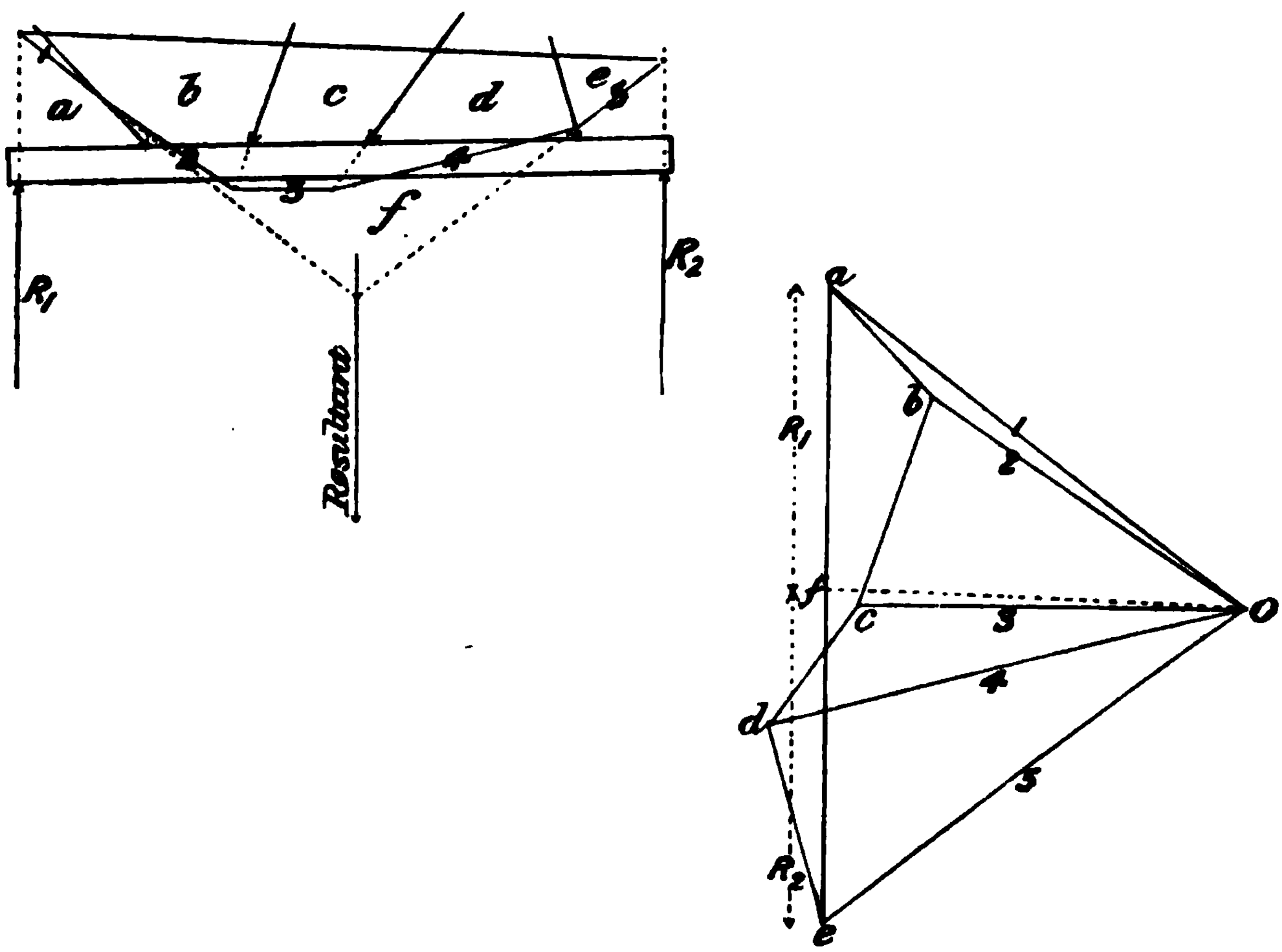


FIG. 408.

The construction of this figure will be clear from the diagram. The closing line of the vector polygon *ae* is the resultant of all the forces acting on the beam, and is the direction of the two reactions, which can be resolved into their vertical and horizontal components. If one end be fixed and the other end be on rollers, the whole of the horizontal component of the resultant must be taken by the abutment at the fixed end. In the case of a beam loaded in this manner, the stress due to bending is not the only stress to be considered (see p. 379). Such a construction is rarely required for a beam ; but we shall require it for roof trusses subjected to wind pressure.

N.B.—It does not follow that the reactions will always be vertical as shown in the figure.

Beam built in at both ends and centrally loaded.

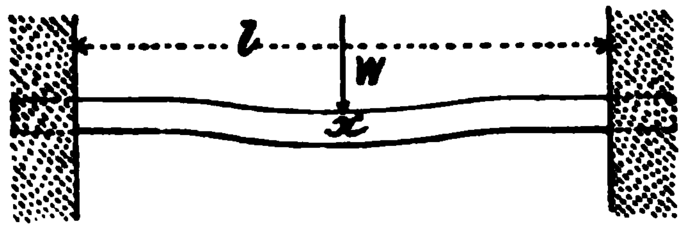


FIG. 409.

Ditto with evenly distributed load.

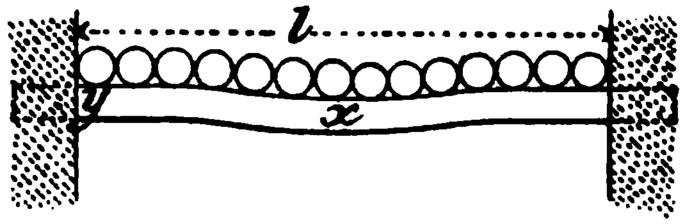


FIG. 410.

Cantilever propped at the outer end with evenly distributed load.

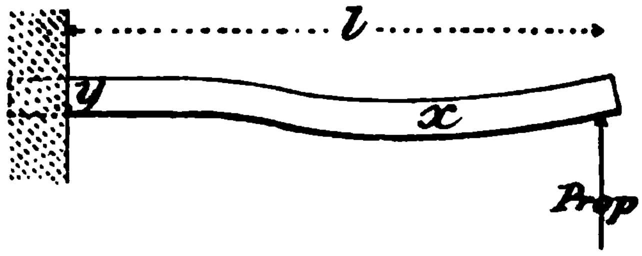


FIG. 411.

Beam built in at both ends, the load applied on one of the ends, which slides parallel to the fixed end.

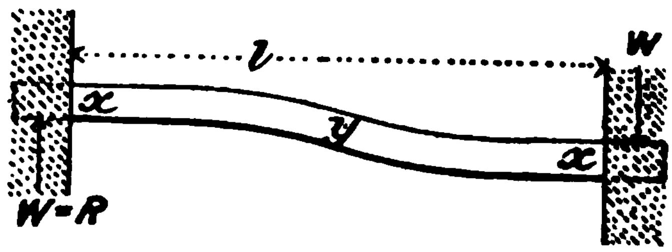


FIG. 412.

Bending moment M in lbs.-inches.	
$M_x = \frac{Wl}{8}$ $M_y = \frac{Wl}{8}$	
$M_x = \frac{wl^2}{24}$ $M_y = \frac{wl^2}{12}$	The determination of these bending moments depends on the elastic properties of the beams, which are fully discussed in Chap. XI. In all these cases the beam is shown built in at both ends. The beams are assumed to be free endwise, and guided so that the ends shall remain horizontal as the beam is bent. If they were rigidly held at both ends, the problem would be much more complex.
$M_x = \frac{wl^2}{16}$ $M_y = \frac{wl^2}{8}$	
$M_x = \frac{Wl}{2}$ $M_y = 0$	

CHAPTER XI.

DEFLECTION OF BEAMS.

Beam bent to the Arc of a Circle.— Let an elastic beam be bent to the arc of a circle, the radius of the neutral axis being ρ. The length of the neutral axis will not alter by the bending. The distance of the skin from the neutral axis $= y$.

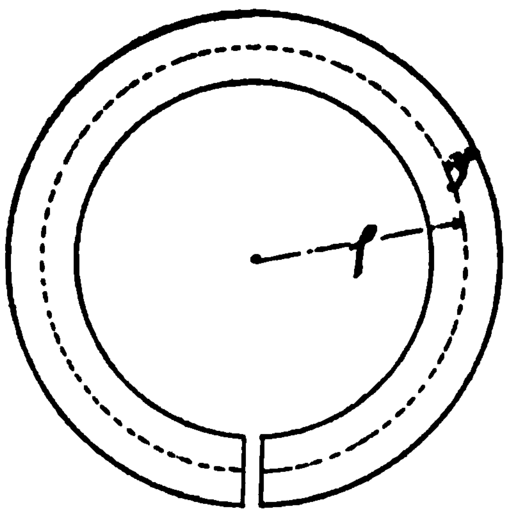


FIG. 413.

$$\left.\begin{array}{l}\text{The original length of}\\ \quad\text{the outer skin}\end{array}\right\} = 2\pi\rho$$

$$\left.\begin{array}{l}\text{the length of the outer}\\ \quad\text{skin after bending}\end{array}\right\} = 2\pi(\rho + y)$$

$$\left.\begin{array}{l}\text{the strain of the skin}\\ \quad\text{due to bending}\end{array}\right\} = 2\pi(\rho + y) - 2\pi\rho = 2\pi y$$

But we have (see p. 280) the following relation :—

$$\frac{\text{strain}}{\text{original length}} = \frac{\text{stress}}{\text{modulus of elasticity}}$$

$$\text{or } \frac{2\pi y}{2\pi\rho} = \frac{y}{\rho} = \frac{f}{\mathrm{E}}$$

But we also have—

$$f = \frac{\mathrm{M}}{\mathrm{Z}}$$

Substituting this value in the above equation—

$$\frac{y}{\rho} = \frac{\mathrm{M}}{\mathrm{EZ}}$$

$$\text{whence } \mathrm{M} = \frac{\mathrm{EZ}y}{\rho}; \text{ or } \mathrm{M} = \frac{\mathrm{EI}}{\rho}$$

Central Deflection of a Beam bent to the Arc of a Circle.—From the figure we have—

$$\rho^2 = (\rho - \delta)^2 + \left(\frac{L}{2}\right)^2$$

$$\text{whence } \delta^2 - 2\rho\delta = -\frac{L^2}{4}$$

The elastic deflection (δ) of a beam is rarely more than $\frac{1}{600}$ of the span (L); hence the δ^2 will not exceed $\frac{L^2}{360,000}$, which is quite negligible;

$$\text{hence } 2\rho\delta = \frac{L^2}{4} \qquad \delta = \frac{L^2}{8\rho}$$

$$\text{But } \rho = \frac{EI}{M}$$

$$\text{hence } \delta = \frac{ML^2}{8EI}$$

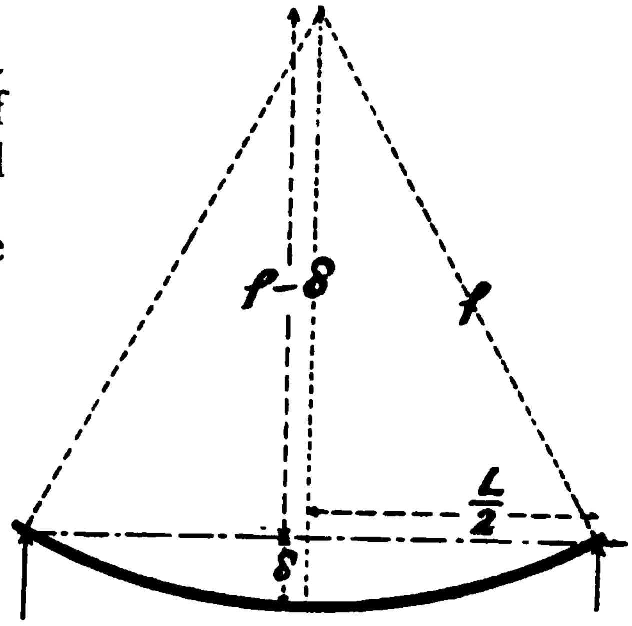


FIG. 414.

We shall shortly give another method for arriving at this result.

General Statement regarding Deflection. — In speaking of the deflection of a cantilever or beam, we always mean the deflection measured from a line drawn tangential to that part of the bent cantilever or beam which remains parallel to its unstrained position. The deflection δ will be seen by referring to the figures shown.

The point t at which the tangent touches the beam we shall term the "tangent point." When dealing with beams, we shall find it convenient to speak of the deflection at the support, *i.e.* the height of the support above the tangent.

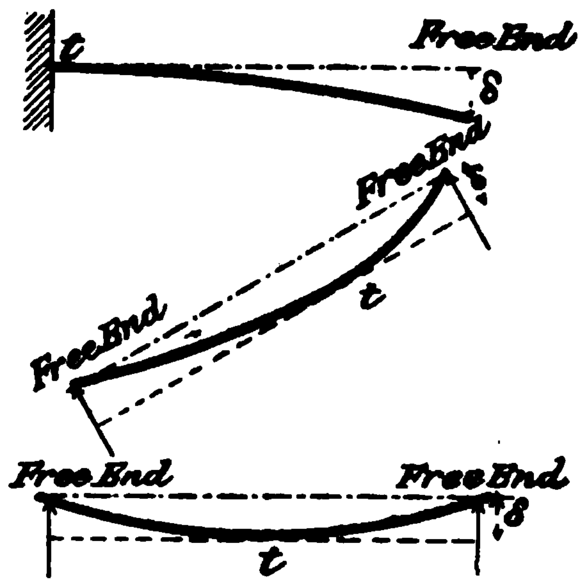


FIG. 415.

Deflection of a Cantilever.—Let the upper diagram (Fig. 416) represent the distribution of bending moment acting on the cantilever, the dark line the bent cantilever, and the straight dotted line the unstrained position of the cantilever. Consider any very small portion yy, distant l_y from the free end of the cantilever. We will suppose the length yy so small that the radius of curvature ρ_y is the same at both points, y, y. Let the angle subtending yy be θ_y (circular measure); then the angle

between the two tangents *ya*, *yb* will also be θ_y. Then the deflection at the extremity of these tangents due to the bending between y, y is—

$$\delta_y = \theta_y \cdot l_y$$

$$\text{But } \theta_y = \frac{\overline{yy}}{\rho_y}$$

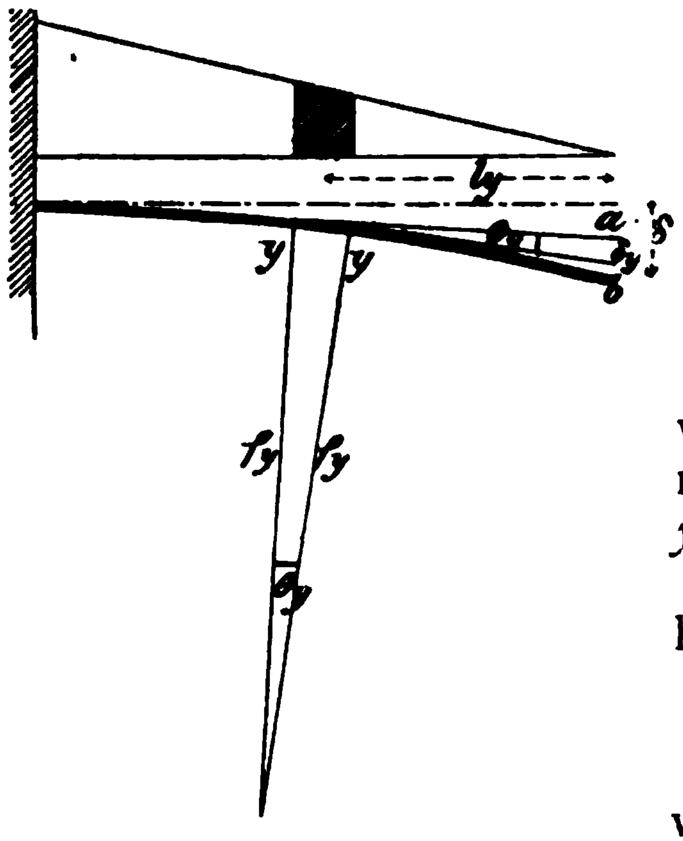


FIG. 416.

and from p. 358, we have—

$$\rho_y = \frac{EI}{M_y}$$

where M_y is the mean bending moment between the points y, y.

Then by substitution, we have—

$$\theta_y = \frac{M_y \cdot \overline{yy}}{EI}$$

where θ_y is the "slope" between the two tangents to the bent beam at $y\ y$;

but $M_y\overline{yy}$ = area (shown shaded) of the bending-moment diagram between y, y

$= A_y$

$$\text{hence } \theta_y = \frac{A_y}{EI}$$

$$\text{and } \delta_y = \frac{A_y l_y}{EI}$$

That is, the deflection at the free end of the cantilever due to the bending between the points y, y is numerically equal to the moment of the portion of the bending-moment diagram over yy about the free end of the cantilever divided by EI. The total deflection at the free end is—

$$\delta = \Sigma(\delta + \delta_x +, \text{etc.})$$

$$\delta = \frac{I}{EI}\Sigma A_y l_y + A_x l_x +, \text{etc.}$$

where the suffix x refers to any other very small portion of the cantilever xx.

Thus the total deflection at the free end of the cantilever is

numerically equal to the sum of the moments of each little element of area of the bending-moment diagram about the free end of the cantilever divided by EI. But, instead of dealing with the moment of each little element of area, we may take the moment of the whole bending-moment diagram about the free end, *i.e.* the area of the diagram × the distance of its centre of gravity from the free end;

$$\text{or } \delta = \frac{AL_c}{EI}$$

where A = the area of the bending-moment diagram;
L_c = the distance of the centre of gravity of the bending-moment diagram from the free end.

To readers familiar with the integral calculus, it will be seen that the length that we have termed *yy* above, is in calculus nomenclature *dl* in the limit, since the total deflection is very small, and the deflection at the free end due to the bending over the elementary length *dl* is—

$$\delta_y = \frac{M_y \cdot l_y \cdot dl}{EI}$$

and the total deflection between points distant L and o from the free end is—

$$\delta = \frac{1}{EI}\int_{l=0}^{l=L} Ml \cdot dl$$

where M = the bending moment at the point distant L from the free end.

In the following cases we shall obtain the deflection by both methods.

CASE I.—*Cantilever with load* W *on free end. Length* L.

$$A = WL \times \frac{L}{2} = \frac{WL^2}{2}$$

$$L_c = \tfrac{2}{3}L$$

$$\delta = \frac{WL^2}{2} \times \tfrac{2}{3}L \times \frac{1}{EI}$$

$$\delta = \frac{WL^3}{3EI}$$

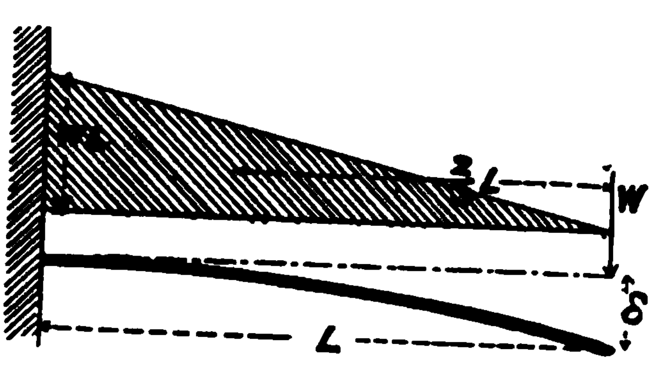


FIG. 417.

or by integration—

$$M = Wl$$

$$\text{hence } \delta = \frac{W}{EI}\int_{l=0}^{l=L} l^2 \, . \, dl = \frac{WL^3}{3EI}$$

CASE II.—*Cantilever with load* W *evenly distributed or w per unit length. Length* L.

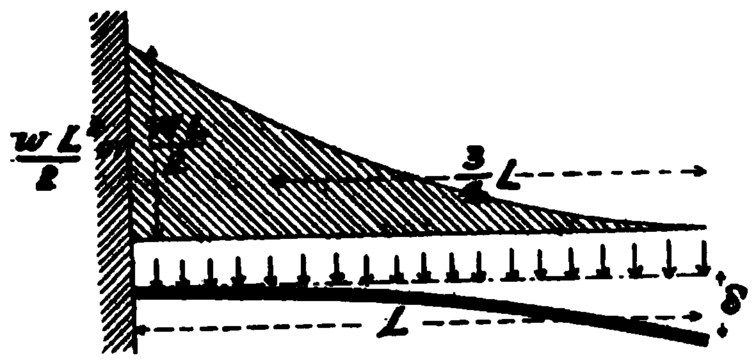


FIG. 418.

$$A = \tfrac{1}{3}\left(\frac{wL^2}{2} \times L\right)$$

$$L_c = \tfrac{3}{4}L$$

$$\delta = \tfrac{1}{3}\left(\frac{wL^2}{2} \times L\right) \times \tfrac{3}{4}L \times \frac{1}{EI}$$

$$\delta = \frac{wL^4}{8EI} = \frac{WL^3}{8EI}$$

or by integration—

$$M = \frac{wl^2}{2}$$

$$\text{hence } \delta = \frac{w}{EI}\int_{l=0}^{l=L} \frac{l^3}{2} dl = \frac{wL^4}{8EI}$$

$$= \frac{WL^3}{8EI}$$

CASE III.—*Cantilever with load unevenly distributed. Length* L.

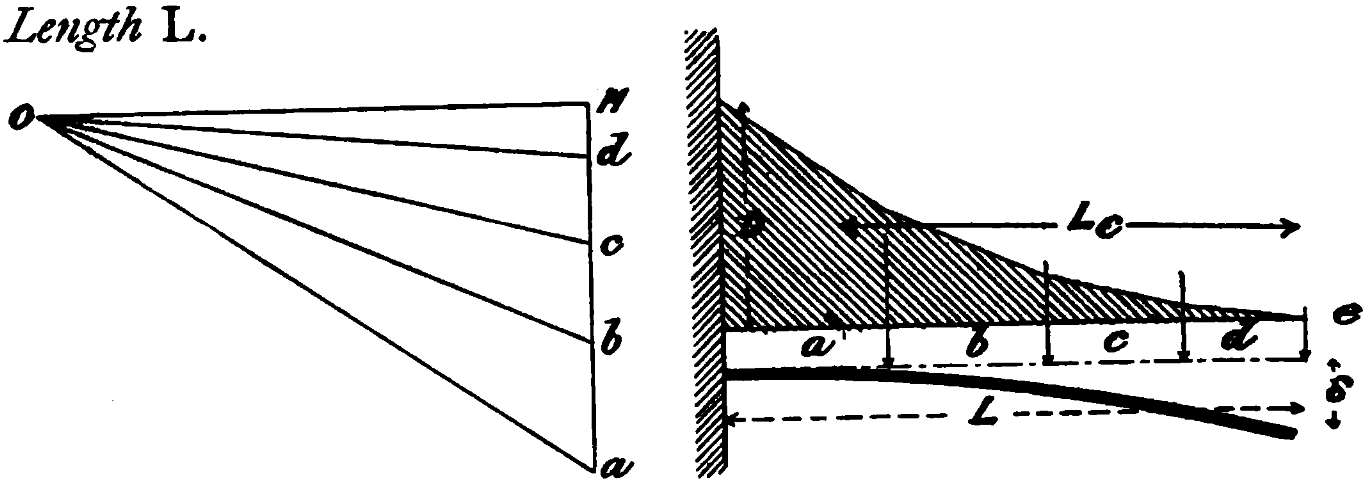


FIG. 419.

Let the bending-moment diagram shown above the cantilever be obtained by the method shown on p. 350.

Then if 1 inch = m lbs. on the load scale;
1 inch = n inches on the length scale;
D = depth of the bending-moment diagram measured in inches;
$\overline{OH}$ = the polar distance in inches;
M = the bending-moment in inch-lbs.;

$$M = m \,.\, n \,.\, D \,.\, \overline{OH};$$

hence 1 inch depth on the bending-moment diagram represents $\frac{M}{D} = m \,.\, n \,.\, \overline{OH}$ inch-lbs.; and 1 square inch of the bending-moment diagram represents $m \,.\, n^2 \,.\, \overline{OH}$ inch-inch-lbs.; hence—

$$\delta = \frac{\left(\begin{array}{c}\text{area of bending-moment}\\ \text{diagram in square inches}\end{array}\right) m \,.\, n^2 . \overline{OH} \times L_c}{EI}$$

The deflection δ found thus will be somewhere between the deflection for a single end load and for an evenly distributed load; generally by inspection it can be seen whether it will approach the one or the other condition. Such a calculation is useful in preventing great errors from creeping in.

In irregularly loaded beams and cantilevers, the deflection cannot conveniently be arrived at by an integration.

Deflection of a Beam freely supported.—Let the lower diagram represent the distribution of bending moment on the beam. The dark line represents the bent beam, and the straight dotted line the unstrained position of the beam. By the same process of reasoning as

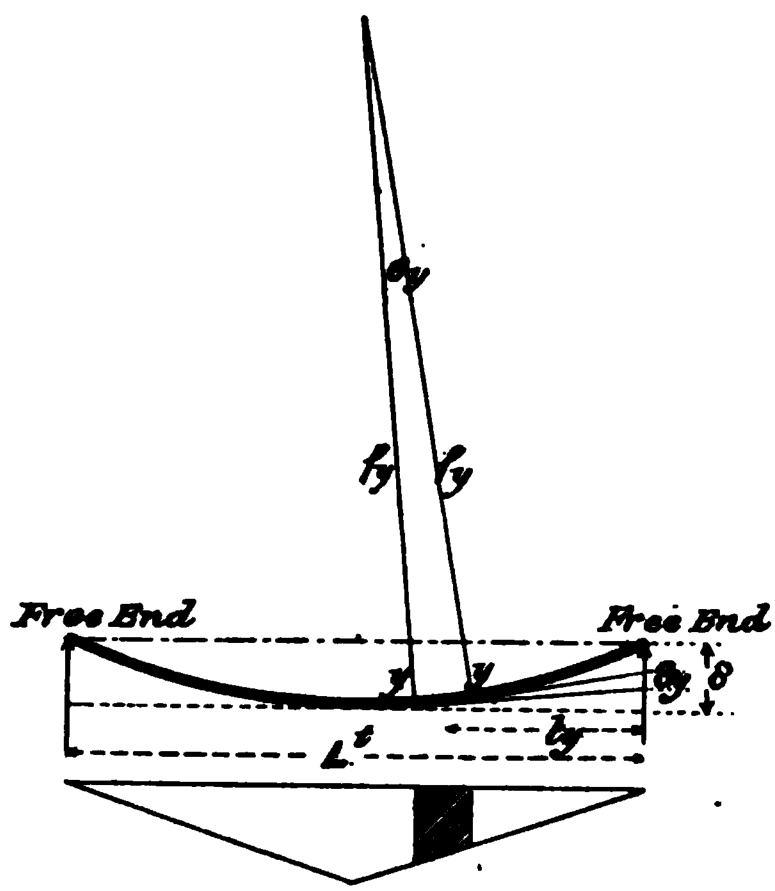


FIG. 420.

in the case of the cantilever, it is readily shown that the deflection of the free end or the support is the sum of the moments of each little area of the bending-moment diagram

between the tangent point and the free end about the free end; or, as before, instead of dealing with the moment of each little area, we may take the moment of the whole area of the bending-moment diagram between the free end and the tangent point, about the free end, *i.e.* the area of the bending-moment diagram between the tangent point and the free end × distance of the centre of gravity of this area from the free end. Then, as before—

$$\delta = \frac{AL_c}{EI}$$

where A and L_c have the slightly modified meanings mentioned above.

CASE IV.—*Beam loaded with central load* W. *Length* L.

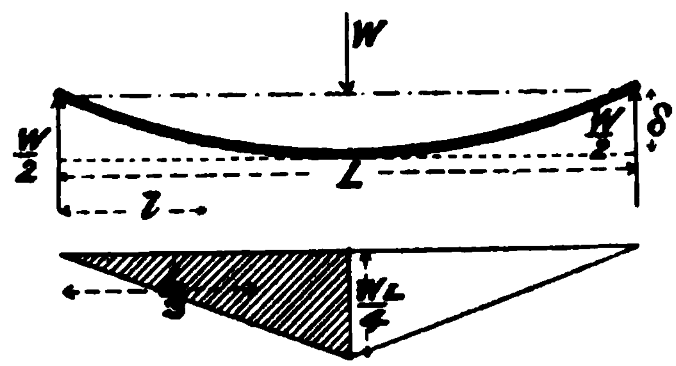


FIG. 421.

$$A = \frac{WL}{4} \times \frac{L}{4}$$

$$L_c = \frac{L}{3}$$

$$\delta = \frac{WL}{4} \times \frac{L}{4} \times \frac{L}{3} \times \frac{1}{EI}$$

$$\delta = \frac{WL^3}{48EI}$$

Or by integration at any point distant l from the support—

$$M = \frac{W}{2}l$$

$$\delta = \frac{W}{EI}\int_0^l \frac{l^2 . dl}{2} = \frac{W}{EI}\left(\frac{l^3}{6}\right)$$

When $l = \frac{L}{2}$, we have—

$$\delta = \frac{WL^3}{2^3 \times 6EI} = \frac{WL^3}{48EI}$$

CASE V.—*Beam loaded with an evenly distributed load w per unit length. Length* L.

$$A = \frac{2}{3}\left(\frac{wL^2}{8} \times \frac{L}{2}\right) = \frac{wL^3}{24}$$

$$L_c = \frac{5L}{16}$$

$$\delta = \frac{wL^3}{24} \times \frac{5L}{16} \times \frac{1}{EI}$$

$$\delta = \frac{5wL^4}{384EI} = \frac{5WL^3}{384EI}$$

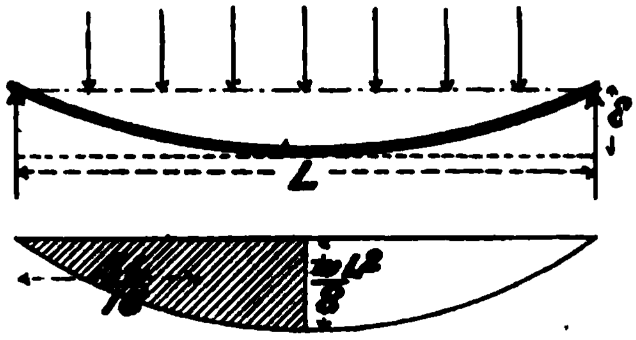


FIG. 422.

Or by integration, at any point distant l from the support (see p. 345)—

$$M = \frac{w}{2}(lL - l^2)$$

$$\delta = \frac{w}{2EI}\int_0^l (l^2L - l^3)dl$$

$$\delta = \frac{w}{2EI}\left(\frac{l^3L}{3} - \frac{l^4}{4}\right)$$

When $l = \frac{L}{2}$, we have—

$$\delta = \frac{w}{2EI}\left(\frac{L^4}{24} - \frac{L^4}{64}\right)$$

$$\delta = \frac{5wL^4}{384EI} = \frac{5WL^3}{384EI}$$

CASE VI. *Beam loaded with two equal weights symmetrically placed.*—By taking the moments of the triangular area *abc* and the rectangle *bced*, the deflection becomes—

$$\delta = \frac{WL_1}{24EI}(12L_1L_2 + 3L_2^2 + 8\,L_1^2)$$

and when $L_1 = L_2 = \frac{L}{3}$, this expression becomes—

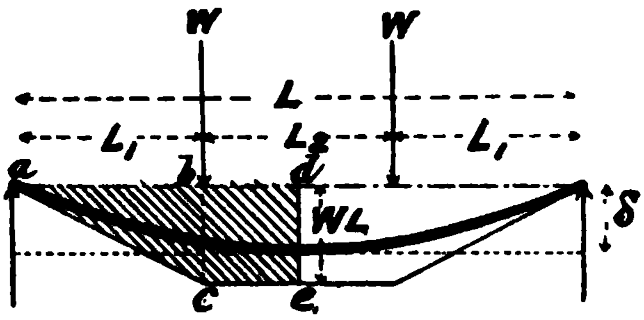


FIG. 423.

$$\delta = \frac{23WL^3}{648EI} = \frac{WL^3}{28EI} \text{ (nearly)}$$

or if W_0 be the *total* load—

$$\delta = \frac{W_0L^3}{56EI} \text{ (nearly)}$$

CASE VII. *Beam unevenly loaded.*—Let the beam be loaded as shown. Construct the bending-moment diagram shown below the beam by the method given on p. 350. Then the bending-moment at any section is $M = m \cdot n \cdot d \cdot \overline{OH}$ inch-lbs., using the same notation as on p. 363. Then 1 inch on the vertical scale of the bending-moment diagram $= \frac{M}{d} = m \cdot n \cdot \overline{OH}$ inch-lbs., and 1 inch on the horizontal scale $= n$ inches. Hence one square inch on the diagram $= m \cdot n^2\overline{OH}$ inch-lbs.

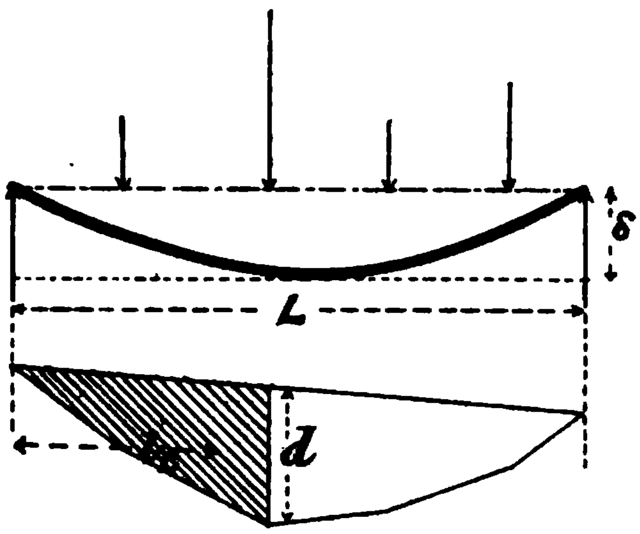


FIG. 424.

Then $A = a \cdot m \cdot n^2\overline{OH}$, where $a =$ the shaded area measured in square inches.

The centre of gravity must be found by one of the methods described in Chap. III. Then $L_c = n \cdot l_c$, where l_c is measured in inches, and the deflection—

$$\delta = \frac{AL_c}{EI} = \frac{a \cdot m \cdot n^3\overline{OH} \cdot l_c}{EI}$$

It is evident that the height of the supports above the tangent is the same at both ends. Hence the moment of the areas about the supports on either side of the tangent point must be the same. The point of maximum deflection must be

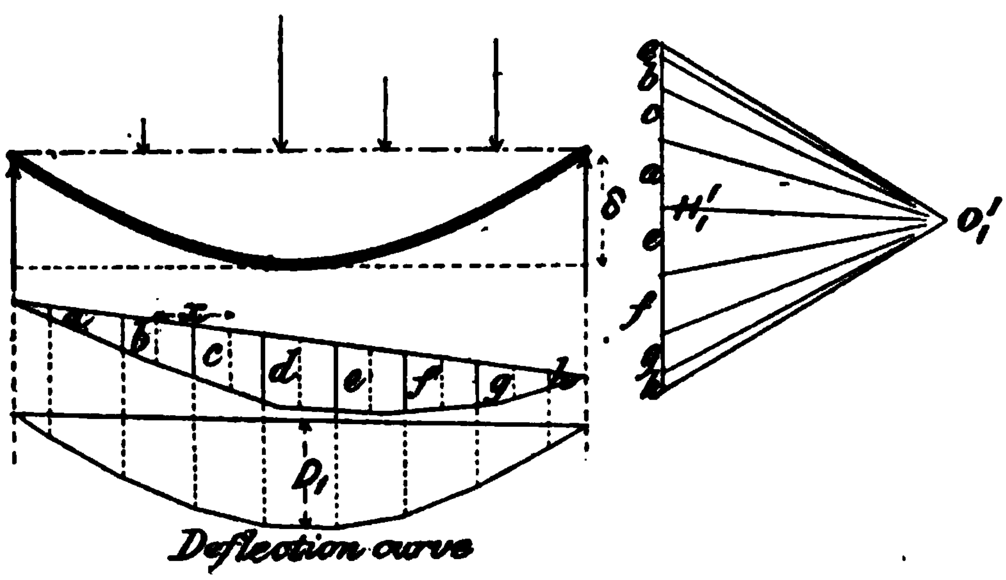


FIG. 425.

found in this way by a series of trials and errors, which is very clumsy.

The deflection may be more conveniently found by a somewhat different process, as in Fig. 425.

We showed above that the deflection is numerically equal to the moment of each little element of area of the bending-moment diagram about the free end $\div$ EI. The moment of each portion of the bending-moment diagram may be found readily by a link-and-vector polygon, similar to that employed for the bending-moment diagram itself.

Treat the bending-moment diagram as a load diagram; split it up into narrow strips of width x, as shown by the dotted lines; draw the middle ordinate of each, as shown in full lines: then any given ordinate $\times$ by x is the area of the strip. Set down these ordinates on a vertical line as shown; choose a pole O′, and complete both polygons as in previous examples. The link polygon thus constructed gives the *form* of the bent beam; this is then reproduced to a horizontal base-line, and gives the bent beam shown in dark lines above. The only point remaining to be determined is the scale of the deflection curve.

We have 1 inch on the load scale of the first bending-moment diagram $\Big\} = m$ lbs.

also 1 inch on the length of the bending-moment diagram $\Big\} = n$ inches

and the bending moment at any point $M = m \,.\, n \,.\, D \,.\, \overline{OH}$

Where D is the depth of the bending-moment diagram at the point in inches, and $\overline{OH}$ is the polar distance, also expressed in inches. Hence 1 inch depth of the bending-moment diagram represents $\dfrac{M}{D} = mn \,.\, \overline{OH}$ inch-lbs., and 1 square inch of the bending-moment diagram represents $mn^2\overline{OH}$ inch-inch-lbs. Hence the area xD represents $xDmn^2\overline{OH}$ inch-inch-lbs.; but as this area is represented on the second vector polygon by D, its scale is $xmn^2\overline{OH}$; hence—

$$\delta = \frac{xmn^2\overline{OH}n\overline{D_1}\,\overline{O_1H_1}}{EI}$$

$$= \frac{xmn^3\overline{D_1}\,\overline{OH}\,\overline{O_1H_1}}{EI}$$

If it be found convenient to reduce the vertical ordinates of the bending-moment diagram when constructing the deflection

vector polygon by say $\frac{1}{r}$, then the above expression must be multiplied by r.

Deflection of Built-in Beams.—When a beam is built in at one end only, it bends down with a convex curvature *upwards* (Fig. 426); but when it is supported at both ends, it bends with a convex curvature *downwards* (Fig. 427);

FIG. 426.

FIG. 427.

and when a beam is built in at both ends (Fig. 428), we get a combined curvature, thus:

FIG. 428.

FIG. 429.

FIG. 430.

Then considering the one kind of curvature as positive and the other kind as negative, the curvature will be zero at the points xx (Fig. 429), at which it changes sign; such are termed "points of contrary flexture." As the beam undergoes no bending at these points, the bending moment is zero. Thus the beam may be regarded as a short central beam with free ends resting on short cantilevers, as shown in Fig. 430.

Hence, in order to determine the strength and deflection of built-in beams, we must calculate first the positions of the points x, x. It is evident that they occur at the points at which the upward slope of the beam is equal to the downward slope of the cantilever.

We showed above that the slope of a beam or cantilever at any point is given by the expression—

$$\text{Slope} = \frac{A}{EI}$$

CASE VIII. *Beam built in at both ends, with central load.*

$$A \text{ for cantilever} = \frac{WL_1}{2} \times \frac{L_1}{2} = \frac{WL_1^2}{4}$$

$$A \text{ for beam} = \frac{WL_2}{2} \times \frac{L_2}{2} = \frac{WL_2^2}{4}$$

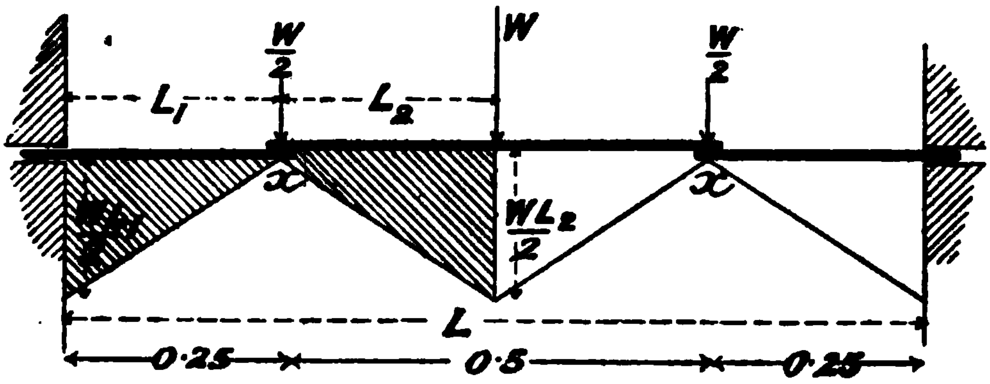


FIG. 431.

Hence, as the slope is the same at the point where the beam joins the cantilever, we have—

$$\frac{WL_1^2}{4} = \frac{WL_2^2}{4}, \text{ or } L_1 = L_2 = \frac{L}{4}$$

Maximum bending moment in middle of central span—

$$\frac{WL_2}{2} = \frac{WL}{8}$$

Maximum bending moment on cantilever spans—

$$\frac{WL_1}{2} = \frac{WL}{8}$$

Deflection of central span—

$$\delta_2 = \frac{W(2L_2)^3}{48EI} = \frac{W\left(\frac{L}{2}\right)}{48EI} = \frac{WL^3}{384EI}$$

Deflection of cantilever—

$$\delta_1 = \frac{\frac{W}{2}L_1^3}{3EI} = \frac{\frac{W}{2}\left(\frac{L}{4}\right)^3}{3EI} = \frac{WL^3}{384EI}$$

Total deflection in middle of central span—

$$\delta = \delta_2 + \delta_1 = \frac{WL^3}{192EI}$$

Thus when the ends are built in, the maximum bending moment is reduced to one-half, and the deflection to one-quarter, of what it would have been with free ends.

CASE IX.—*Beam built in at both ends, with a uniformly distributed load.*

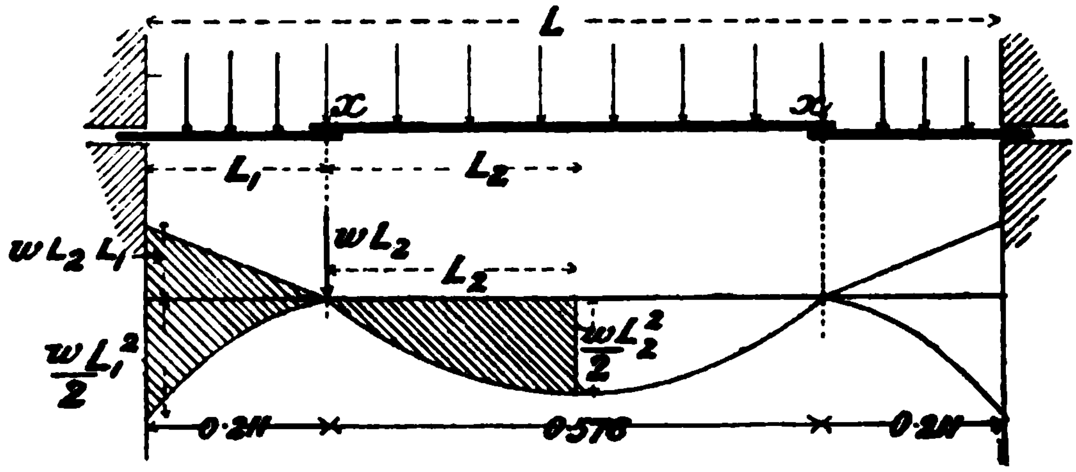


FIG. 432.

A for cantilever—

$$\left(\frac{wL_2L_1}{2} + \frac{wL_1^2}{6}\right)L_1$$

A for beam—

$$\frac{wL_2^2}{2} \times \frac{2}{3}L_2 = \frac{wL_2^3}{3}$$

These must be equal, as explained above—

$$\frac{wL_2^3}{3} = \frac{wL_1^2L_2}{2} + \frac{wL_1^3}{6}$$

Let $L_1 = nL_2$.

$$\text{Then } \frac{L_2^3}{3} = \frac{n^2L_2^3}{2} + \frac{n^3L_2^3}{6}$$

$$2 = 3n^2 + n^3$$

which on solving gives us $n = 0·732$.

We also have—

$$L_1 + L_2 = \frac{L}{2}$$

$$\text{or } 1·732L_2 = \frac{L}{2}$$

$$L_2 = 0·289L$$

$$\text{and } L_1 = 0·732 \times 0·289L = 0·211L$$

Maximum bending moment in middle of central span—

$$\frac{wL_2^2}{2} = \frac{w \times 0{\cdot}289^2L^2}{2} = \frac{wL^2}{24}$$

Maximum bending moment on cantilever spans—

$$wL_2L_1 + \frac{wL_1^2}{2} = w \times 0{\cdot}289L \times 0{\cdot}211L + \frac{w \times 0{\cdot}211^2L^2}{2}$$
$$= \frac{wL^2}{12}$$

Deflection of central span—

$$\delta_2 = \frac{5w(0{\cdot}578L)^4}{384EI} = \frac{WL^4}{689EI}$$

Deflection of cantilevers due to distributed load—

$$\delta_1 = \frac{w(0{\cdot}211L^4)}{8EI} = \frac{wL^4}{4038EI}$$

Deflection due to half-load on central part—

$$\delta_0 = \frac{wL_2 \times L_1^3}{3EI} = \frac{w \times 0{\cdot}289L \times 0{\cdot}211^3L^3}{3EI}$$
$$= \frac{wL^4}{1105EI}$$

Total deflection in middle of central span—

$$\delta = \delta_2 + \delta_1 + \delta_0 = \frac{wL^4}{384EI}$$

Thus, when the ends are built in and free to slide sideways, the maximum bending moment on a uniformly loaded beam is reduced to $\frac{8}{12} = \frac{2}{3}$, and the deflection to $\frac{1}{5}$ of what it would have been with free ends.

Beams supported at more than Two Points.—When a beam rests on three or more supports, it is termed a continuous beam. We shall only treat a few of the simplest cases in order to show the principle involved.

CASE X. *Beam resting on three supports, load evenly distributed.*—The proportion of the load carried by each support entirely depends upon their relative heights. If the central support or prop be so low that it only just touches the

beam, the end supports will take the whole of the load. Likewise, if it be so high that the ends of the beam only just touch the end supports, the central support will take the whole of the load.

The deflection of an elastic beam is strictly proportional to the load. Hence from the deflection we can readily find the load.

$$\left.\begin{array}{r}\text{The deflection in the middle}\\ \text{when not propped}\end{array}\right\} = \delta = \frac{5WL^3}{384EI}$$

Let W_1 be the load on the central prop.

$$\text{Then the } \textit{upward} \text{ deflection due to } W_1 = \delta_1 = \frac{W_1L^3}{48EI}$$

If the top of the three supports be in one straight line, the upward deflection due to W_1 must be equal to the downward deflection due to W, the distributed load; then we have—

$$\frac{5WL^3}{384EI} = \frac{W_1L^3}{48EI}$$

$$\text{whence } W_1 = \tfrac{5}{8}W$$

Thus the central support or prop takes $\frac{5}{8}$ of the whole

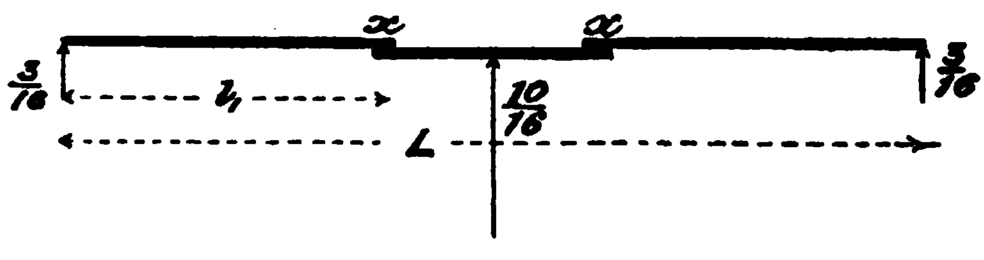


FIG. 433.

load; and as the load is evenly distributed, each of the end supports takes one-half of the remainder, viz. $\frac{3}{16}$ of the load.

The bending moment at any point x distant l_1 from the end support is—

$$M_x = \tfrac{3}{16}wLl_1 - wl_1 \times \frac{l_1}{2}$$

$$= wl_1\left(\tfrac{3}{16}L - \frac{l_1}{2}\right) = \frac{wl_1}{16}(3L - 8l_1)$$

The points of contrary flexure occur at the points where the bending moment is zero, *i.e.* when—

$$\frac{wl_1}{16}(3L - 8l_1) = 0$$
$$\text{or when } 3L = 8l_1$$
$$\text{or } l_1 = \tfrac{3}{8}L$$

Thus the length of the middle span is $\frac{L}{4}$. It is readily shown, by the methods used in previous paragraphs, that the maximum bending-moment occurs over the middle prop, and is there $\frac{WL^2}{32}$, or $\frac{1}{4}$ as great as when not propped.

The load on the prop may also be arrived at by another method, which is also applicable to cases of more than one prop, where both the spans and loads are uneven, and, further, where the supports are not all on a level.

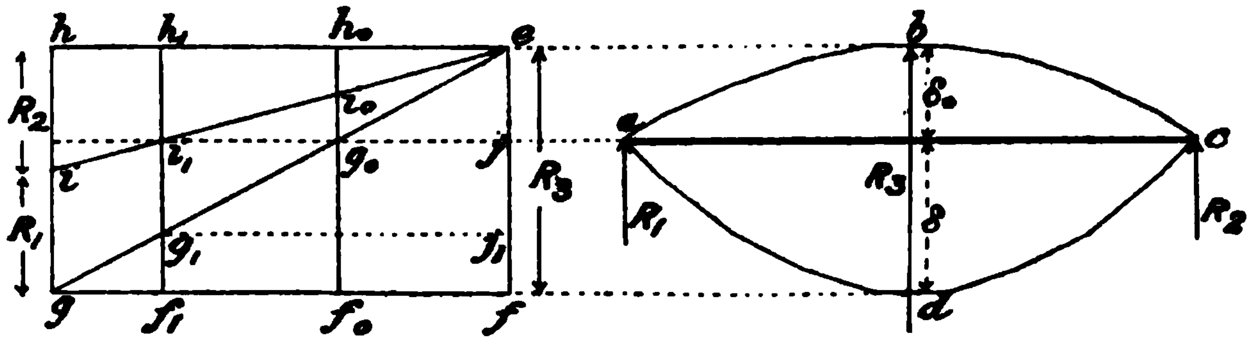


FIG. 434.

In Fig. 434, let *abc* represent to an exaggerated scale of deflection the form of the beam when resting on the central prop only.

$$\text{The deflection } \delta_0 = \frac{w\left(\frac{L}{2}\right)^4}{8EI} = \frac{wL^4}{128EI}$$

Likewise, let *adc* represent on the same scale the form of the beam when supported at the two ends only.

$$\text{The deflection } \delta = \frac{5wL^4}{384EI}$$
$$\text{hence } \delta_0 = \tfrac{3}{5}\delta$$

If the central prop be raised above the end supports by an

amount δ_0, it will take the whole of the load, and will entirely relieve the end supports. On the other hand, if the central prop be lowered below the end supports by an amount δ, it will be entirely relieved of all load, and the two end supports will take the whole of the load—one-half each, as the load is evenly distributed. At intermediate positions the load will be distributed over the three supports. The proportion carried by the central prop will entirely depend upon its height relatively to the others; this proportion is readily obtained by the diagram to the left. The deflections δ and δ_0 are projected on to a vertical line ef, which represents the load supported by R_3 when the prop is pushed up to the point e. Horizontal lines of any convenient length are drawn from e and f, and the rectangle $efgh$ completed. Then gh is bisected in i, and each half represents the proportion of the load carried by the end supports when the prop is removed or lowered to f. If the prop be pushed up until it is level with the end supports, viz. to j, from j draw a horizontal line to meet ge in g_0; at g_0 erect a perpendicular: then the proportion of the load on the prop is f_0g_0, and on the end supports g_0i_0 and i_0h_0, where f_0h_0 is the whole load.

We have shown that $\delta_0 = \frac{3}{5}\delta$; hence—

$$g_0h_0 = \frac{3}{5}f_0g_0, \text{ or } f_0g_0 = \frac{5}{8}f_0h_0$$

or $\frac{5}{8}$ of the whole load is taken by the prop, and $\frac{3}{16}$ of the load by the end supports.

If the prop be lowered to j_1, then $\dfrac{f_1g_1}{f_1h_1}$ is the proportion of the load on the prop, and $\dfrac{g_1i_1}{f_1h_1}$ or $\dfrac{i_1h_1}{f_1h_1}$ the proportion on the end supports. Similarly for any other height of prop.

There is in reality no need to put in the curves abc, adc; all that is required is the ratio $\dfrac{\delta}{\delta_0}$. Then any line ef is divided in this ratio to find the point j corresponding to all three supports being on one level.

CASE XI. *Beam with the load unevenly distributed, with an uncentral prop.*—Construct the bending moment and deflection curves for the beam when supported at the ends only (Fig. 435).

Then, retaining the same scales, construct similar curves for the beam when supported by the prop only (Fig. 436). If, due

to the uneven distribution of the load, the beam does not balance on its prop, we must find what force must be applied at one end of the beam in order to balance it. The unbalanced moment is

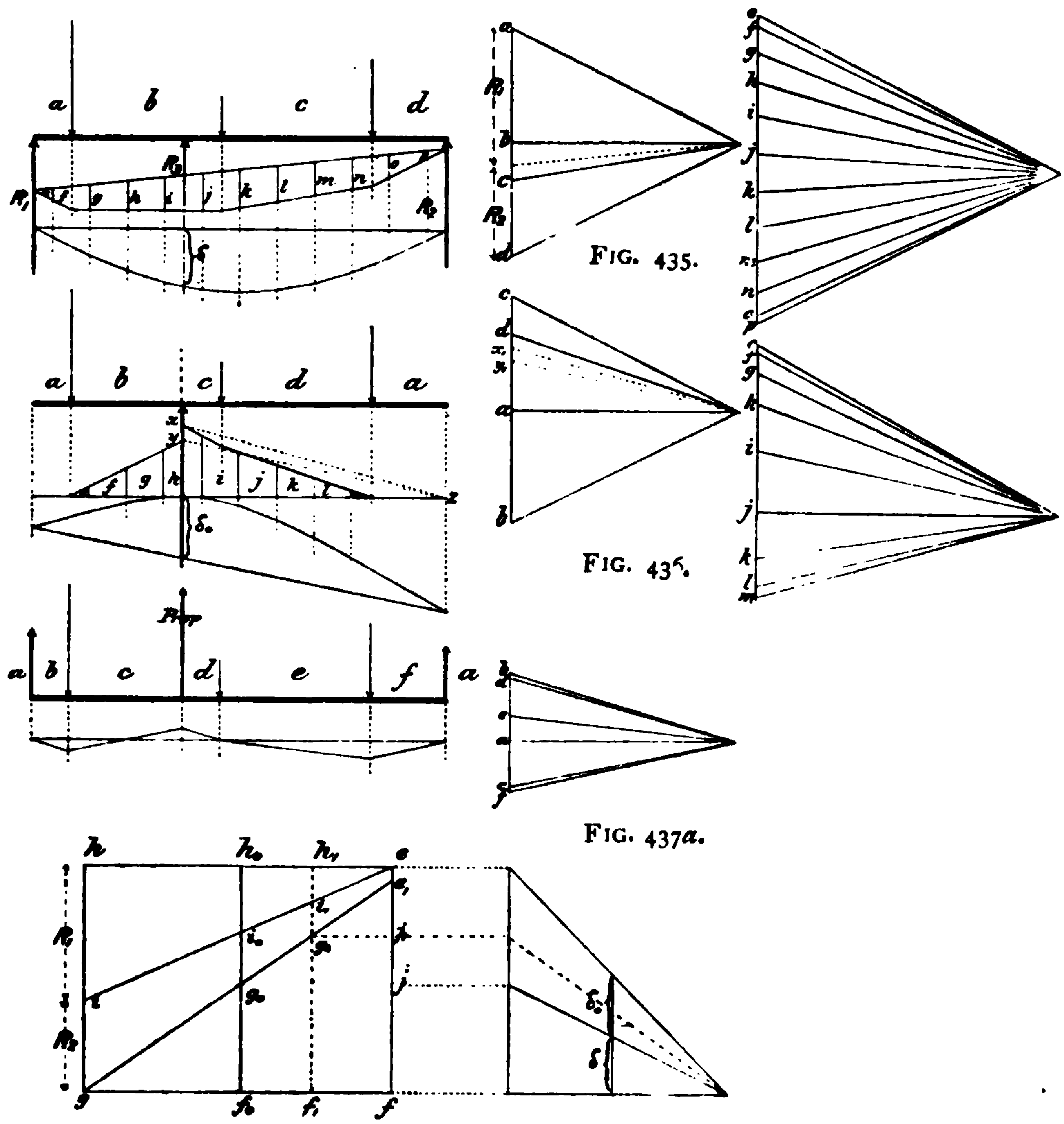


FIG. 435.

FIG. 436.

FIG. 437*a*.

FIG. 437.

shown by xy (Fig. 436). In order to find the force required at z to balance this, join xz and yz, and from the pole of the vector polygon draw lines parallel to them; then the intercept $x_1y_1 = W_1$ on the vertical load line gives the required force *acting upwards* (in this case).

In Fig. 437 set off δ and δ_0 on a vertical. If too small to be conveniently dealt with, increase by the method shown to ej, jf, and construct the rectangle $efgh$ as described in the last case. If the prop be lowered so that the beam only just touches it,

the whole load will come on the end supports; the proportion on each is obtained from R_1 and R_2 in Fig. 435. Divide gh in i in this proportion.

As the prop is pushed up, the two ends keep on the end supports until the deflection becomes $\delta + \delta_0$; at that instant the reaction R_1 becomes zero just as the beam end is about to lift off the support, but the other reaction R_2 supports the unbalanced force W_1. This is shown in the diagram by $ee_1 = W_1$ to same scale as R_1 and R_2.

Join ie and ge_1; then, if the three supports be level, the prop will be at the height j. Draw a horizontal from j to meet ge_1 in g_0; erect a perpendicular. Then the proportion of the load taken by the prop is $\frac{f_0g_0}{f_0h_0}$, by the support R_1 is $\frac{i_0h_0}{f_0h_0}$, by the support R_2 is $\frac{g_0i_0}{f_0h_0}$.

Likewise, if the prop be raised to a height corresponding to j_1, the proportions will be as above, with the altered suffixes to the letters.

In Fig. 437, *a*, we have the final bending-moment diagram for the propped beam when all the supports are level; comparing it with Fig. 435 it will be seen how greatly a prop assists in reducing the bending moment.

It should be noted that in the above constructions there is no need to trouble about the scale of the deflections in the case of level supports, only when the prop is raised or lowered above or below the end supports.

This method, which the author believes to be new, is equally applicable to continuous beams of any number of spans, but space will not allow of any further cases being given.

Stiffness of Beams.—The ratio $\frac{\text{deflection}}{\text{span}}$ is termed the "stiffness" of a beam. This ratio varies from about $\frac{1}{2000}$ in the best English practice for bridge work; it is often as low as $\frac{1}{500}$ for small girders and rolled joists.

By comparing the formulas given above for the deflection, it will be seen that it may be expressed thus:

$$\delta = \frac{ML^2}{nEI}$$

where M is the bending moment, and n is a constant depending on the method of loading.

In the above equation we may substitute fZ for M and Zy for I; then—

$$\delta = \frac{fZL^2}{nEZy} = \frac{fL^2}{nEy}$$

Hence for a stiffness of $\frac{1}{2000}$, we have—

$$\frac{\delta}{L} = \frac{1}{2000} = \frac{fL}{nEy}$$

$$\text{or } 2000\,fL = nEy$$

Let $f = 15{,}000$ lbs. square inch;
$E = 30{,}000{,}000$ „ „

$$\text{Then } ny = L$$

$$\text{But } y = \frac{d}{2}$$

where d = depth of section (for symmetrical sections); then—

$$nd = 2L$$

$$\text{and } \frac{d}{L} = \frac{2}{n}$$

VALUES OF n.

	Beam.	Cantilever.
(*a*) Central load	12	—
End load	—	3
(*b*) Evenly distributed load	9·6	4
(*c*) Two equal symmetrically placed loads dividing beam into three equal parts	9·3	—
(*d*) Irregular loading (approx.)	11	3·5

VALUES OF $\frac{L}{d}$

	Stiffness. $\frac{1}{2000}$.	$\frac{1}{1000}$	$\frac{1}{500}$
Beam, central load	6	12	24
Cantilever, end load	1·5	3	6
Beam, evenly distributed load	4·8	9·6	19·2
Cantilever, „ „	2	4	8
Beam, two symmetrically placed loads, as in Fig. 423	4·65	9·3	18·6
Beam, irregular loading (approx.)	5·5	11	22
Cantilever, „ „	1·75	3·5	7

This table shows the relation that must be observed between the span and the depth of the section for a given stiffness.

The stress can be found direct from the deflection of a given beam if the modulus of elasticity be known; as this does not vary much for any given material, a fairly accurate estimate of the stress can be made. We have above—

$$\delta = \frac{2fL^2}{nEd}$$

$$\text{hence } f = \frac{nEd\delta}{2L^2}$$

The system of loading being known, the value of n can be found from the table above. The value of E must be assumed for the material in the beam. The depth of the section d can readily be measured, also δ and L.

The above method is extremely convenient for finding approximately the stress in any given beam. The error cannot well exceed 10 per cent., and usually will not amount to more than 5 per cent.

CHAPTER XII.

COMBINED BENDING AND DIRECT STRESSES.

In the figure, let a weight W be supported by two bars, 1 and 2, whose sectional areas are respectively A_1 and A_2, and the corresponding loads on the bars R_1 and R_2; then, in order that the stress may be the same in each, W must be so placed that R_1 and R_2 must be proportional to the sectional areas of the bars, or $\frac{R_1}{R_2} = \frac{A_1}{A_2}$. But $R_1x = R_2z$, or $A_1x = A_2z$; hence W passes through the centre of gravity of the two bars when the stress is equal on all parts of the section. This relation holds, however many bars may be taken, even if taken so close together as to form a solid section; hence, *in order to obtain a direct stress of uniform intensity all over a section, the external force must be so applied that it passes through the centre of gravity of the section.*

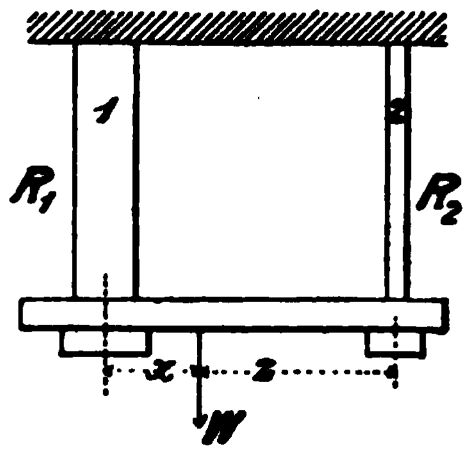


FIG. 438.

If W be not placed at the centre of gravity of the section, but at a distance y from it, we shall have—

$$R_2(x + z) = W(x + y)$$

and when W is at the centre of gravity—

$$R_2(x + z) = Wx$$

Thus when W is not placed at the centre of gravity of the section, the section is subjected to a bending moment W_y in addition to the direct force W. Thus—

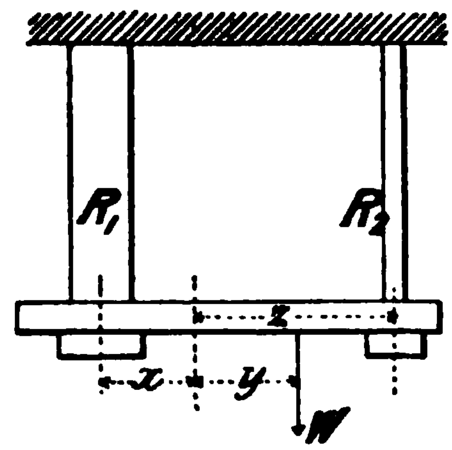


FIG. 439.

If an external force W *acts on a section at a distance y from its centre of gravity, it will be subjected to a direct force* W *acting*

uniformly all over the section and a bending moment W_y. The intensity of stress on any part of the section will be the sum of the direct stress and the stress due to bending, tension and compression being regarded as stresses of opposite sign.

In the figure let the bar be subjected to both a direct stress (+), say tension, and bending stresses. The direct stress acting uniformly all over the section may be represented by the diagram *abcd*, where *ab* or *cd* is the intensity of the tensile stress (+), then if the intensity of tensile stress due to bending be represented by *be* (+), and the compressive stress (−) by *fc*, we shall have—

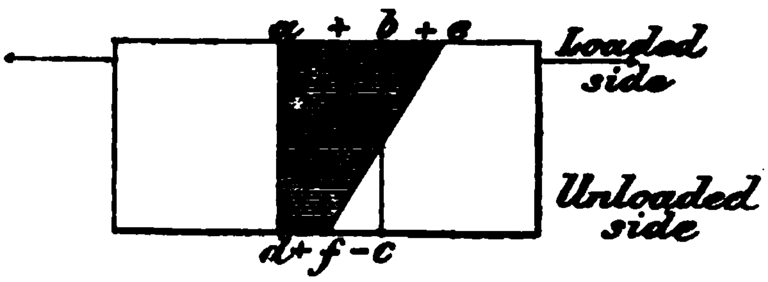


FIG. 440.

The total tensile stress on the outer skin $= ab + be = ae$
" compressive " " $= dc - fc = df$

If the bending moment had been still greater, as shown in Fig. 441, the stress *df* would be −, *i.e.* one side of the bar would have been in compression.

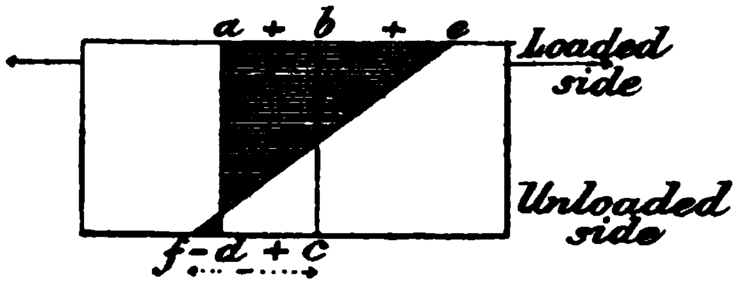


FIG. 441.

Stresses on Bars loaded out of the Centre.—

Let W = the load carried by the bar producing either direct tensile or compressive stresses;
A = the sectional area of the bar;
Z = the modulus of the section in bending;
y = the eccentricity of the load, *i.e.* the distance of the point of application of the load from the centre of gravity of the section;
f'_t = the direct tensile stress acting evenly over the section;
f'_c = the direct compressive stress acting evenly over the section;

f_t = the tensile stress due to bending;
f_c = the compressive stress due to bending;
M = the bending moment on the section.

$$\text{Then } \frac{W}{A} = f'_c \text{ or } f'_t \text{ or } f' \text{ (direct stress)}$$

$$\frac{M}{Z} = \frac{Wy}{Z} = f_t \text{ or } f_c \text{ or } f \text{ (bending stress)}$$

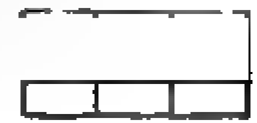
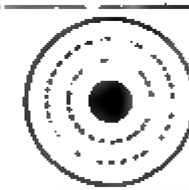
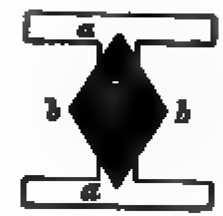
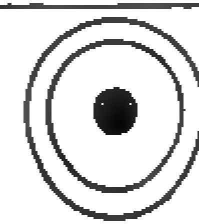
Then the maximum stress on the skin of the section on the loaded side $\Big\} = f' + f = \frac{W}{A} + \frac{Wy}{Z}$

$$= W\left(\frac{1}{A} + \frac{y}{Z}\right)$$

Then the maximum stress on the skin of the section on the unloaded side $\Big\} = f' - f = W\left(\frac{1}{A} - \frac{y}{Z}\right)$

In order that the stress on the unloaded side may not be of opposite sign to the direct stress, the quantity $\frac{1}{A}$ must be greater than $\frac{y}{Z}$. When they are equal, the stress will be zero on the unloaded side, and of twice the intensity of the direct stress on the loaded side; then we have $\frac{1}{A} = \frac{y}{Z}$, or $\frac{Z}{A} = y$. Hence, in order that the stress may not change sign or that there may be no reversal of stress in a section, the line of action of the external force must not be situated at a greater distance than $\frac{Z}{A}$ from the neutral axis.

For convenience of reference, we give various values of $\frac{Z}{A}$ in the following table :—

Section.	Z.	A.	$y = \frac{Z}{A}$	For no change of sign in the stress, the force must act within the	or the shaded area
Rectangular	$\frac{BH^2}{6}$	BH	$\frac{H}{6}$	Middle third	 Fig. 442.
Circular	$\frac{\pi D^3}{32}$	$\frac{\pi D^2}{4}$	$\frac{D}{8}$	Middle fourth	 Fig. 443.
Hollow rectangle or **I** section	$\frac{BH^3 - bh^3}{6H}$	$BH - bh$	$\frac{BH^3 - bh^3}{6H(BH - bh)}$	Points a, a	 Fig. 444.
Do. sideways ⊢⊣	$\frac{bH^3 + Bh^3}{6H}$ (Using the symbols of Fig. 354)	$Bh + bH$	$\frac{bH^3 + Bh^3}{6H(Bh + bH)}$	Points b, b	
Hollow circular	$\frac{\pi(D^4 - D_i^4)}{32D}$	$\frac{\pi(D^2 - D_i^2)}{4}$	$\frac{D^2 + D_i^2}{8D}$		 Fig. 445.

General Case of Eccentric Loading.—In the above instances we have only dealt with sections symmetrical about the neutral axis, and we showed that the skin stress was much greater on the one side than the other. In order to equalize the skin stress, we frequently use unsymmetrical sections.

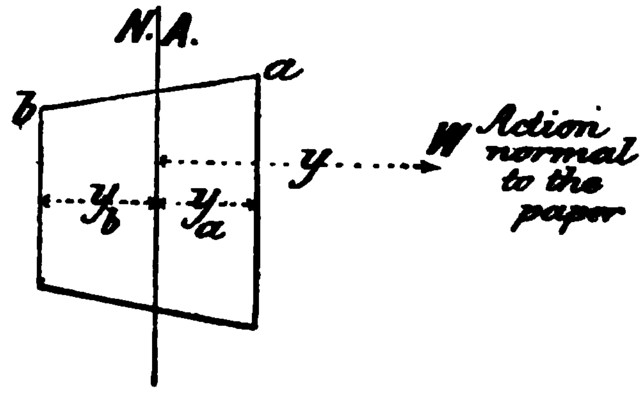


FIG. 446.

Let the skin stress at a due to bending and direct stress $= f_a$; likewise at $b = f_b$.

The direct stress all over the section $= f' = \dfrac{W}{A}$.

For bending we have—

$$M = fZ = \frac{fI}{y_a} \text{ or } \frac{fI}{y_b}$$

according to the side we are considering.

$$\text{or } Wy = \frac{fI}{y_a} \text{ or } \frac{fI}{y_b}$$

$$\text{hence } f = \frac{Wyy_a}{I}$$

$$\text{and } f_a = f' + f = \frac{W}{A} + \frac{Wy \cdot y_a}{I}$$

$$\text{also } f_b = f' - f = \frac{W}{A} - \frac{Wy \cdot y_b}{I}$$

When $y_a = y_b$, the expression becomes the same as we had above.

Cranked Tie-bar.—Occasionally tie bars and rods have

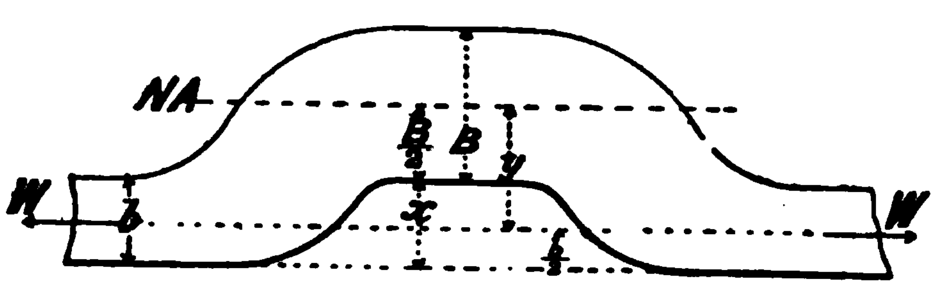


FIG. 447.

to be cranked in order to give clearance or for other reasons, but are very rarely properly designed, and therefore are a source of constant trouble.

The normal width of the tie-bar is b; the width in the cranked part must be greater as it is subjected to bending as well as to tension. We will calculate the width B to satisfy the condition that the maximum intensity of stress in the wide part shall not be greater than that due to direct tension in the narrow part.

Let the thickness of the bar be t.

Then, using the same notation as before—

$$y = \frac{B}{2} + x - \frac{b}{2}$$

$$\left.\begin{array}{l}\text{the direct stress on the}\\ \text{wide part of the bar}\end{array}\right\} = \frac{W}{Bt} = f'$$

$$\left.\begin{array}{l}\text{the bending stress on the}\\ \text{wide part of the bar}\end{array}\right\} = \frac{Wy}{Z} = \frac{6W\left(\frac{B}{2} + x - \frac{b}{2}\right)}{B^2t}$$

$$\left.\begin{array}{l}\text{the maximum skin stress}\\ \text{due to both}\end{array}\right\} = \frac{W}{Bt} + \frac{6W\left(\frac{B}{2} + x - \frac{b}{2}\right)}{B^2t}$$

But as the stress on the wide part of the bar has to be made equal to the stress on the narrow part, we have—

$$\frac{W}{bt} = \frac{W}{Bt} + \frac{6W(B + 2x - b)}{2B^2t}$$

Then dividing both sides of the equation by $\frac{W}{t}$, and solving, we get—

$$B = \sqrt{6bx + b^2 + 2b}$$

Both b and x are known for any given case, hence the width B is readily arrived at. If a rectangular section be retained, the stress on the inner side will be much greater than on the outer. The actual values are easily calculated by the methods given above, hence there will be a considerable waste of material. For economy of material, the section should be tapered off at the back to form a trapezium section. Such a section may be assumed, and the stresses calculated by the method given in the last paragraph; if still unequal, the correct section can be arrived at by one or two trials. An expression can be got out to give the form of the section at once, but it is very cumbersome and more trouble in the end to use than the trial and error method.

Hooks.—A hook may be regarded as a special case of a cranked tie-bar, and if a rectangular section be retained, as

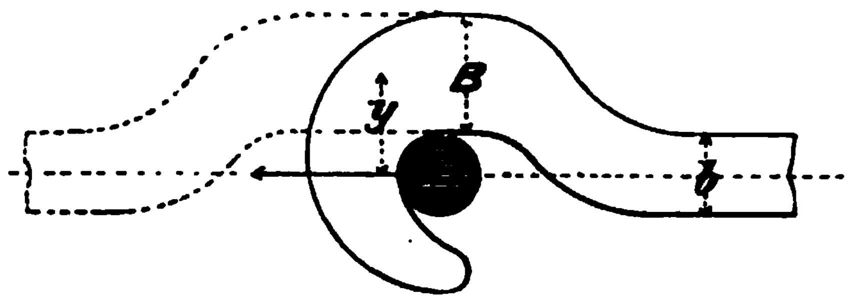


FIG. 448.

in a railway drawbar hook, the equation given above will serve for finding the width B. Crane hooks, however, are always made on more economical lines; the section where the bending is greatest is tapered in order to make the stress of equal intensity on the two sides. The stresses can be calculated by the formulæ given on p. 383.

Inclined Beam.—Many cases of inclined beams occur in practice, such as in roofs, etc.; they are in reality members subject to combined bending and direct stresses. In Fig. 449, resolve W into two components, W_1 acting normal to the beam, and P acting parallel with the beam; then the bending moment at the section $x = W_1 l_1$.

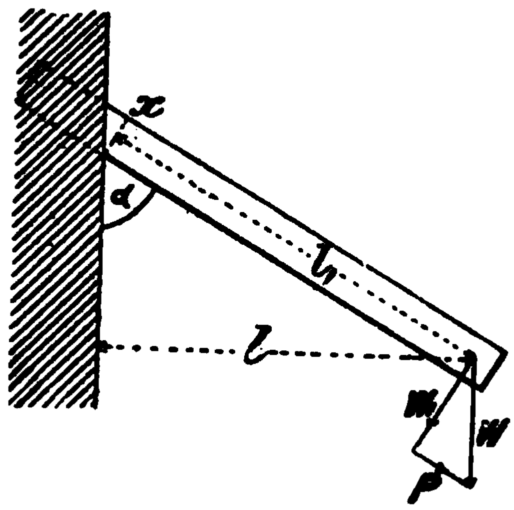


FIG. 449.

$$\text{But } W_1 = W \sin \alpha$$

$$\text{and } l_1 = \frac{l}{\sin \alpha}$$

$$\text{hence } M_x = W \sin \alpha \times \frac{l}{\sin \alpha}$$

$$M_x = Wl = fZ$$

$$f = \frac{Wl}{Z}$$

The tension acting all over the section $= \dfrac{P}{A} = \dfrac{W \cos \alpha}{A}$

$$\text{hence } f \text{ max.} = \frac{W \cos \alpha}{A} + \frac{Wl}{Z} = W\left(\frac{\cos \alpha}{A} + \frac{l}{Z}\right)$$

$$\text{and } f \text{ min.} = \frac{W \cos \alpha}{A} - \frac{Wl}{Z} = W\left(\frac{\cos \alpha}{A} - \frac{l}{Z}\right)$$

N.B.—The Z is for the section x taken normal to the beam, *not* a vertical section.

Machine Frames subjected to Bending and Direct Stresses.—Many machine frames which have a gap, such as punching and shearing machines, riveters, etc., are subject to both bending and direct stresses. Take, for example, a punching-machine with a box-shaped section through AB.

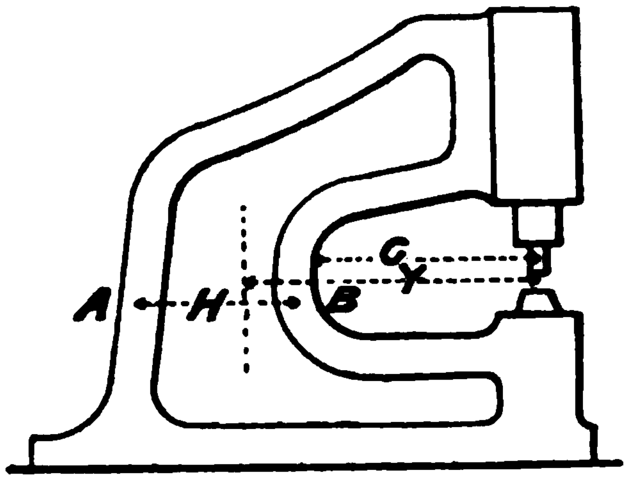


Fig. 450.

Let the load on the punch $= W$, and the distance of the punch from the centre of gravity of the section $= Y$. Y is at present unknown, unless a section has been assumed, but if not a fairly close approximation can be obtained thus: We must first of all fix roughly upon the ratio of the compressive to the tensile stress due to bending; the actual ratio will be somewhat less, on account of the uniform tension all over the section, which will diminish the compression and increase the tension. Let the ratio be, say, 3 to 1; then, neglecting the strength of the web, our section will be somewhat as follows:—

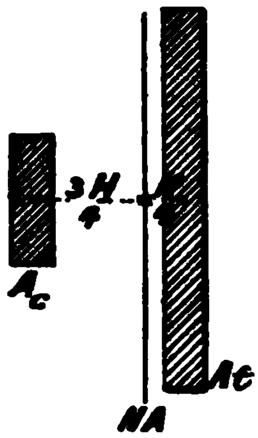


Fig. 451.

$$\text{Make } A_c = \tfrac{1}{3}A_t$$

$$\text{then } Y = G + \frac{H}{4} \text{ approx.}$$

$$Z = \frac{I}{y} = \frac{A_c\left(\frac{3}{4}H\right)^2 + 3A_c\left(\frac{H}{4}\right)^2}{\frac{H}{4}} \text{ (approx.)}$$

$$Z = 3A_cH \text{ (for tension)}$$

$$\text{but } WY = fZ \ (f \text{ being the tensile stress})$$

$$W\left(G + \frac{H}{4}\right) = 3A_cHf$$

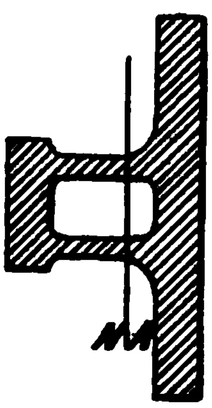


Fig. 452.

$$A_c = \frac{W\left(G + \frac{H}{4}\right)}{3Hf} \text{ or } \frac{W\left(G + \frac{H}{n+1}\right)}{nHf}$$

where n is the ratio of the compressive to the tensile stress.

$$\text{and } A_t = nA_c$$

Having thus approximately obtained the sectional areas of the flanges, complete the section as shown in Fig. 452; and as a check on the work, calculate the stresses accurately by finding the centre of gravity of the complete section, also the Z or the I, and apply the formula given on p. 383.

CHAPTER XIII.

STRUTS.

General Statement.—The manner in which short compression pieces fail is shown in Chapter VIII.; but when their length is great in proportion to their diameter, they bend laterally, unless they are initially absolutely straight, exactly centrally loaded, and of perfectly uniform material—three conditions which are never fulfilled in practice. The nature of the stresses occurring in a strut is, therefore, that of a bar subjected to both bending and compressive stresses. In Chapter XIII. it was shown that if the load deviated but very slightly from the centre of gravity of the section, it very greatly increased the stress in the material; thus, in the case of a circular section, if the load only deviated by an amount equal to one-eighth diameter from the centre, the stress was doubled; hence a very slight initial bend in a compression member very seriously affects its strength.

Effects of Imperfect Loading.—Even if a strut be initially straight before loading, it does not follow that it will remain so when loaded; either or both of the following causes may set up bending—

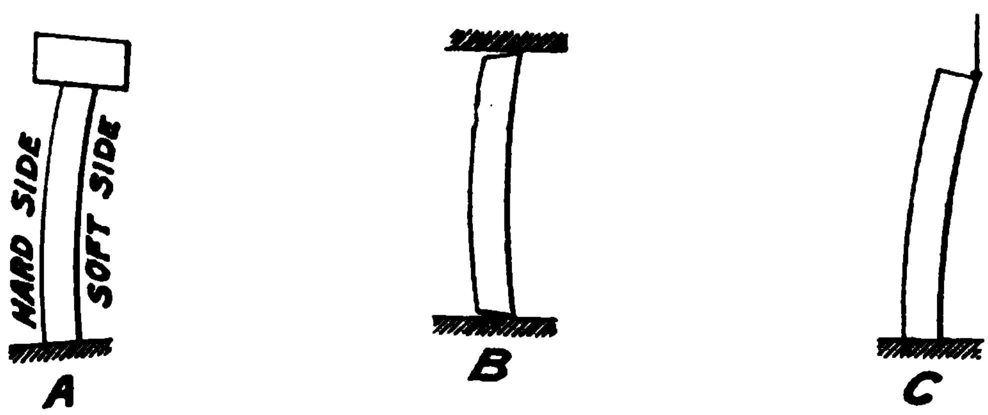


FIG. 453.

(1) The one side of the strut may be harder and stiffer than the other; and consequently the soft side will yield most, and the strut will bend as shown in A, Fig. 453.

(2) The load may not be perfectly centrally applied, either through the ends not being true as shown in B, or through the load acting on one side, as in C.

Possible Discrepancies between Theory and Practice.—We have shown that a very slight amount of bending makes a serious difference in the strength of struts; hence such accidental circumstances as we have just mentioned may not only make a serious discrepancy between theory and experiment, but also between experiment and experiment. Then, again, the theoretical determination of the strength of struts does not rest on a very satisfactory basis, as in all the theories advanced somewhat questionable assumptions have to be made; but, in spite of it, the calculated buckling loads agree fairly well with experiments.

Bending of Long Struts.—The bending moment at the middle of the bent strut shown in Fig. 454 is evidently $W\delta$.

Then $W\delta = fZ$, using the same notation as in the preceding chapters.

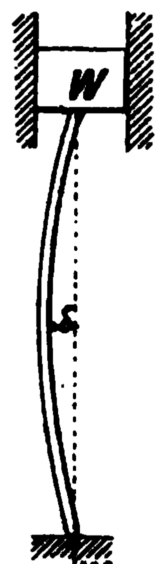


FIG. 454.

If we increase the deflection we shall correspondingly increase the bending moment, and consequently the stress.

From above we have—

$$W = \frac{f}{\delta}Z \text{ or } \frac{f_1}{\delta_1}Z, \text{ and so on}$$

but as f varies with δ, $\frac{f}{\delta}$ = a constant, say K—

$$\text{then } W = KZ$$

but Z for any given strut does not vary when the strut bends; hence there is only one value of W that will satisfy the equation.

When the strut is thus loaded, let an external bending-moment M, indicated by the arrow (Fig. 455), be applied to it until the deflection is δ_1, and its stress f_1—

$$\text{Then } W\delta_1 + M = f_1Z$$
$$\text{but } W\delta_1 = f_1Z$$

Therefore $M = 0$; that is to say, that no external bending moment M is required to keep the strut in its bent position, or the strut, when thus loaded, is in a state of neutral equilibrium

and will remain when left alone in any position in which it may be placed; this condition, of course, only holds so long as the strut is elastic, *i.e.* before the elastic limit is reached. This state of neutral equilibrium may be proved experimentally, if a long thin piece of elastic material be loaded as shown.

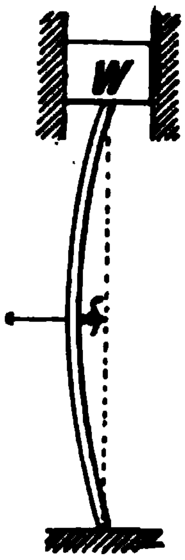


FIG. 455.

Now, place a load W_1 *less* than W on the strut, say $W = W_1 + w$, and let it again be bent by an external bending moment M till its deflection is δ_1 and the stress f_1; then we have, as before—

$$W_1\delta_1 + M = f_1Z = W\delta_1 = W_1\delta_1 + w\delta_1$$
$$\text{hence } M = w\delta_1$$

Thus, in order to keep the strut in its bent position with a deflection δ_1, we must subject it to a + bending moment M, *i.e.* one which tends to bend the strut in the *same* direction as $W_1\delta_1$; hence, if we remove the bending moment M, the deflection will become zero, *i.e.* the strut will straighten itself.

Now, let a load W_2 *greater* than W be placed on the strut, say $W = W_2 - w$, and let it again be bent until its deflection $= \delta_1$, and the stress f_1 by an external bending moment M; then we have as before—

$$W_2\delta_1 + M = f_1Z = W_2\delta_1 - w\delta_1$$
$$\text{hence } M = -w\delta_1$$

Thus, in order to keep the strut in its bent position with a deflection δ_1, we must subject it to a − bending moment M, *i.e.* one which tends to bend the strut in the *opposite* direction to $W_2\delta_1$; hence, if we remove the bending moment M, the deflection will go on increasing, and ere long the elastic limit will be reached when the strain will increase suddenly and much more rapidly than the stress, consequently the deflection will suddenly increase and the strut will buckle.

Thus, the strut may be in one of three conditions—

Condition.	When slightly bent by an external bending moment M, on being released, the strut will—	When supporting a load—
i.	Remain bent	W.
ii.	Straighten itself	*less* than W.
iii.	Bend still further and ultimately buckle	*greater* than W.

Condition ii. is, of course, the only one in which a strut can exist for practical purposes; how much the working load must be less than W is determined by a suitable factor of safety. W is termed the buckling load of the strut; but at present we are unable to determine it, as neither δ nor f are known.

Buckling Load of Long Thin Struts. Euler's Formula.—The results arrived at in the paragraph above refer only to very long thin struts; we will now proceed to determine the value of W for such struts. If the deflection were entirely due to the eccentricity x of the load, then the bending moment at every section of the strut would be constant and equal to Wx, and the strut would then bend to the arc of a circle (see p. 358); but as soon as the strut began to bend this condition would fail, for x would vary from point to point. However, for the present we will assume that struts do bend to an arc of a circle; we shall return to this point later on, and then give a more exact result.

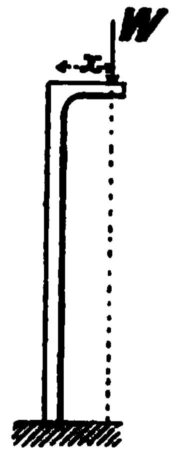


FIG. 456.

Let l = the effective length of the strut (see Fig. 458);
E = Young's modulus of elasticity;
I = moment of inertia of a section of the strut (assumed to be of constant cross-section).

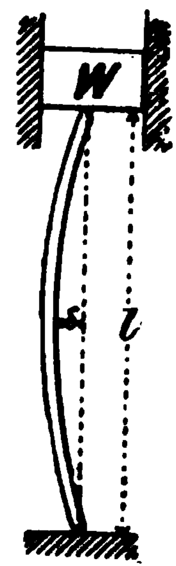


FIG. 457.

Then for a strut loaded thus—

$$\delta = \frac{Ml^2}{8EI} \text{ (see p. 359)}$$

$$\delta = \frac{W\delta l^2}{8EI}$$

$$\text{or } W = \frac{8EI}{l^2}$$

As the strut is very long and the deflection small, the length l remains practically constant, and the other quantities 8, E, I are also constant for any given strut; thus, W is equal to a constant, which we have previously shown must be the case.

We mentioned above that once the strut has begun to bend it cannot remain a circular arc, because the bending moment no longer remains constant at every section, but it will vary directly as the distance of any given section from the line of application of the load. Under these conditions it is readily shown that the strut bends to a curve of sines (see Fidler's

"Practical Treatise on Bridge Construction," p. 157), and that the deflection in the middle is then $\frac{ML^2}{\pi^2 EI}$, whence we get—

$$W = \frac{\pi^2 EI}{l^2} = \frac{10EI}{l^2} \text{ (nearly)}$$

The I in this formula is the *least* moment of inertia of the section.

Effect of End holding on the Buckling Load.—In the case we have just considered the strut was supposed to be free or pivoted at the ends, but if the ends are not free the strut behaves in a different manner, as shown in the accompanying diagram.

DIAGRAM SHOWING STRUTS OF EQUAL STRENGTH.

One end free, the other fixed.	Both ends pivoted or rounded.	One end rounded or pivoted, the other end built in or fixed.	Both ends fixed or built in.
$l = 2L$	$l = L$	$l = \frac{L}{\sqrt{2}}$	$l = \frac{L}{2}$

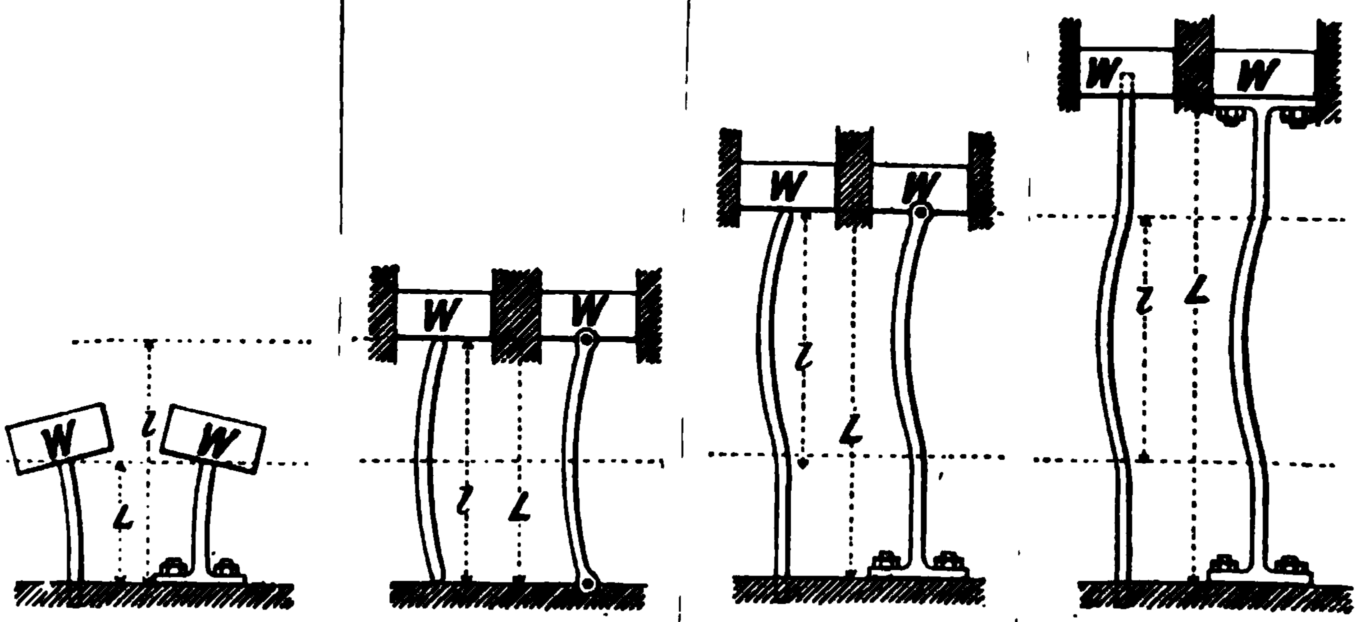


FIG. 458.

$W = \frac{2 \cdot 5EI}{L^2}$	$W = \frac{10EI}{L^2}$	$W = \frac{20EI}{L^2}$	$W = \frac{40EI}{L^2}$
$P = \frac{2 \cdot 5E\rho^2}{L^2}$	$P = \frac{10E\rho^2}{L^2}$	$P = \frac{20E\rho^2}{L^2}$	$P = \frac{40E\rho^2}{L^2}$

Each strut is supposed to be of the same section, and loaded with the same weight W.

Let $P =$ the buckling stress of the strut, *i.e.* $\frac{W}{A}$, where

$W =$ the buckling load of the strut;
$A =$ the sectional area of the strut.

We also have $\frac{I}{A} = \rho^2$ (see p. 78), where ρ is the radius of gyration of the section.

Substituting these values in the above equation, we have—

$$P = \frac{\pi^2 E \rho^2}{l^2}$$

The "effective" or "virtual" length l, shown in the diagram, is found by the methods given in Chapter XI., for finding the virtual length of built-in beams.

The square-ended struts in the diagrams are shown bolted down to emphasize the importance of rigidly fixing the ends; if the ends merely rested on flat flanges without any means of fixing, they much more nearly approximate round-ended struts.

It will be observed that Euler's formula takes no account of the compressive stress on the material; it simply aims at giving the load which will produce neutral equilibrium as regards bending in a long bar, and even this it only does imperfectly, for when a bar is subjected to both direct and bending stresses, the neutral axis no longer passes through the centre of gravity of the section. We have shown above that when the line of application of the load is shifted but one-eighth of the diameter from the centre of a round bar, the neutral axis has shifted to the outermost edge of the bar. In the case of a strut subject to bending, the neutral axis shifts away from the line of application of the load; thus the bending moment increases more rapidly than Euler's hypothesis assumes it to do, consequently his formula gives too high results; but in very long columns in which the compressive stress is small compared with the stress due to bending, the error may not be serious. But if the formula be applied to short struts, the result will be absurd. Take, for example, an iron strut of circular section, say 4 inches diameter and 40 inches long; we get $P = \frac{10 \times 29000000 \times 1}{1600} = 181{,}000$ lbs. per square inch, which is far higher than the crushing strength of a short specimen of the material, and obviously absurd.

If Euler's formula be employed, it must be used exclusively

for long struts, whose length l is not less than 30 diameters for wrought iron and steel, or 12 for cast iron and wood.

Notwithstanding the unsatisfactory basis on which it rests, many high authorities, such as Unwin, Reuleaux, Bovey, Thurston, and others, prefer it to Gordon's, which we will shortly consider. For a full discussion of the whole question of struts, the reader is referred to Todhunter and Pearson's "History of the Theory of Elasticity."

Gordon's Strut Formula rationalized.—Gordon's strut formula, as usually given, contained empirical constants obtained from experiments by Hodgkinson and others; the author, however, has succeeded in arriving at these constants rationally, which will shortly be seen to agree remarkably well with the constants found by experiment.

Gordon's formula certainly has this advantage, that it agrees far better with experiments on the ultimate resistance of columns than does the formula propounded by Euler; and, moreover, it is applicable to columns of any length, short or long, which we have seen above, is not the case with Euler's formula. The elastic conditions assumed by Euler cease to hold when the elastic limit is passed, hence a long strut always fails at or possibly before that point is reached; but in the case of a short strut, in which the bending stress is small compared with the compressive stress, it does not at all follow that the strut will fail when the elastic limit in compression is reached—indeed, experiments show conclusively that such is not the case. A formula for struts of any length must therefore cover both cases, and be equally applicable to short struts that fail by crushing and to long struts that fail by bending. In constructing this formula we assume that the strut fails either by buckling or by crushing,[1] when the sum of the direct compressive stress and the skin stress, due to bending, are equal to the crushing strength of the material; in using the term "crushing strength" for ductile materials, we mean the stress at which the material becomes plastic. This assumption, we know, is not strictly true, but it cannot be far from the truth, or our calculated values of the constant (a), shortly to be considered, would not agree so well with the experimental values.

[1] Mons. Considère and others have found that for long columns the resistance does not vary directly as the crushing resistance of the material, but for short columns, which fail by crushing and not by bending, the resistance does of course entirely depend upon it, and therefore must appear in any formula professing to cover struts of all lengths.

Let S = the crushing (or plastic) strength of a short specimen of the material;

C = the direct compressive stress on the section of the strut;

then, adopting our former notation, we have—

$$C = \frac{W}{A} \text{ and } f = \frac{W\delta}{Z}$$

$$\text{then } S = C + f$$

$$S = \frac{W}{A} + \frac{W\delta}{Z}$$

We have shown above that, on Euler's hypothesis, the maximum deflection of a strut is—

$$\delta = \frac{Ml^2}{10EI} = \frac{fZl^2}{10EZy}$$

where y is the distance of the most strained skin from the centre of gravity of the section, or from the assumed position of the neutral axis. We shall assume that the same expression holds in the present case. In symmetrical sections $y = \frac{d}{2}$, where d is the *least* diameter of the strut section.

By substitution, we have—

$$\delta = \frac{fl^2}{5Ed}$$

$$\text{also } S = \frac{W}{A} + \frac{Wfl^2}{5EdZ} \qquad \text{(i.).}$$

$$= \frac{W}{A}\left(1 + \frac{Afl^2}{5EdZ}\right)$$

$$= \frac{W}{A}\left(1 + \frac{Afd}{5EZ} \times \frac{l^2}{d^2}\right)$$

If W be the buckling load, we may replace $\frac{W}{A}$ by P.

Let $\frac{Afd}{5EZ} = a$.

$$\text{Then } S = P\left(1 + a\frac{l^2}{d^2}\right)$$

$$\text{or } P = \frac{S}{1 + a\frac{l^2}{d^2}} = \frac{S}{1 + ar^2}$$

where $r = \frac{l}{d}$, which is a modification of "Gordon's Strut Formula."

P may be termed the buckling stress of the strut.

The d in the above formula is the least dimension of the section, thus—

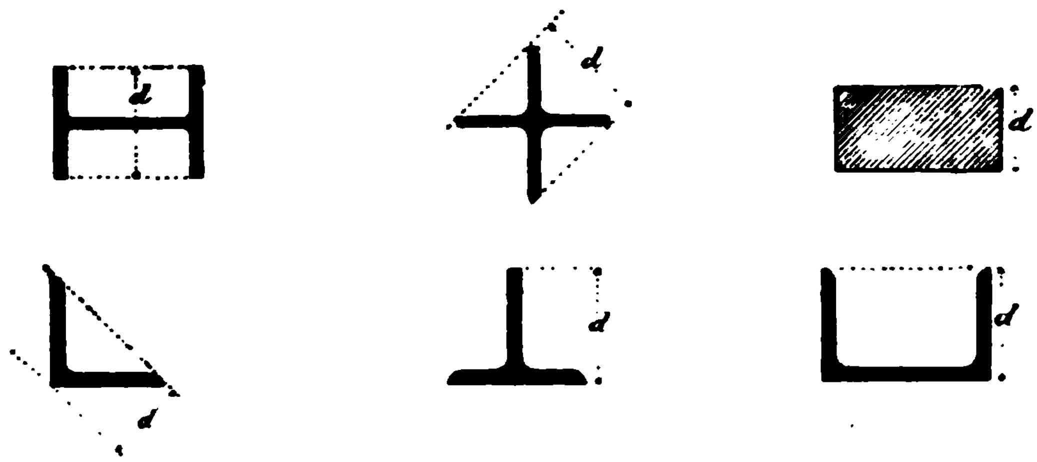


FIG. 459.

It now remains to be seen how the values of the constant a agree with those found by experiment; it, of course, depends upon the values we choose for f and E. The latter presents no difficulty, as it is well known for all materials; but the former is not so obvious at first. In equation (i.), the first term provides for the crushing resistance of the material irrespective of any stress set up by bending; and the second term provides for the bending resistance of the strut. We have already shown that the strut buckles when the elastic limit is reached, hence, we may reasonably take f as the elastic limit of the material. Values of S, f, and E, are given in the table below; they must be taken as fair average values, to be used in the absence of more precise data.

Material.	Pounds per square inch.		
	S	f	E
Soft wrought iron	40,000	28,000	25,000,000
Hard ,, ...	48,000	32,000	29,000,000
Mild steel	67,000	45,000	30,000,000
Hard ,, ...	110,000	75,000	32,000,000
Cast iron	80,000 (no marked limit)	80,000	13,000,000
,, (hard close-grained metal)	130,000	130,000	22,000,000
Pitchpine and oak	8,000	8,000	900,000

Material.	Form of section.	$a = \frac{Afd}{5EZ}$	a by experiment.
Wrought iron ...	▬	$\frac{1}{40}$ to $\frac{1}{750}$	$\frac{1}{730}$
	●	$\frac{1}{60}$ to $\frac{1}{370}$	$\frac{1}{360}$
	○	$\frac{1}{930}$ to $\frac{1}{940}$	$\frac{1}{1370}$ [1]
	H L + ⊥ ⊔	$\frac{1}{450}$ to $\frac{1}{460}$	$\frac{1}{460}$ to $\frac{1}{500}$ (author)
Mild steel ...	▬	$\frac{1}{340}$	$\frac{1}{500}$
	●	$\frac{1}{400}$	$\frac{1}{350}$
	○	$\frac{1}{670}$	$\frac{1}{630}$
	H L + ⊥ ⊔	$\frac{1}{320}$	$\frac{1}{300}$ to $\frac{1}{350}$ (author)
Hard steel ...	▬	$\frac{1}{350}$	$\frac{1}{350}$
	●	$\frac{1}{260}$	$\frac{1}{250}$
	○	$\frac{1}{410}$	$\frac{1}{380}$ [1]
	H L + ⊥ ⊔	$\frac{1}{210}$	—
Cast iron	▬	$\frac{1}{130}$ to $\frac{1}{140}$	$\frac{1}{112}$
	●	$\frac{1}{100}$ to $\frac{1}{110}$	$\frac{1}{133}$
	○	$\frac{1}{150}$ to $\frac{1}{170}$	$\frac{1}{130}$ (Rankine) $\frac{1}{200}$
	H L + ⊥ ⊔	$\frac{1}{70}$ to $\frac{1}{80}$	$\frac{1}{31}$ [1]
Pitchpine and oak	▬	$\frac{1}{90}$	$\frac{1}{62}$
	●	$\frac{1}{70}$	$\frac{1}{62}$ (author) $\frac{1}{80}$

N.B.—The values of a given in the last column are four times as great as those usually given, due to the l used in our formula being taken equal to L for rounded ends, whereas some other writers take it for square or fixed ends.

[1] The discrepancies in these cases may be due to the section being thicker or thinner than the one assumed in calculating the value of a. In cases of sections other than those given above the value of a should be worked out.

The values of the constant *a* have been worked out for the various materials, and are given in tabulated form below; also values found by experiment as given in Rankine's "Applied Mechanics," and by Bovey, "Theory of Structures and Strength of Materials" (Wiley, New York).

The following tables have been worked out by the formula given above to the nearest 100 lbs. per square inch. For those who are constantly designing struts, it will be found convenient to plot them to a large scale, in the same manner as shown in Fig. 460. In order better to compare the results obtained by Euler's and by Gordon's formula, curves representing both are given, from which it will be seen that they agree fairly well for very long struts, but that Euler's is quite out of it for short struts.

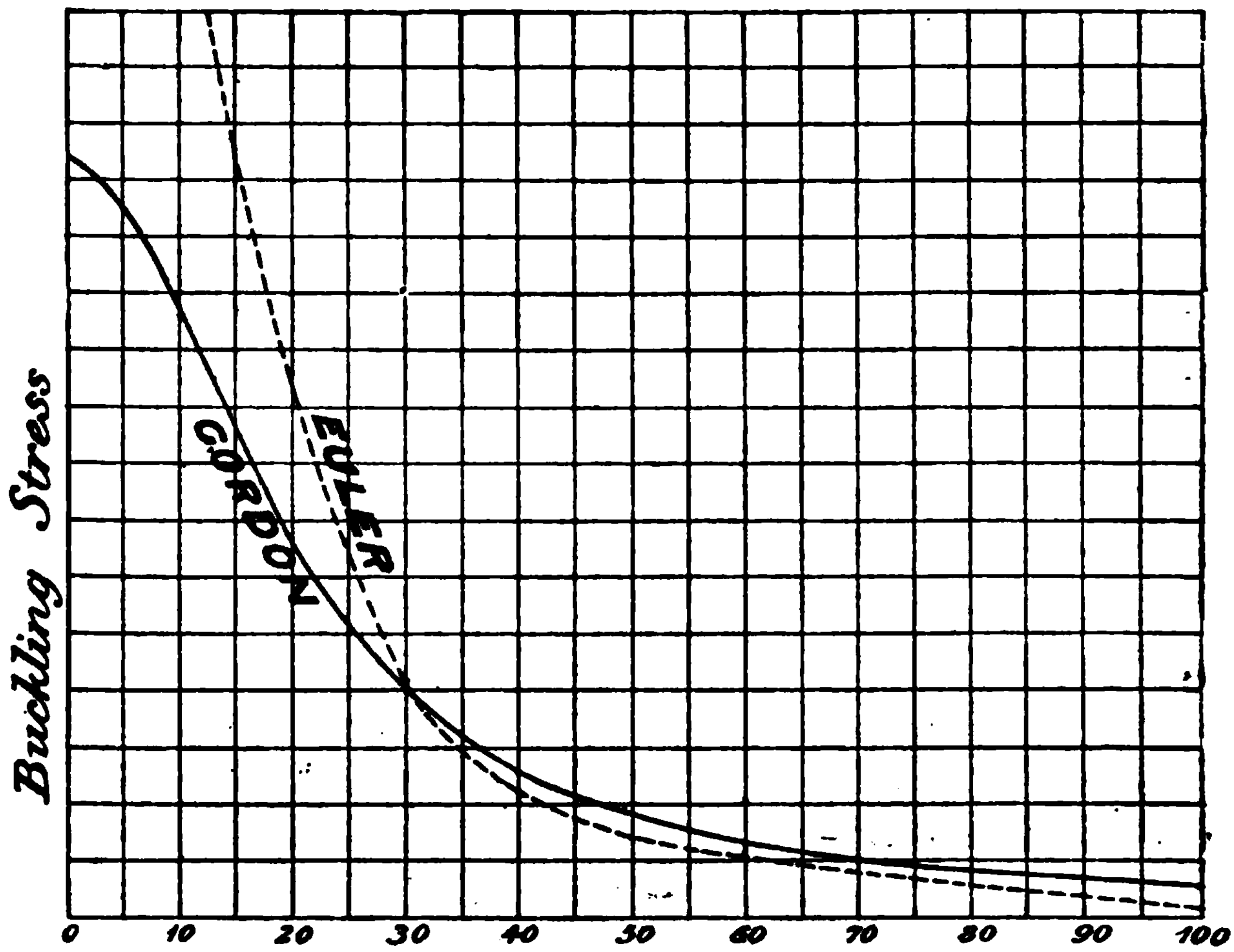


FIG. 460.

NOTE.—The two curves in many cases practically coincide after 40 diameters. In the figure the Gordon curve has been shifted bodily up, to better show the relation.

BUCKLING LOAD OF STRUTS IN POUNDS PER SQUARE INCH OF SECTION.

r or $\frac{l}{d}$	▬	●	○	H L + ⊥ U	▬	●	○	H L + ⊥ U
	Wrought iron.				Mild steel.			
5	42,600	42,000	43,000	41,700	64,100	63,100	64,700	62,000
10	38,800	37,200	39,900	36,000	56,600	53,600	58,300	51,000
20	28,800	25,600	30,900	23,200	38,500	33,500	41,900	29,800
30	20,000	16,800	22,200	14,700	25,000	20,600	28,600	17,500
40	14,000	11,400	16,300	9,600	17,000	13,400	19,700	11,100
50	10,400	8,000	12,400	6,700	11,900	9,200	14,100	7,800
60	7,600	5,900	9,100	4,900	8,700	6,700	10,500	5,400
70	5,800	4,500	7,100	3,700	6,700	5,000	8,100	4,100
80	4,600	3,500	5,600	2,900	5,200	3,900	6,300	3,200
90	3,700	2,800	4,500	2,100	4,200	3,100	5,100	2,500
100	3,100	2,300	3,800	1,900	3,400	2,500	4,200	2,100
	Hard steel.				Cast iron (soft).			
5	102,000	100,300	104,000	98,600	67,100	64,000	69,200	58,900
10	85,500	79,400	89,600	74,500	45,200	40,000	49,200	32,900
20	51,500	43,300	57,500	37,900	19,600	16,000	22,900	11,900
30	30,800	24,600	36,100	20,400	10,100	8,000	12,100	5,800
40	19,700	15,400	23,700	12,700	6,000	4,700	7,300	3,400
50	13,500	10,300	16,400	8,500	3,900	3,100	4,800	2,200
60	9,750	7,400	11,900	6,100	2,800	2,200	3,400	1,500
70	7,300	5,500	9,100	4,500	2,100	1,600	2,500	1,100
80	5,700	4,300	7,100	3,500	1,600	1,200	2,000	870
90	4,600	3,400	5,700	2,800	1,300	980	1,600	690
100	3,700	2,800	4,600	2,200	1,000	790	1,300	560

r or $\frac{l}{d}$	Cast iron (hard close-grained).				Pitchpine and oak.	
5	109,000	104,000	112,000	95,700	6,300	5,900
10	73,500	65,000	80,000	53,500	3,800	3,300
20	31,800	26,000	37,200	19,300	1,500	1,200
30	16,400	13,000	19,700	9,400	730	580
40	9,700	7,600	11,900	5,500	430	340
50	6,300	5,000	7,800	3,600	280	220
60	4,600	3,600	5,500	2,400	200	150
70	3,400	2,600	4,100	1,800	140	110
80	2,600	1,900	3,300	1,400	110	90
90	2,100	1,600	2,600	1,100	90	70
100	1,600	1,300	2,100	900	70	60

Straight-line Strut Formula.—For struts having an effective length of from 10 to 30 diameters, a straight-line formula of the form $M - N\frac{l}{d}$, where M is a constant depending upon the strength of the material, and N a constant depending on the form of section and elasticity of the strut, is often used, and preferred by many. Space, however, forbids our going into the matter, especially as we have gone so fully into Gordon's formula.

Columns loaded on Side Brackets.—The barbarous practice of loading columns on side brackets is unfortunately far too common. As usually carried out, the practice reduces the strength of the column to one-tenth[1] of its strength when centrally loaded.

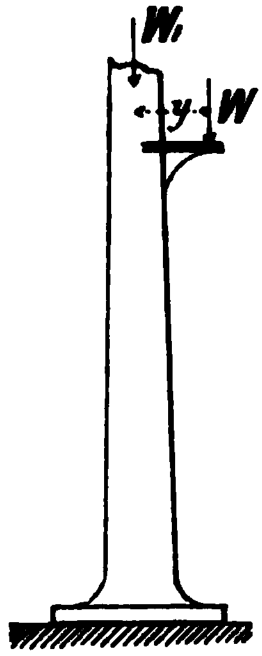


FIG. 461.

Let f_c = the maximum compressive stress on the material due to both direct and bending stresses;

f_t = the maximum tensile stress on the material due to both direct and bending stresses;

f = the skin stress due to bending;

C = the compressive stress acting all over the section due to the weight W;

A = the sectional area of the column.

$$\text{Then } f_c = f + C = \frac{Wy}{Z} + \frac{W}{A}$$

$$\text{and } f_t = \frac{Wy}{Z} - \frac{W}{A}$$

If the column also carries a central load W_1, the above become—

$$f_c = \frac{Wy}{Z} + \frac{W + W_1}{A}$$

$$f_t = \frac{Wy}{Z} - \frac{W + W_1}{A}$$

Columns loaded thus, almost invariably fail in tension, therefore the strength must be calculated on the f_t basis; we

[1] The *ten* is not used with any special significance here, may be one-tenth or even one-twentieth.

have neglected the deflection due to loading (Fig. 462), which makes matters still worse; the tensile stress then becomes—

$$f_t = \frac{W(y+\delta)}{Z} - \frac{W+W_1}{A}$$

The author knows of an instance of a public building in which a column is loaded as shown in Fig. 463; the deflections given were taken when the gallery was empty, and no wind on the roof. The deflections are so serious that when the gallery

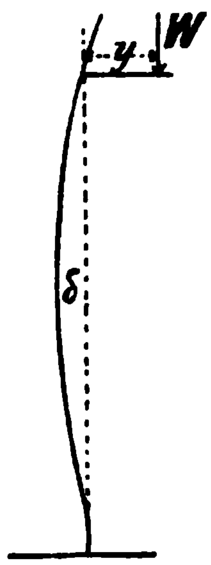


FIG. 462.

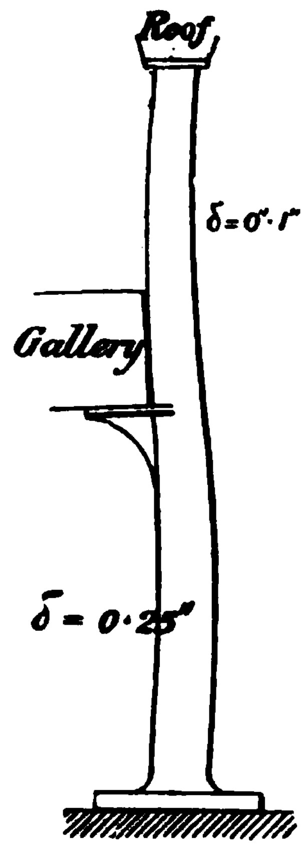


FIG. 463.

is full, an experienced eye immediately detects them on entering the building.

The following test of a column by the author will serve to emphasize the folly of loading columns in this manner.

ESTIMATED BUCKLING LOAD IF CENTRALLY LOADED, ABOUT
1000 tons.

Length 10 feet, end flat, not fixed.

Sectional area of metal at fracture ...	34·3 sq. inches
Modulus of section at fracture	75·0
Distance of point of application of load from centre of column, neglecting slight amount of deflection when loaded ...	17 inches
Breaking load applied at edge of bracket	**65·5 tons**
Bending moment on section when fracture occurred	1114 tons-inches

Compressive stress all over section when fracture occurred...	1·91	tons per	sq. inches
Skin stress on the material due to bending, assuming the bending formula to hold up to the breaking point	14·85	,,	,,
Total tensile stress on material due to combined bending and compression[1] ...	12·94	,,	,,
Total compressive stress on material due to combined bending and compression	16·76	,,	,,
Tensile strength of material as ascertained from subsequent tests	8·45	,,	,,
Compressive strength of material as ascertained from subsequent tests	30·4	,,	,,

Thus we see that the column failed by tension in the material on the off side, *i.e.* the side remote from the load.

FACTOR OF SAFETY FOR STRUTS.

	Dead loads.	Live loads.
Wrought iron and steel	4	8
Cast iron	6	12
Timber	5	10

[1] The discrepancy between this and the tensile strength is due to the bending formula not holding good at the breaking point, as previously explained.

CHAPTER XIV.

TORSION. GENERAL THEORY.

LET Fig. 464 represent two pieces of shafting provided with

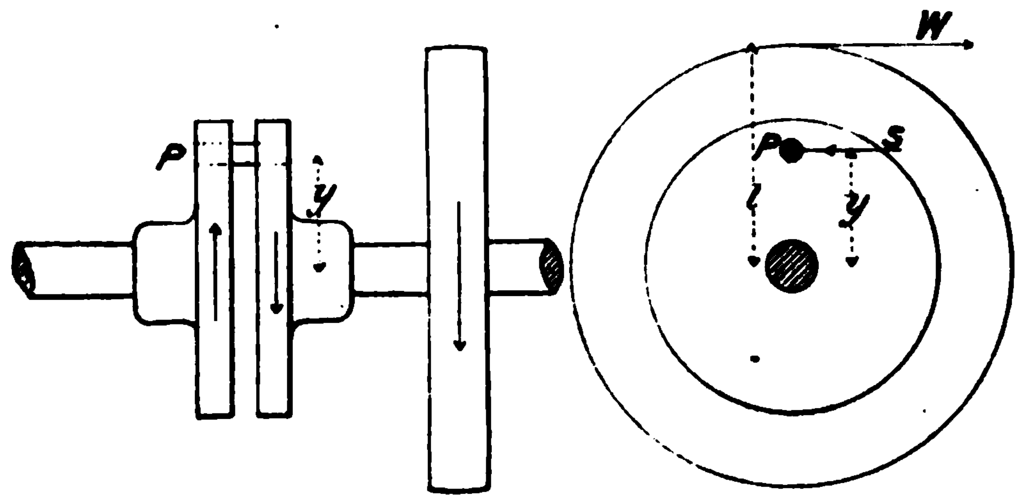


FIG. 464.

disc couplings as shown, the one being driven from the other through the pin P, which is evidently in shear.

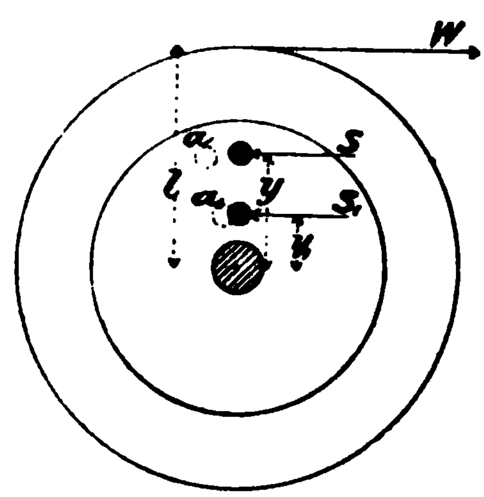


FIG. 465.

Let S = the shearing resistance of the pin.

Then we have $Wl = Sy$

Let the area of the pin $= a$, and the shear stress on the pin be f_s. Then we may write the above equation—

$$Wl = f_s a y$$

Now consider the case in which there are two pins, then—

$$Wl = Sy + S_1 y_1 = f_s a y + f_{s1} a_1 y_1$$

The dotted holes in the figure are supposed to represent the pin-holes in the other disc coupling. Before W was applied the pin-holes were exactly opposite one another, but after the application of W the yielding or the shear of the pins caused a

slight movement of the one disc relatively to the other, but shown very much exaggerated in the figure. It will be seen that the yielding or the strain varies directly as the distance from the axis of revolution (the centre of the shaft). When the material is elastic, the stress varies directly as the strain; hence—

$$\frac{f_{s1}}{f_s} = \frac{y_1}{y}, \text{ or } f_{s1} = \frac{f_s y_1}{y}$$

Substituting this value in the equation above, we have—

$$Wl = f_s a y + \frac{f_s a_1 y_1^2}{y}$$

$$= \frac{f_s}{y}(ay^2 + a_1 y_1^2)$$

Then, if $a = a_1$, and say $y_1 = \frac{y}{2}$

$$Wl = f_s a y + \frac{f_s a y}{4} = f_s a y\,(1 + \tfrac{1}{4})$$

Thus the inner pin, as in the beam (see p. 292), has only increased the strength by $\frac{1}{4}$. Now consider a similar arrangement with a great number of pins, such a number as to form a hollow or a solid section, the areas of each little pin or element being a, a_1, a_2, etc., distant y, y_1, y_2, etc., respectively from the axis of revolution. Then, as before, we have—

$$Wl = \frac{f_s}{y}(ay^2 + a_1 y_1^2 + a_2 y_2^2 +, \text{etc.})$$

But the quantity in brackets, viz. each little area multiplied by the square of its distance from the axis of revolution, is the polar moment of inertia of the section (see p. 77), which we will term I_p. Then—

$$Wl = \frac{f_s I_p}{y} = f_s Z_p$$

The Wl is termed the twisting moment, M_t. f_s is the skin shear stress on the material furthest from the centre, and is therefore the maximum stress on the material, often termed the skin stress.

y is the distance of the skin from the axis of revolution.

$\frac{I_p}{y}$ = the modulus of the section = Z_p. To prevent confusion, we shall use the suffix p to indicate that it is the polar modulus of the section, and not the modulus for bending.

Thus we have $M_t = f_s Z_p$
or the twisting moment = the skin stress × the polar modulus of the section

Shafts subject to Torsion.—To return to the shaft couplings. When power is transmitted from one disc to the other, the pin will evidently be in shear, and will be distorted

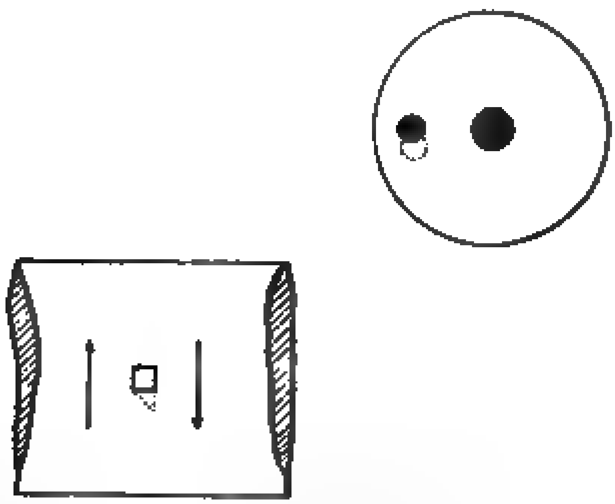


FIG. 466.

as shown (exaggerated). Likewise, if a small square be marked on the surface of a shaft, when the shaft is twisted it will also become a rhombus, as shown dotted on the shaft below.

In Chapter VIII. we showed that when an element was distorted by shear, as shown in Fig. 467 (*a*), it was

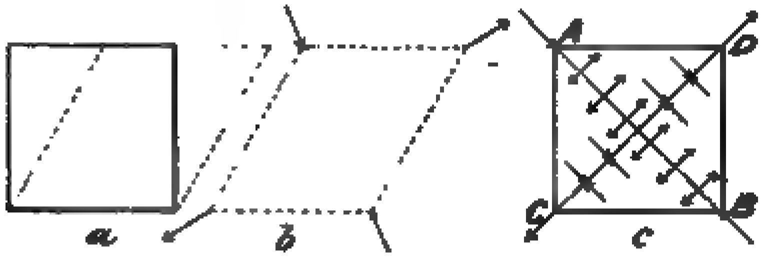


FIG. 467.

equivalent to the element being pulled out at two opposite corners and pushed in at the others, as shown in Fig. 467, (*b*) and (*c*), hence all along the diagonal section AB there

is a tension tending to pull the two triangles ADB, ACB apart; similarly there is a compression along the diagonal CD. These diagonals make an angle of 45° with their sides. Thus, if two lines be marked on a shaft at an angle of 45° with the axis, there will be a tension normal to the one diagonal, and a compression normal to the other. That this is the case can be shown very clearly by getting a piece of thin tube and sawing a diagonal slot along it at an angle of 45°. When the outer end is twisted in the direction of the arrow A, there will be compression normal to the slot, shown by a full line, and the slot will close; but if it be twisted in the direction of the arrow B, there will be tension normal to the slot, and will cause it to open.

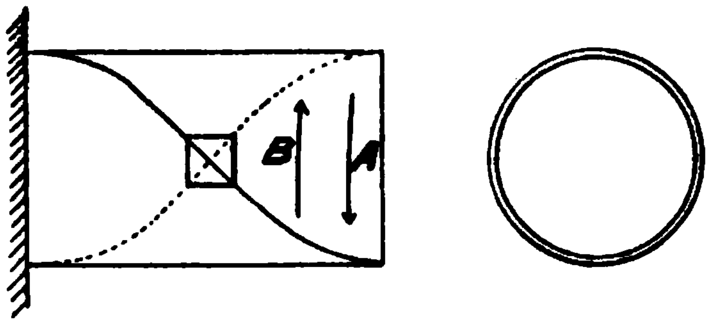


FIG. 468.

Graphical Method of finding the Polar Modulus for a Circular Section.—The method of graphically finding the polar modulus of the section is precisely similar in principle to that given for bending (see Chap. IX.), hence we shall not do more than briefly indicate the construction of the modulus figure. It is of very limited application, as it is *only true for circular sections.*

As in the beam modulus figure, we want to construct a figure to show the distribution of stress in the section.

Consider a small piece of a circular section as shown, with two blocks equivalent to the pins we used in the disc couplings above. The stress on the inner block $= f_{s1}$, and on the outer block $= f_s$; then $\frac{f_{s1}}{f_s} = \frac{y_1}{y}$. Then by projecting the width of the inner block on to the outer circle, and joining down to the centre of the circle, it is evident, from similar triangles, that we reduce the width and area of the inner block in the ratio $\frac{y_1}{y}$, or in the ratio of $\frac{f_{s1}}{f_s}$. The reduced area of the inner block, shown shaded, we will now term a_1', where $\frac{a_1'}{a_1} = \frac{f_{s1}}{f_s} = \frac{y_1}{y}$, or $a_1'f_s = a_1 f_{s1}$.

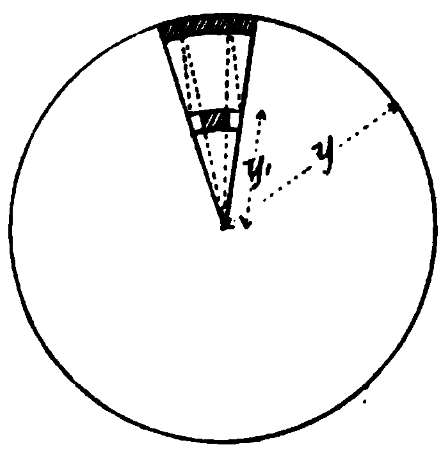


FIG. 469.

Then the *magnitude* of the resultant force acting on the two blocks $= af_s + a_1 f_{s1}$

$$= af_s + a_1' f_s = f_s (a + a_1')$$

$$= f_s \text{ (shaded area or area of modulus figure)}$$

And the *position* of the resultant is distant y_0 from the centre, where—

$$y_0 = \frac{ay + a_1' y_1}{a + a_1'}$$

i.e. at the centre of gravity of the blocks.

$$\text{Then } f_s Z_p = f_s(a + a_1') \times \frac{ay + a_1' y_1}{a + a_1'}$$

$$= f_s(ay + a_1' y_1) = \frac{f_s}{y}(ay^2 + a_1 y_1^2)$$

which is the same result as we had before for Wl, thus proving the correctness of the graphical method.

In the figure above we have only taken a small portion of a circle; we will now use the same method to find the Z_p for a

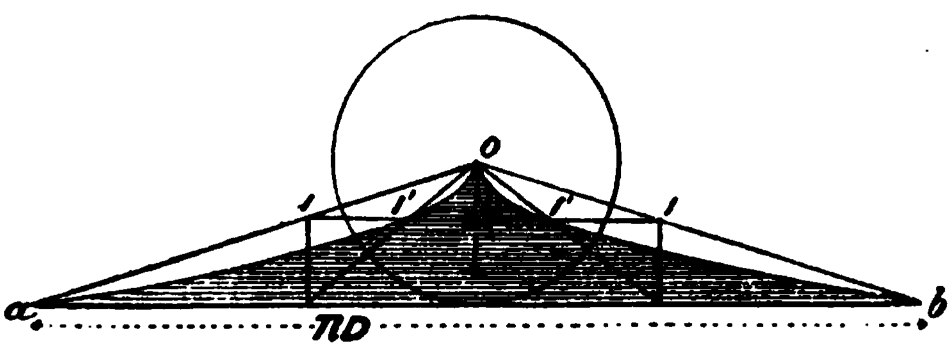


FIG. 470.

circle. For convenience in working, we will set it off on a straight base thus: Draw a tangent *ab* to the circle, making the length $= \pi D$; join the ends to the centre O; draw a series of lines parallel to the tangent; then their lengths intercepted between *ao* and *bo* are equal to the circumference of circles of radii O_1, O_2, etc. Thus the triangle O*ab* represents the circle rolled out to a straight base. Project each of these lines on to the tangent, and join up to the centre; then the width of the line 1′1′, etc., represents the stress in the metal at that layer in precisely the same manner as in the beam modulus figures. Then—

$$\left.\begin{array}{l}\text{The polar modulus of}\\ \text{the section } Z_p\end{array}\right\} = \left\{\begin{array}{l}\text{area of modulus figure} \times \text{distance of}\\ \text{c. of g. of modulus figure from centre}\\ \text{of circle}\end{array}\right.$$

$$\text{or } Z_p = Ay_c$$

The construction for a hollow circle is precisely the same as for the solid circle. It is given for the sake of graphically illustrating the very small amount that a shaft is weakened by making it hollow.

This construction can be applied to any form of section,

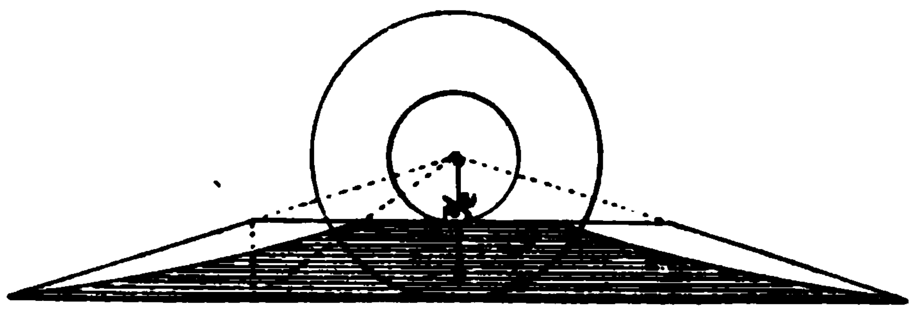


FIG. 471.

but the strengths of shafts other than circular do not vary as their polar moments of inertia or moduli of their sections; serious errors will be involved if they are even taken to be approximately correct. The calculation of the stresses in irregular figures in torsion involves fairly high mathematical work. The results of such calculations by St. Venant and Lord Kelvin will be given in tabulated form later on in this chapter.

Strength of Circular Shafts in Torsion.—We have shown above that the strength of a cylindrical shaft varies as $\frac{I_p}{y} = Z_p$. In Chapter III., we showed that $I_p = \frac{\pi D^4}{32}$, where D is the diameter, and y in this case $= \frac{D}{2}$; hence—

$$Z_p = \frac{\frac{\pi D^4}{32}}{\frac{D}{2}} = \frac{\pi D^3}{16} = \frac{D^3}{5\cdot 1} = 0\cdot 196 D^3$$

which, it will be noticed, is just twice the value of the Z for bending. In order to recollect which is which, it should be remembered that the material in a circular shaft is in the very best form to resist torsion, but in a very bad form to resist bending; hence the torsion modulus will be greater than the bending modulus.

For a hollow shaft—

$$I_p = \frac{\pi(D^4 - D_i^4)}{32}, \text{ where } D_i = \text{the internal diameter}$$

$$\text{hence } Z_p = \frac{\frac{\pi(D^4 - D_i^4)}{32}}{\frac{D}{2}} = \frac{\pi(D^4 - D_i^4)}{16D}$$

$$= 0{\cdot}196\left(\frac{D^4 - D_i^4}{D}\right)$$

Hollow shafts for marine work are nearly always made with the internal diameter equal to one-half the external;

$$\text{or } D_i = \frac{D}{2}$$

Then—

$$Z_p = 0{\cdot}196\left(\frac{D^4 - \frac{D^4}{16}}{D}\right)$$

$$= 0{\cdot}196\tfrac{15}{16}(D^3) = 0{\cdot}184D^3$$

Thus by removing one-quarter the metal from the centre of the shaft, the strength has only been reduced by one-sixteenth; similarly, it can be shown that by removing one-half the metal, the strength will be reduced by one-fourth; or, generally, if $\frac{1}{n}$ of the metal be removed from the centre of the shaft, we have—

$$\text{The internal area} = \frac{\text{the external area}}{n}$$

$$\text{or } \frac{\pi D_i^2}{4} = \frac{\pi D^2}{4n}$$

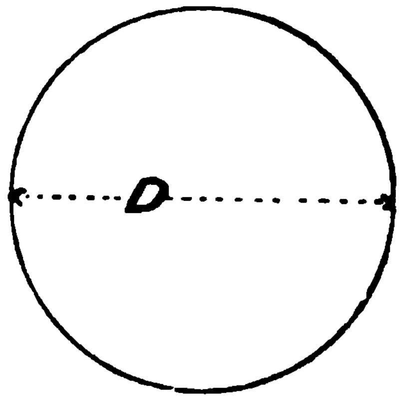
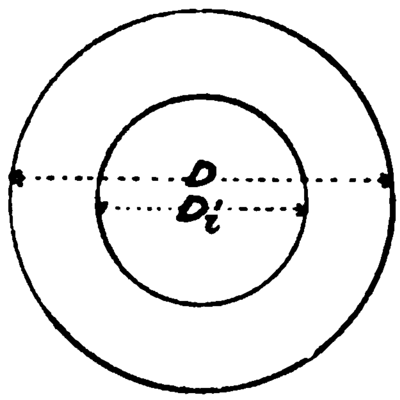
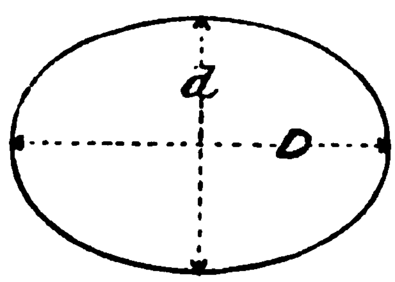
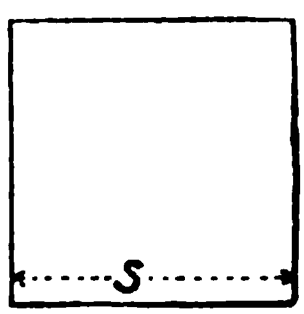
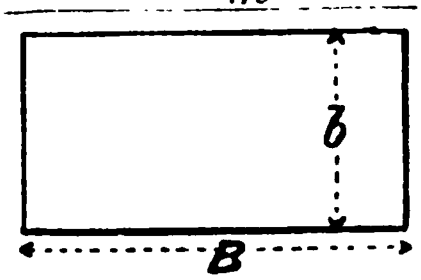
$$\text{or } D_i^2 = \frac{D^2}{n}, \text{ or } D_i^4 = \frac{D^4}{n^2}$$

$$\text{and } Z_p = 0{\cdot}196\left(\frac{D^4 - \frac{D^4}{n^2}}{D}\right)$$

$$= 0{\cdot}196\left(D^3 - \frac{D^3}{n}\right)$$

$$Z_p = 0{\cdot}196D^3\left(1 - \frac{1}{n^2}\right)$$

Strength of Shafts of Various Sections.

 FIG. 472.	$Z_p = \frac{\pi D^3}{16}$, or $Z_p = \frac{D^3}{5 \cdot 1}$, or $Z_p = 0 \cdot 196 D^3$	$\theta = \frac{584 M_t l}{G D^4}$
 FIG. 473.	$Z_p = \frac{D^4 - D_i^4}{5 \cdot 1 D}$, or $Z_p = 0 \cdot 196 \left(\frac{D^4 - D_i^4}{D} \right)$ If $D_i = \frac{D}{m}$ $Z = 0 \cdot 196 D^3 \left(1 - \frac{1}{m^4} \right)$	$\theta = \frac{584 M_t l}{G(D^4 - D_i^4)}$
 FIG 474.	$Z_p = \frac{\pi D d^2}{16}$, or $Z_p = \frac{D d^2}{5 \cdot 1}$, or $Z_p = 0 \cdot 196\ D d$	$\theta = \frac{292 M_t l (d^2 + D^2)}{G D^3 d^3}$
 FIG. 475.	$Z_p = 0 \cdot 208 S^3$	$\theta = \frac{377 M_t l}{S^4 G}$
 FIG. 476.	$Z_p = \frac{B b^2}{3 + 1 \cdot 8 m}$ where $m = \frac{b}{B}$	$\theta = \frac{188 \cdot 5 M_t l (b^2 + B^2)}{b^3 B^3 G}$
Any section not containing re-entrant angles (due to St. Venant).	$Z_p = \frac{A^4}{40 I_p y}$ (approx.) where A = area of section; I_p = polar moment of inertia of section; y = distance of furthest edge from centre of section.	

Twist of Shafts.—In Chapter VIII., we showed that when an element was sheared, the amount of slide x bore the following relation :—

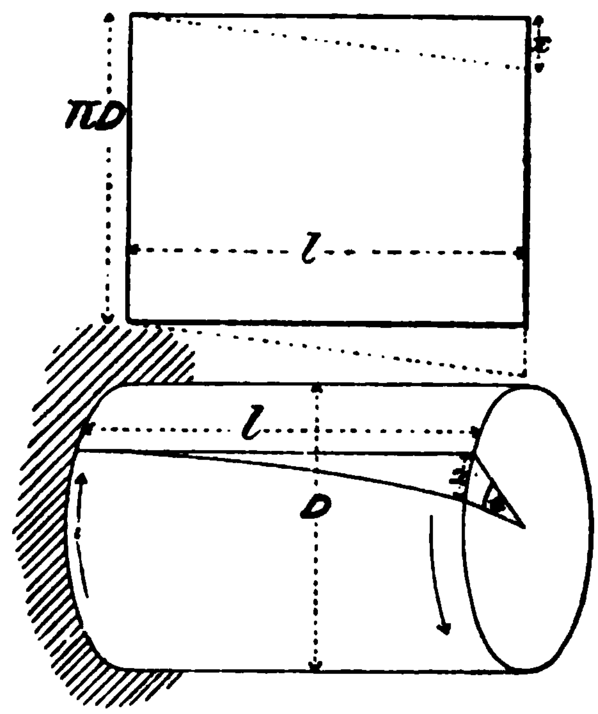


FIG. 477.

$$\frac{x}{l} = \frac{f_s}{G} \quad . \quad . \quad . \quad \text{(i.)}$$

where f_s is the shear stress on the material;
G is the coefficient of rigidity.

In the case of a shaft, the x is measured on the curved surface. It will be more convenient if we express it in terms of the angle of twist.

$$\text{The circumference of the shaft} = \pi D$$

$$\text{The arc subtending } 1^\circ = \frac{\pi D}{360}$$

$$\text{,, \quad ,, } \quad \theta^\circ = \frac{\pi D\theta}{360} = x$$

Substituting the value of x in equation (i.), we have—

$$\frac{\pi D\theta}{360 l} = \frac{f_s}{G}, \text{ or } \theta = \frac{360 f_s l}{\pi G D}$$

$$\text{But } M_t = f_s Z_p = f_s \frac{\pi D^3}{16}, \text{ and } f_s = \frac{16 M_t}{\pi D_3}$$

$$\text{hence } \theta = \frac{360 \times 16 \times M_t l}{\pi^2 G D^4} = \frac{584 M_t l}{G D^4}$$

for solid circular shafts. Substituting the value of Z_p for a hollow shaft in the above, we get—

$$\theta = \frac{584 M_t l}{G(D^4 - D_i^4)}$$

for hollow circular shafts.

N.B.—The twist of a hollow shaft is the difference of the twists of two solid shafts whose diameters are respectively the outer and inner diameters of the hollow shaft.

When it is desired to keep the twist or spring of shafts

within narrow limits, the stress has to be correspondingly reduced. Long shafts are frequently made very much stronger than they need be in order to reduce the spring. A common limit to the amount of spring is 1° in 20 diameters; the stress corresponding to this is arrived at thus—

$$\text{We have above } \theta = \frac{360 f_s l}{\pi G D}$$

$$\text{but when } \theta = 1°, \; l = 20D$$

$$\text{then } f_s = \frac{\pi G D}{360 \times 20D} = \frac{G}{2292}$$

For steel, G = 13,000,000; f_s = 5670 lbs. per sq. inch
Wrought iron, G = 11,000,000; f_s = 4800 „ „
Cast iron, G = 6,000,000; f_s = 2620 „ „

In the case of short shafts, in which the spring is of no importance, the following stresses may be allowed:—

Steel, f_s = 10,000 lbs. per sq. inch
Wrought iron, f_s = 8000 „ „
Cast iron, f_s = 3000 „ „

Horse-power transmitted by Shafts.—Let a force of P lbs. act at a distance r inches from the centre of a shaft; then—

$$\left.\begin{array}{l}\text{The twisting moment on the shaft}\\ \text{in lbs.-inches}\end{array}\right\} = \text{P (lbs.)} \times r \text{ (inches)}$$

$$\left.\begin{array}{l}\text{The work done per}\\ \text{revolution in foot-lbs.}\end{array}\right\} = \frac{\text{P (lbs.)} \times r \text{ (inches)} \times 2\pi}{12}$$

$$\left.\begin{array}{l}\text{The work done per}\\ \text{minute in foot-lbs.}\end{array}\right\} = \frac{\text{P (lbs.)} \times r \text{ (inches)} \times 2\pi \text{N (revs.)}}{12}$$

where N = number of revolutions per minute.

$$\text{The horse-power transmitted} = \frac{2\pi P r N}{12 \times 3300} = \text{H.P.}$$

then—

$$Pr = \frac{12 \times 33000 \times \text{H.P.}}{2\pi N} = \frac{f_s D^3}{5.1}$$

$$D^3 = \frac{12 \times 33000 \times \text{H.P.} \times 5.1}{2\pi N f_s} = \frac{321400 \text{ H.P.}}{N f_s}$$

$$= \frac{64.3 \text{ H.P.}}{N}$$

taking f_s at 5000 lbs. per square inch.

$$D = 4\sqrt[3]{\frac{\text{H.P.}}{\text{N}}} \text{ (nearly) for 5000 lbs. per sq. inch}$$

$$= 3\cdot5\sqrt[3]{\frac{\text{H.P.}}{\text{N}}} \text{ for 7500 lbs. per sq. inch}$$

$$= 3\sqrt[3]{\frac{\text{H.P.}}{\text{N}}} \text{ for 12,000 lbs. per sq. inch}$$

Taking the value of $Z_p = 0\cdot184D^3$ for hollow shafts having the internal diameter equal to half the external, we get—

$$D = 4\cdot1\sqrt[3]{\frac{\text{H.P.}}{\text{N}}} \text{ for 5000 lbs. per sq. inch}$$

$$= 3\cdot56\sqrt[3]{\frac{\text{H.P.}}{\text{N}}} \text{ for 7500 lbs. per sq. inch}$$

$$= 3\cdot05\sqrt[3]{\frac{\text{H.P.}}{\text{N}}} \text{ for 12,000 lbs. per sq. inch}$$

Combined Torsion and Bending.—In Fig. 478 a shaft is shown subjected to torsion only. We have previously seen

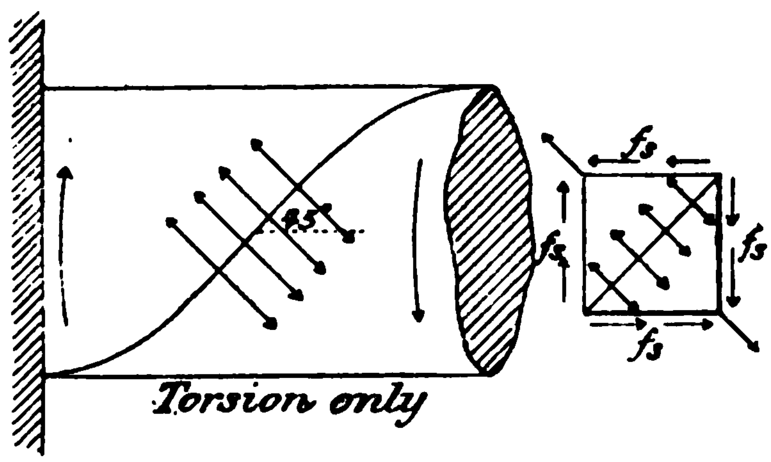


FIG. 478.

(Chap. VIII.) that in such a case there is a tension acting

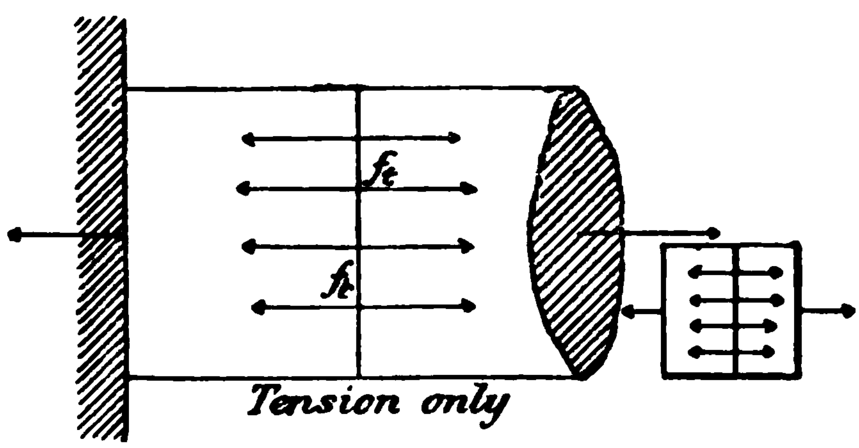


FIG. 479.

normal to a diagonal drawn at an angle of 45° with the axis of the shaft, as shown by the arrows in the figure. In Fig. 479 a shaft is shown subjected to tension only. In this case the

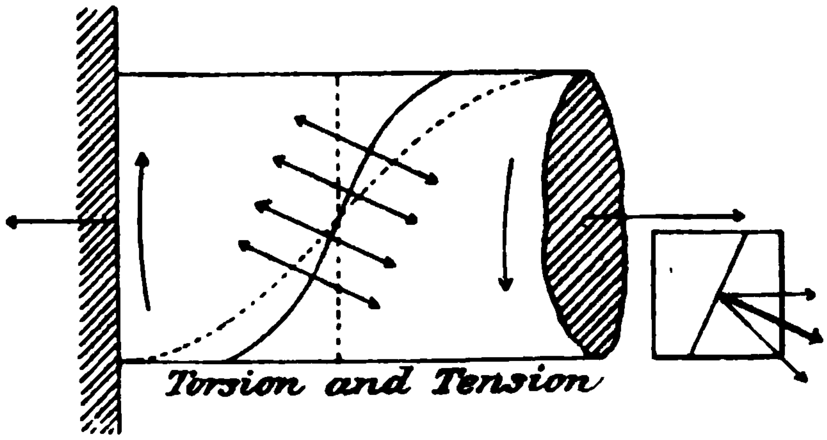


FIG. 480.

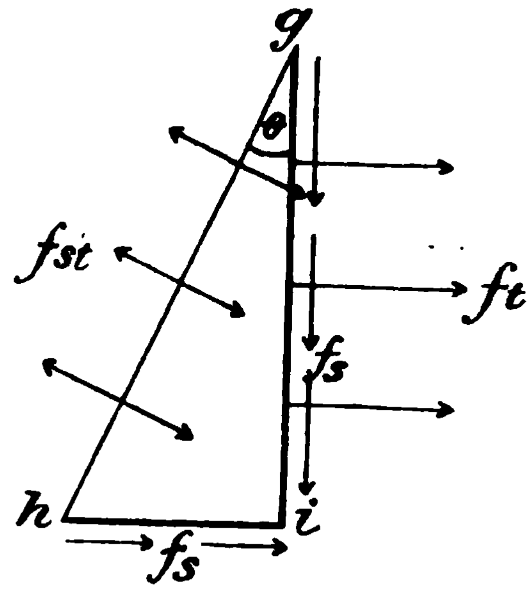


FIG. 481.

tension acts normally to a face at 90° with the axis. In Fig. 480 a shaft is shown subjected to both torsion and tension; the face over which there is the greatest tension will therefore lie between the two faces mentioned above, and the tension on this face will be greater than the tension on either of the other faces, when acted upon only by torsion or tension.

We have shown in Chapter VIII. that the stress f_{st} normal to the face *gh* due to combined tension and shear is—

$$f_{st} = \frac{f_t}{2} + \sqrt{\frac{f_t^2}{4} + f_s^2}$$

If the tension be produced by bending, we have—

$$f_t = \frac{M}{Z}$$

Likewise, if the shear be produced by twisting—

$$f_s = \frac{M_t}{Z_p}$$

$$\text{But } Z_p = 2Z$$

$$\text{then } f_s = \frac{M_t}{2Z}$$

Substituting these values in the above equation—

$$f_{st} = \frac{M}{2Z} + \sqrt{\frac{M^2}{4Z^2} + \frac{M_t^2}{4Z^2}}$$

$$= \frac{1}{2Z}(M + \sqrt{M^2 + M_t^2})$$

$$f_s Z = M_e = \frac{M + \sqrt{M^2 + M_t^2}}{2}$$

$$\text{also } f_s \times 2Z = f_s Z_p = M_{et} = M + \sqrt{M^2 + M_t^2}$$

The M_e is termed the equivalent bending moment, the M_{et} the equivalent twisting moment, that would produce the same intensity of stress in the material as the combined bending and twisting.

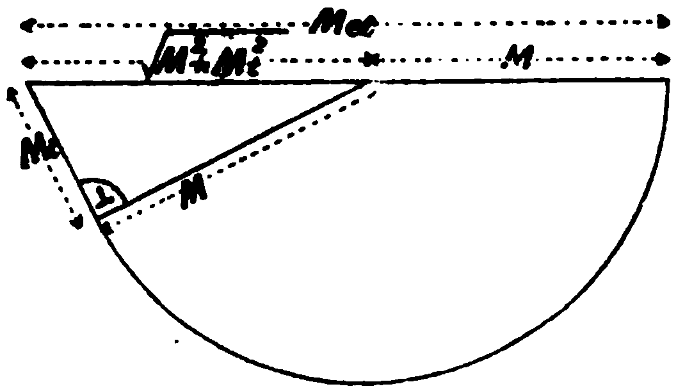


FIG. 482.

The construction shown in Fig. 482 is a convenient graphical method of finding M_{et}:—

In Chapter VIII. we also showed that—

$$p\overline{gi} = p_0\overline{gh}\sin\theta$$

$$\frac{p\overline{gi}}{p_0\overline{gh}} = \frac{p}{p_0}\cos\theta = \sin\theta$$

$$\frac{p}{p_0} = \frac{\sin\theta}{\cos\theta} = \tan\theta$$

Or, using the notation above—

$$\frac{f_s}{f_{st}} = \tan\theta$$

From this expression we can find the angle of greatest stress θ, and therefore the angle at which fracture will probably occur.

In Fig. 483 we show the fractures of two cast-iron torsion test-pieces, the one broken by pure torsion, the other by combined torsion and bending. Around each a spiral piece of paper has been wrapped in order to show how the angle of

fracture agreed with the theoretical angle θ; the agreement is remarkably close.

Whirling of Shafts.—A horizontal shaft sags between its bearings due to its own weight and that of the pulleys, hence during each revolution it is bent to and fro.

If the period of this disturbing action approximates to the

Pure torsion

Combined torsion and bending.

FIG. 483.

natural period of vibration of the shaft, the amplitude of the vibrations will continue to increase, and set up a violent whipping action known as whirling; if the speed of rotation be increased beyond this whirling speed, the shaft will again become steady.

The treatment of an unloaded shaft, *i.e.* one without pulleys, is tolerably simple (see Rankine's "Machinery and Millwork," p. 549), but it becomes very complex in the case of a loaded shaft. A very thorough and able investigation of this question has been undertaken by Professor Dunkerley, of Greenwich, who has not only shown mathematically the speed at which

whirling takes place, but has proved the correctness of his deductions by a series of most careful and interesting experiments.

It would be impossible to do the barest justice to his work without occupying many pages of our already too limited space; the reader is therefore referred to his paper on "The Whirling and Vibration of Shafts" (*Phil. Trans.*, vol. 185A, pp. 279–360), or to his paper read before the Liverpool Engineering Society, 1894–5, which is more suited to the average engineer.

Helical Springs.—The wire in a helical spring is, to all intents and purposes, subjected to pure torsion, hence we can readily determine the amount such a spring will stretch or compress under a given load, and the load it will safely carry.

We may regard a helical spring as a long thin shaft coiled into a helix, hence we may represent our helical spring thus:

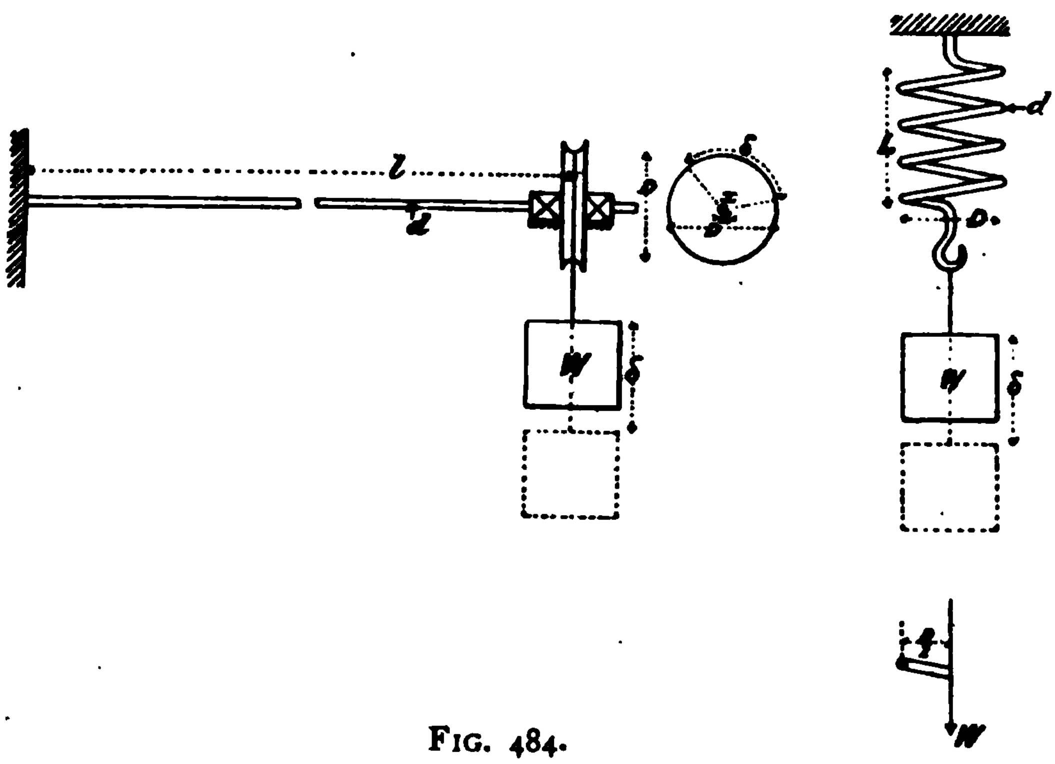


FIG. 484.

In the figure to the left we have the wire of the helical spring straightened out into a shaft, and provided with a grooved pulley of diameter D, *i.e.* the mean diameter of the coils in the spring; hence the twisting moment upon it is $\frac{WD}{2}$. That the twisting moment on the wire when coiled into a helix is also $\frac{WD}{2}$ will be clear from the bottom right-hand figure. The length of wire in the spring (not including the ends and hook) is equal to l. Let $n =$ the number of coils; then $l = \pi Dn$ nearly, or more accurately $l = \sqrt{(\pi Dn)^2 + L^2}$, Fig. 485, a refinement which is quite unnecessary for springs as ordinarily made.

When the load W is applied, the end of the shaft twists, so that a point on the surface moves through a distance x, and a point on the rim of the pulley moves through a distance δ,

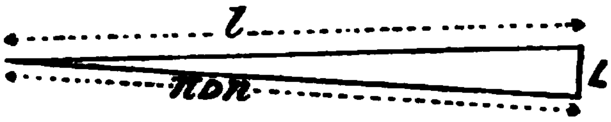


Fig. 485.

where $\frac{\delta}{D} = \frac{x}{d}$, and $\delta = \frac{Dx}{d}$.

$$\text{But we have } \frac{x}{l} = \frac{f_s}{G} \quad \therefore \quad x = \frac{lf_s}{G}$$

$$\text{hence } \delta = \frac{Dlf_s}{Gd} \quad \ldots \ldots \quad \text{(i.)}$$

$$\text{Also } \frac{WD}{2} = f_s Z_p = \frac{f_s \pi d^3}{16}$$

$$f_s = \frac{16WD}{2\pi d^3} = \frac{2{\cdot}55WD}{d^3} \quad \ldots \quad \text{(ii.)}$$

$$\text{then } \delta = \frac{2{\cdot}55D^2lW}{Gd^4} \text{ (from i. and ii.)}$$

Substituting the value of $l = n\pi D$—

$$\delta = \frac{2{\cdot}55D^2Wn\pi D}{Gd^4} = \frac{8D^3Wn}{Gd^4}$$

$$\text{G for steel} = 12{,}000{,}000 \qquad \delta = \frac{D^3Wn}{1{,}500{,}000d^4}$$

$$\text{G for hard brass} = 5{,}000{,}000 \qquad \delta = \frac{D^3Wn}{625{,}000d^4}$$

The load a spring will support with a given deflection is—

$$W = \frac{Gd^4\delta}{8D^3n} = \frac{1{,}500{,}000d^4\delta}{D^3n} \text{ for steel}$$

$$W = \frac{625{,}000d^4\delta}{D^3n} \text{ for brass}$$

Safe load.—From equation (ii.), we have—

$$W = \frac{f_s d^3}{2{\cdot}55D} = \frac{27{,}500d^3}{D} \text{ for 70,000 lbs. per square inch} \quad . \quad \text{(iii.)}$$

$$= \frac{23{,}500d^3}{D} \text{ for 60,000 lbs. per square inch}$$

$$= \frac{19{,}600d^3}{D} \text{ for 50,000 lbs. per square inch}$$

Experiments by Mr. Wilson Hartnell show that for steel wire the following stresses are the maxima consistent with safety :—

Diameter of wire.	Safe stress.
$\frac{1}{4}$ inch	70,000 lbs. per square inch
$\frac{3}{8}$,, ...	60,000 lbs. ,, ,,
$\frac{1}{2}$,, ...	50,000 lbs. , ,,

Taking a mean value, we may say—

$$W = \frac{24{,}000d^3}{D}$$

Work stored in Springs.

$$\left.\begin{array}{r}\text{The work done in stretching}\\ \text{or compressing a spring}\end{array}\right\} = \frac{W\delta}{2} \text{ (see page 452)}$$

$$= \frac{f_s d^3 \times Dlf_s}{2 \times 2{\cdot}55D \times Gd} \text{ (from ii. and iii.)}$$

$$= \frac{f_s^2 d^2 l}{5{\cdot}1G}$$

$$\text{(substituting the value of } l\text{)} \quad = \frac{f_s^2 d^2 nD}{1{\cdot}62G} \text{ (inch-lbs.)}$$

$$= \frac{f_s^2 d^2 nD}{19{,}500{,}000}$$

$$\text{putting } G = 12{,}000{,}000$$

Weight of Spring.—Taking the weight of 1 cub. inch of steel = 0·28 lb., then—

The weight of the spring $w = 0{\cdot}785d^2l \times 0{\cdot}28 = 0{\cdot}22d^2l$

Substituting the value of l, we have—

$$w = 0{\cdot}69nd^2D$$

Height a Steel Spring will lift itself (h).

$$h = \frac{\text{work stored in spring}}{\text{weight of spring}}$$

$$h = \frac{f_s^2 d^2 nD}{1{\cdot}62G \times 0{\cdot}69nd^2D} = \frac{f_s^2}{1{\cdot}12G} \text{ inches}$$

$$= \frac{f_s^2}{13{\cdot}4G} = \frac{f_s^2}{161{,}000{,}000} \text{ feet}$$

The value of h is given in the following table corresponding to various values of f:—

f_s (lbs. per square inch) ...	30,000	60,000	90,000	120,000	150,000
h (feet)	5·56	22·4	50·3	89·5	139·8

These figures are of interest in showing what a very small amount of energy can be stored in springs.

All the quantities given above are for springs made of wire of circular section; for wire of square section of side S they become—

$$\delta = \frac{D^3Wn}{2,320,000S^4} \text{ (steel springs)}$$

$$\delta = \frac{D^3Wn}{967,000S^4} \text{ (brass springs)}$$

$$W = \frac{2,320,000S^4\delta}{D^3n} \text{ steel}$$

$$W = \frac{967,000S^4\delta}{D^3n} \text{ brass}$$

Safe load.

$$W = \frac{29,120S^3}{D} \text{ stress } 70,000 \text{ lbs. per square inch}$$

$$= \frac{24,960S^3}{D} \quad \text{,,} \quad 60,000 \text{ lbs.} \quad \text{,,} \quad \text{,,}$$

$$= \frac{20,800S^3}{D} \quad \text{,,} \quad 50,000 \text{ lbs.} \quad \text{,,} \quad \text{,,}$$

Taking a mean value we have—

$$W = \frac{25,000S^3}{D}$$

$$\text{work stored} = \frac{f_s^2S^2nD}{26,840,000} \text{ (inch-lbs.) steel}$$

$$\text{weight} = 0·88nS^2D \text{ (steel)}$$

$$\left.\begin{array}{l}\text{Height a square section spring}\\ \text{will lift itself (steel)}\end{array}\right\} h = \frac{f_s^2}{283,680,000} \text{ (feet)}$$

f_s (lbs. per square inch) ...	30,000	60,000	90,000	120,000	150,000
h (feet)	3·17	12·7	28·6	50·8	79·3

It will be observed that in no respect is a square-section spring so economical in material as a spring of circular section.

CHAPTER XV.

STRUCTURES.

Wind Pressures.—All structures, more or less, are exposed to wind pressure. In many instances, the force of the wind is the greatest force a structure ever has to withstand.

Let v = velocity of the wind in feet per second;
V = velocity of the wind in miles per hour;
m = mass of air delivered per square foot per second;
W = weight of air delivered per square foot per second (pounds);
w = weight of 1 cubic foot of air (say 0·0807 lb.);
P = pressure of wind per square foot of surface exposed (pounds).

Then, when the wind impinges on a flat surface, the momentum destroyed per square foot per second is—

$$mv = P$$

$$\text{or } \frac{Wv}{g} = P$$

$$\text{But } W = wv$$

$$\text{hence } \frac{wv^2}{g} = P$$

or expressed in miles per hour by substituting $v = 1{\cdot}466V$, and putting in the value of w, we have—

$$P = \frac{0{\cdot}0807 \times 1{\cdot}466^2 \times V^2}{32{\cdot}2} = 0{\cdot}0054V^2$$

Smeaton, based on Rouse's experiments, gives—

$$P = 0{\cdot}005V^2$$

Others have found somewhat lower results, due partly to the density of the air (w) being less at higher temperatures, and to the fact that they have assumed the pressure to be

constant over large areas of gauge surface, which we show below to be incorrect.

When a wind blows horizontally on a flat, inclined surface, the pressure in horizontal, vertical, and normal directions may be arrived at thus—

$$\begin{aligned} \text{the normal pressure } P_n &= P \,.\, \sin\theta \\ \text{the horizontal pressure } P_h &= P_n \sin\theta \\ &= P \sin^2\theta \\ \text{the vertical pressure } P_v &= P_n \cos\theta \\ &= P \,.\, \cos\theta \,.\, \sin\theta \end{aligned}$$

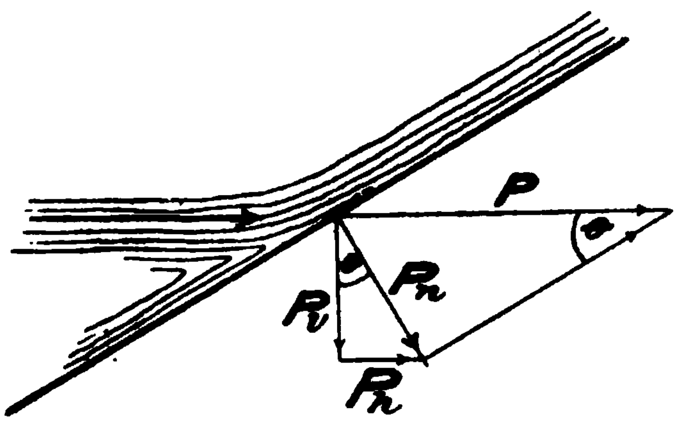


FIG. 486.

In the above we have neglected the friction of the air moving over the inclined surface, which will probably account for the discrepancy between the calculated pressure and that found by experiment. The following table will enable a comparison to be made. The experimental values have been reduced to a *horizontal* pressure of 40 lbs. per square foot.

Angle of roof.	Normal pressure.		Vertical pressure.		Horizontal pressure.	
	Experiment.[1]	P sin θ.	Experiment.	P cos θ sin θ.	Experiment.	P $\sin^2\theta$.
10°	9·7	7·0	9·6	6·9	1·7	1·2
20°	18·1	13·7	17·0	12·9	6·2	4·7
30°	26·4	20·0	22·8	17·3	13·2	10·0
40°	33·3	25·7	25·5	19·7	21·4	16·5
50°	38·1	30·6	24·5	19·7	29·2	23·5
60°	40·0	34·6	20·0	17·3	34·0	30·0
70°	41·0	37·6	14·0	12·9	38·5	35·4

When the wind blows upon a surface other than plane, the pressure on the projected area depends upon the form of the surface. The following table gives some idea of the relative wind-resistance of various surfaces, as found by various experimenters :—

[1] Deduced from Hutton's experiments by Unwin (see "Iron Roofs and Bridges").

Flat plate	1
Parachute (concave surface), depth = $\frac{\text{diameter}}{3}$ ...	about 2
Sphere	0·36–0·41
Elongated projectile	0·5
Cylinder	0·54–0·57
Wedge (base to wind)	0·97
„ (edge to wind), vertex angle 90°	0·6–0·7
Cone (base to wind)	0·95
„ (apex to wind), vertex angle 90°	0·69–0·72
„ „ „ 60°	0·54

The pressure and velocity of the wind increase very much as the height above the ground increases (Stevenson's experiments).

Feet above ground	5	9	15	25	52
Velocity in miles per hour ...	4	6	6	6·5	7·5
	7	17	18	21	23
	13	23	25	30	32
	19	28	31	35	40
	26	32	34	37	43

The wind pressure measured by small gauges is always higher than when measured with gauges offering a large surface to the wind, probably because the highest pressures are only confined to very small areas, and are much greater than the mean taken on a larger surface.

In the experiments at the Forth Bridge, this was very clearly shown. For example—

Date.	Small revolving gauge.	Small fixed gauge.	Large fixed gauge, 15′ × 20′.		
			Mean.	Centre.	Corner.
Mar. 31, 1886	26	31	19	28½	22
Jan. 25, 1890	27	24	18	23½	22

In designing structures, it is usual to allow for a pressure of 40 lbs. per square foot. In very exposed positions this may not be excessive, but for inland structures, unless exceptionally exposed, 40 lbs. is unquestionably far too high an estimate.

For further information on this question the reader should refer to special works on the subject, such as Walmisley's "Iron Roofs."

Weight of Roof Coverings.—For preliminary estimates, the weights of various coverings may be taken as—

Covering.	Weight per sq. foot in pounds.
Slates ...	8–9
Tiles (flat)	12–20
Corrugated iron	$1\frac{1}{2}$–$3\frac{1}{2}$
Asphalted felt	2–4
Lead ...	5–8
Copper	1–$1\frac{1}{2}$
Snow [1]	5

Weight of Roof Structures.—For preliminary estimates, the following formulas will give a fair idea of the probable weight of the ironwork in a roof.

Let W = weight of ironwork per square foot of *covered area* (*i.e.* floor area) in pounds;
D = distance apart of principals in feet;
S = span of roof in feet.

Then for trusses—

$$W = \frac{\sqrt{DS}}{8} + 3$$

and for arched roofs—

$$W = \frac{\sqrt{DS}}{4} + 3$$

Distribution of Load on a Roof.—It will often save trouble and errors if a sketch be made of the load distribution on a roof in this manner.

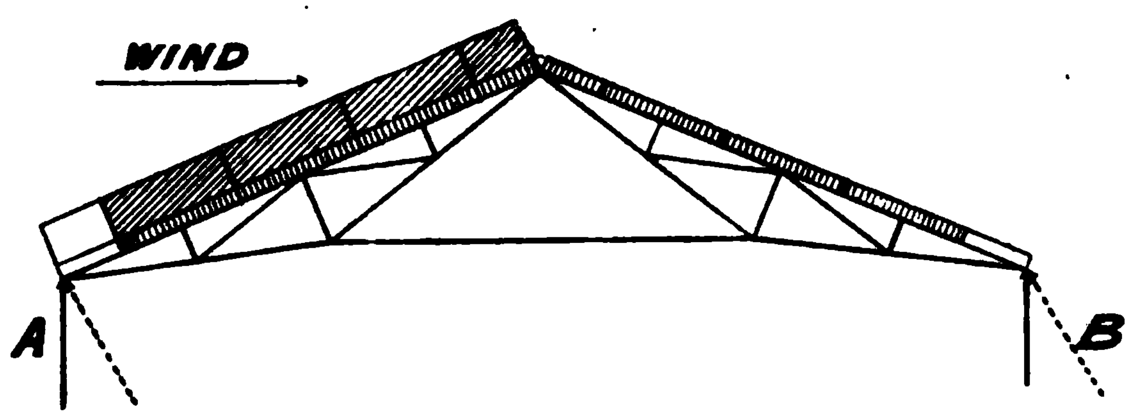


FIG. 487.

The height of the diagram shaded normal to the roof is the weight of the covering and ironwork (assumed uniformly

[1] Many authorities consider that the weight of snow should not be taken into account, because the snow will not settle if a strong wind be blowing, and if the wind be allowed for, the weight of the snow is negligible.

distributed). The height of the diagonally shaded diagram represents the wind pressure on the one side. The lowest section of the diagram on each side is left unshaded, to indicate that if both ends of the structure are rigidly fixed to the supporting walls, that portion of the load may be neglected as far as the structure is concerned. But if A be on rollers, and B be fixed, then the wind-load only on the A side must be taken into account.

When the slope of the roof varies, as in curved roofs, the height of the wind diagram must be altered accordingly. An instance of this will be given shortly.

The whole of the covering and wind-load must be concentrated at the joints, otherwise bending stresses will be set up in the bars.

Method of Sections.—Sometimes it is convenient to check the force acting on a bar by a method known as the method of sections—usually attributed to Ritter, but really due to Rankine—termed the method of sections, because the structure is supposed to be cut in two, and the forces required to keep it in equilibrium are calculated by taking moments.

Suppose it be required to find the force acting along the bar *pq*. Take a section through the structure *ab*; then three forces, *pa*, *qe*, *pq*, must be applied to the cut bars to keep the structure in equilibrium. Take moments about the point O. The forces *pa* and *qe* pass through O, and therefore have no moment about it; but *pq* has a moment $pq \times y$ about O.

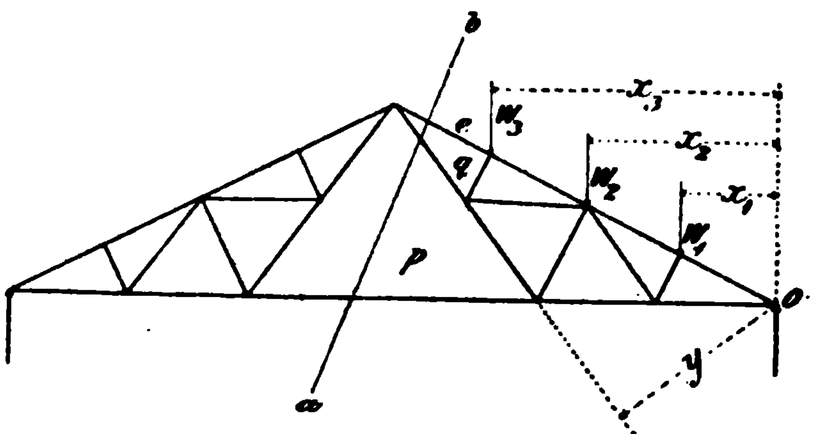


FIG. 488.

$$\overline{pq} \times y = W_1x_1 + W_2x_2 + W_3x_3$$

$$\overline{pq} = \frac{W_1x_1 + W_2x_2 + W_3x_3}{y}$$

By this method forces may often be arrived at which are difficult by other methods.

Forces in Roof Structures.—We have already shown in Chapter IV. how to construct force or reciprocal diagrams for simple roof structures. Space will only allow of our now dealing with one or two cases in which difficulties may arise.

In the truss shown (Fig. 489), a difficulty arises after the force in the bar *tu* has been found. Some writers, in order to get over the difficulty, assume that the force in the bar *rs* is the same as in *ut*. This may be the case when the structure is evenly loaded, but it certainly is not so when wind is acting on one side of the structure. We have taken the simplest case of loading possible, in order to show clearly the special method of dealing with such a case.

The method of drawing the reciprocal diagram has already been described. We go ahead in the ordinary way till we reach

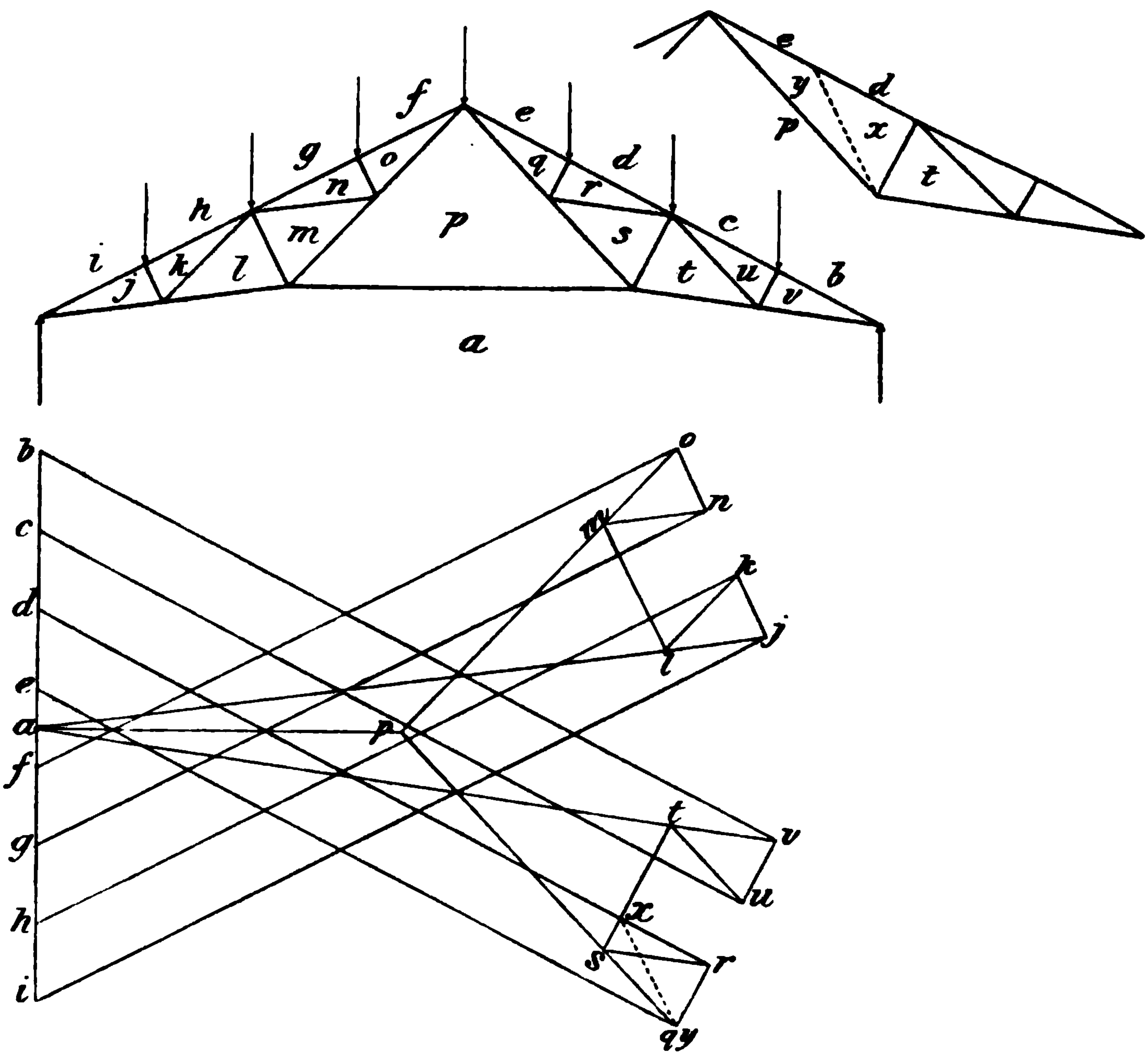


FIG. 489.

the bar *st* (Fig. 489). In the place of *sr* and *rq* substitute a temporary bar *xy*, shown dotted in the side figure. With this alteration we can now get the force *ey* or *eq*; then *qr*, *rs*, etc., follow quite readily; also the other half of the structure.

There are other methods of solving this problem, but the one given is believed to be the best and simplest. The author

is indebted to Professor Barr, of Glasgow University, for this method.

When the wind acts on a structure, having one side fixed and the other on rollers, the only difficulty is in finding the reactions. The method of doing this by means of a funicular polygon is shown in Fig. 490. The funicular polygon has been fully described in Chapters IV. and X, hence no further description is necessary. The direction of the reaction at the fixed support

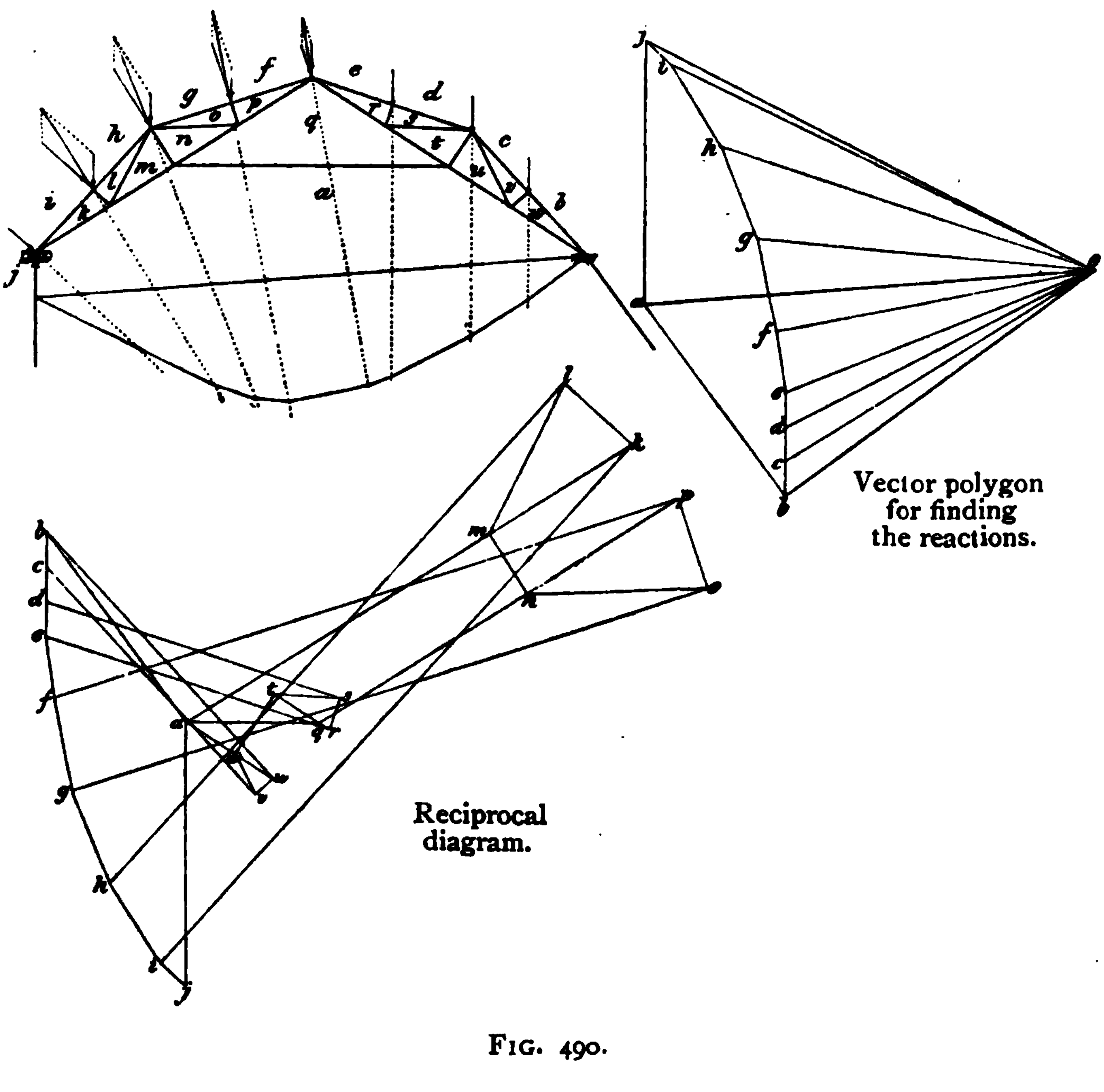


FIG. 490.

is unknown, but as it must pass through the point where the roof is fixed, the funicular polygon should be started from this point.

The direction of the reaction at the roller end is vertical, hence from *j* in the vector polygon a perpendicular is dropped to meet the ray drawn parallel to the closing line of the funicular polygon. This gives us the point *a*; then, joining *ba*, we get the direction of the fixed reaction. The reciprocal diagram is also constructed; it presents no difficulties beyond that mentioned in the last paragraph.

In the figure, the vertical forces represent the dead weight on the structure, and the inclined forces the wind. The two are combined by the parallelogram of forces.

In designing a structure, a reciprocal diagram must be drawn for the structure, both when the wind is on the roller and on the fixed side of the structure, and each member of the structure must be designed for the greatest load.

The nature of the forces, whether compressive or tensional, must be obtained by the method described in Chapter IV.

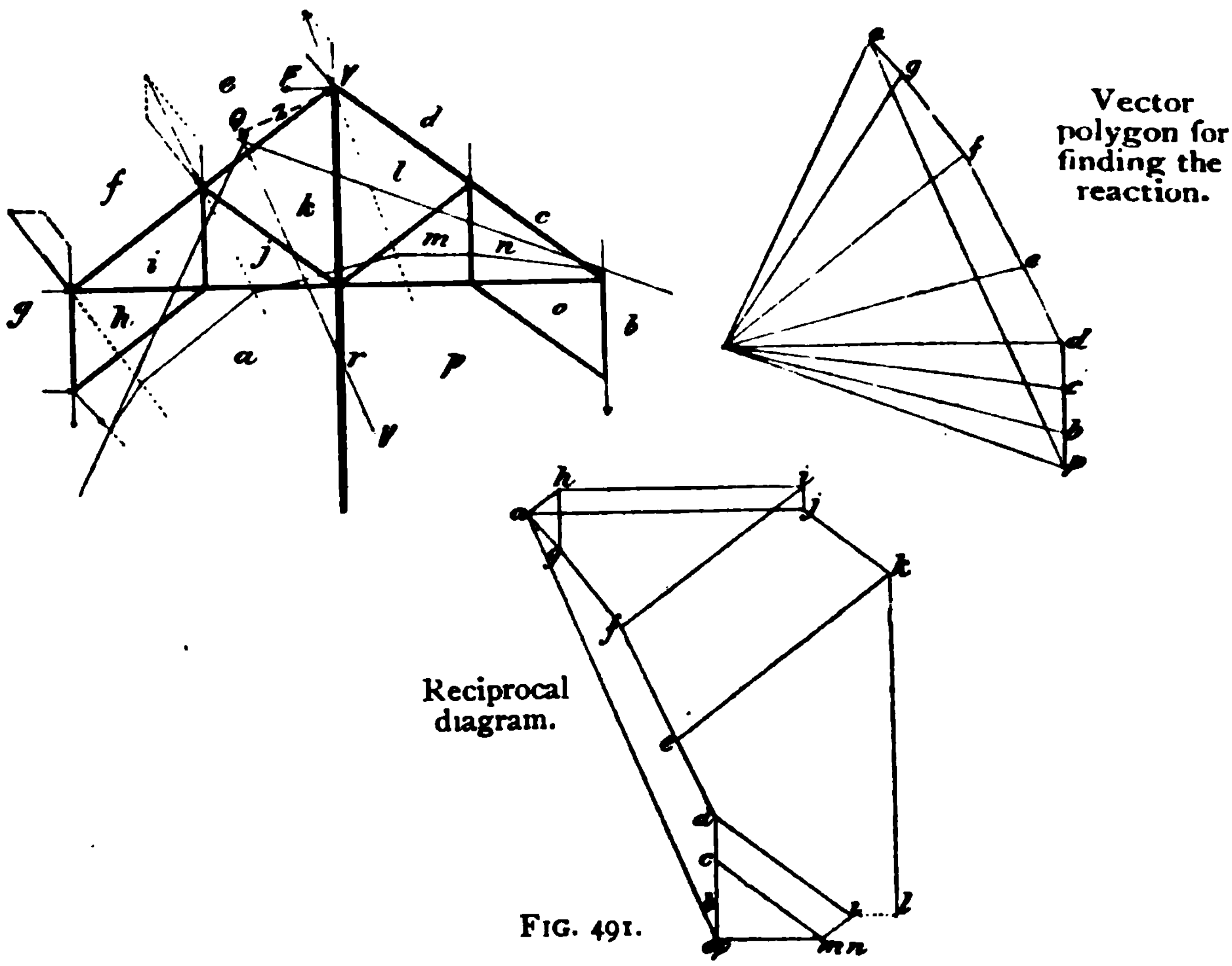


FIG. 491.

Island Station Roof.—This roof presents one or two interesting problems, especially the stresses in the main post. The determination of the resultant of the wind and dead load at each joint is a simple matter. The resultant of all the forces is given by *pa* on the vector polygon in magnitude and direction (Fig. 491). Its position on the structure must then be determined. This has been done by constructing a funicular polygon in the usual way, and producing the first and last links to meet in the point Q. Through Q a line is drawn parallel to *pa* in the vector polygon. This resultant cuts the post in *r'*, and may be resolved into its horizontal and vertical components, the horizontal components producing bending moments of different sign, thus giving the post a double curvature (Fig. 492).

The bending moment on the post is obtained by the product $\overline{pa} \times Z$, where Z is the perpendicular distance of Q from the centre line of the post.

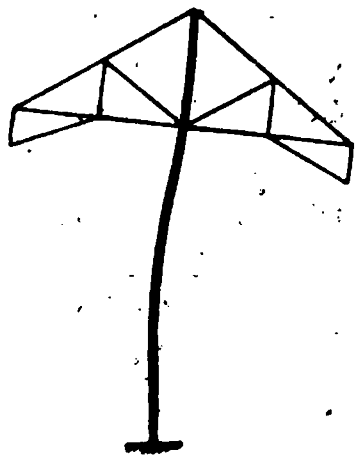


FIG. 492.

When using reciprocal diagrams for determining the stresses in structures, we can only deal with direct tensions and compressions. But in the present instance, where there is bending in one of the members, we must introduce an imaginary external force to prevent this bending action. It will be convenient to assume that the structure is pivoted at the virtual joint r, and that an external horizontal force F is introduced at the apex Y to keep the structure in equilibrium. The value of F is readily found thus. Taking moments about r, we have—

$$F \times rY = \overline{pa} \times z$$

$$\text{or } F = \frac{\overline{pa} . z}{rY}$$

z is the perpendicular distance from the point Y to the resultant.

On drawing the reciprocal diagram, neglecting F, it will be found that it will not close. This force is shown dotted on the reciprocal diagram $l\,i$, and on measurement will be found to be equal to F.

Dead and Live Loads on Bridges.—The dead loads consist of the weight of the main and cross girders, floor,

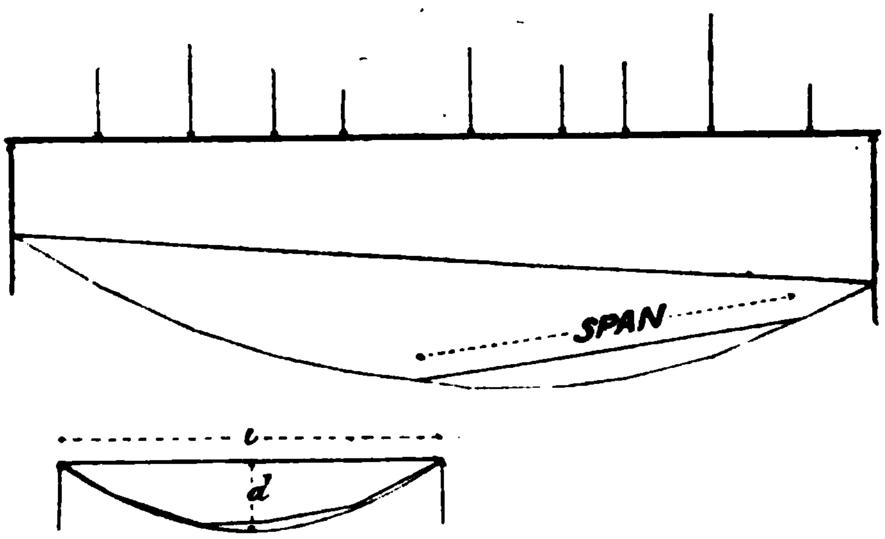


FIG. 493.

ballast, etc., and, if a railway bridge, the permanent way; and the live loads consist of the train or other traffic passing over the bridge, and the wind pressure.

The determination of the amount of the dead loads and the resulting bending moment is generally quite a simple matter. In order to simplify matters, it is usual to assume (in small bridges) that the dead load is evenly distributed, and consequently that the bending-moment diagram is parabolic.

In arriving at the bending moment on railway bridges, an equivalent evenly distributed load is often taken to represent the actual but somewhat unevenly distributed load due to a passing train. The maximum bending moment produced by a train which covers a bridge (treated as a standing load) can be arrived at thus (Fig. 493). Take a span greater than twice the actual span, so as to get every possible combination of loads that may come on the structure. Construct a bending-moment diagram in the ordinary way, then find by trial where the greatest bending moment occurs, by fitting in a line whose length is equal to the span. A parabola may then be drawn to enclose this diagram as shown in the lower figure; then, if d = depth of this parabola to proper scale, we have $\frac{wl^2}{8} = d$, where w is the equivalent evenly distributed load due to the train. (The small diagram is not to scale in this case.)

Let W_r = total rolling load in tons distributed on each pair of rails;

S = span in feet.

Then for English railways $W_r = 1{\cdot}6S + 20$.

Maximum Shear due to a Rolling Load. *Concentrated Rolling Load.*—The shear at any section is equal to the algebraic sum of the forces acting to the right or left of any section, hence the shear between the load and either abutment is equal to the reaction at the abutment. We have above—

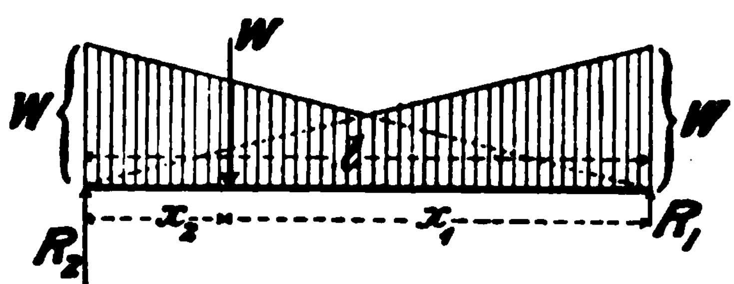


FIG. 494.

$$R_1 l = Wx_2$$

$$R_1 = W\frac{x_2}{l}$$

$$\text{likewise } R_2 = W\frac{x_1}{l}$$

When the load reaches the abutment, the shear becomes W,

and when in the middle of the span the shear is $\frac{W}{2}$. The shear diagram is shown shaded vertically.

Uniform Rolling Load, whose Length exceeds the Span.—Let w = the uniform load per foot run.

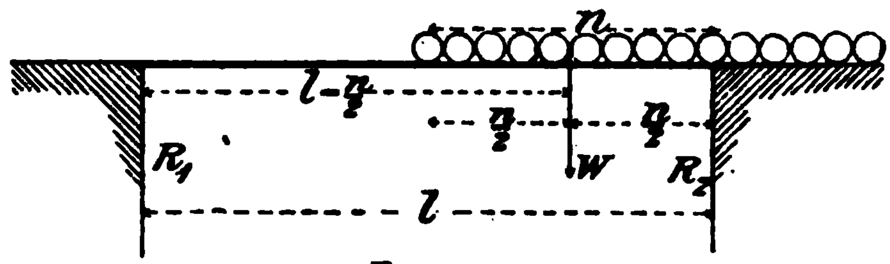


FIG. 495.

The total load on the structure $= wn = W$

$$\text{Then } R_1\left(l - \frac{n}{2}\right) = R_2\left(\frac{n}{2}\right)$$

$$R_1 l = \frac{n}{2}(R_1 + R_2) = \frac{wn^2}{2}$$

$$R_1 = \frac{w}{2l}n^2 = Kn^2$$

where K is a constant for any given case. But as the train passes from the right to the left abutment, the shear between the head of the train and the left abutment is equal to the left reaction R_1, and this varies as the square of the covered length of the bridge, hence the curve of shear is a parabola.

When the train reaches the abutment, $l = n$;

$$\text{then } R_1 = \frac{wn}{2} = \frac{W}{2}$$

The curves of shear for both dead and live loads are shown in Fig. 496. When a train passes from right to left over the point f, the shear is reversed in sign, because the one shear is positive and the other negative. The distance x between the two points f, f is known as the "focal distance" of the bridge.

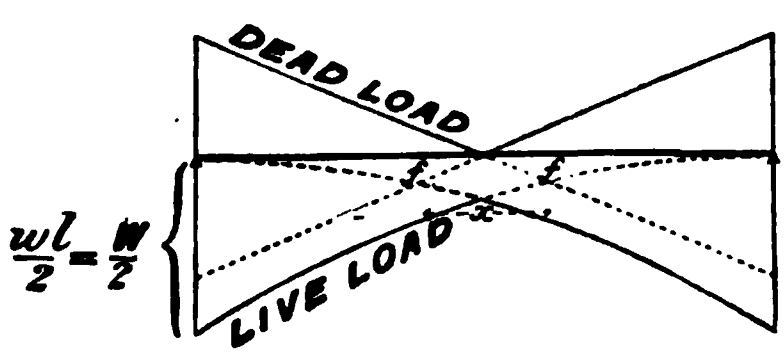


FIG. 496.

Determination of Forces acting on the Members of a Girder.—The small

forces on the top and bottom joints represent the **dead** load

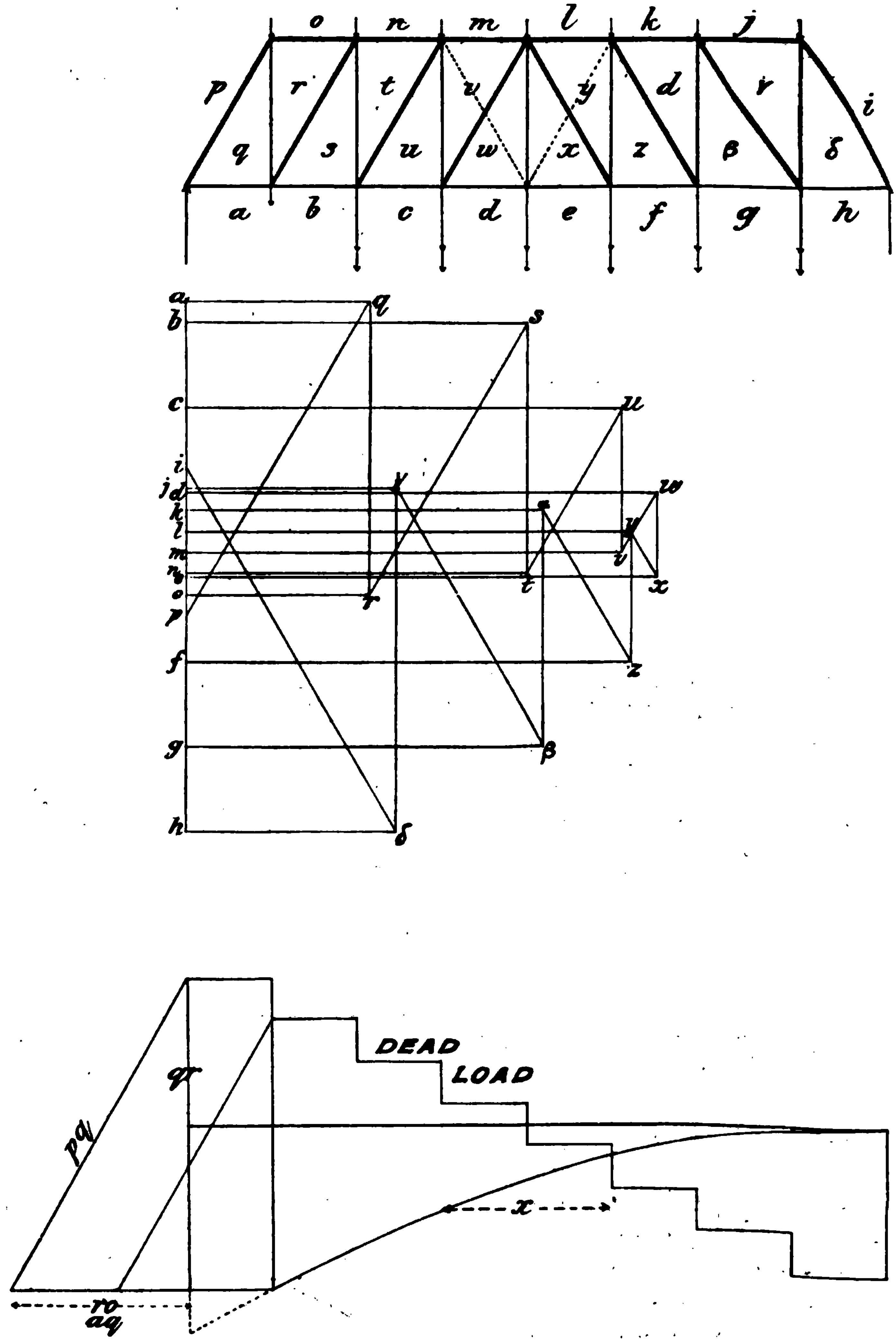


FIG. 497.

on the structure. The larger ones represent the **live** load

which covers the bridge from the right abutment to the panel *b*. The forces can be obtained direct from the reciprocal diagram or from the shear diagram, which is constructed by the method given in the last paragraph; the manner in which the forces can be obtained from it will be apparent by an inspection of the diagram. Inside the focal length x it is usual to use double bracing. The stress in the bracing of these panels is reversed as the train passes over. It is, of course, possible to make single bracing strong enough to withstand the reversal of stress, but from economic reasons it is better to use the double bracing. For a treatment of these and various other points in practical bridge design, the reader must consult special works on the subject.

Girder with a Double System of Triangulation.—Most girders with double triangulation are statically indeterminate, and have to be treated by special methods. They can, however, generally be treated by reciprocal diagrams without any material error (Fig. 498). We will take one simple case to illustrate two methods of treatment. In the first each system is treated separately, and where the members overlap, the forces must be added; in the second the whole diagram will be constructed in one operation. The same result will, of course, be obtained by both methods.

In dealing with the second method, the forces *mn*, *hg* acting on the two end verticals are simply the reactions of Fig. A. There is less liability to error if they are treated as two upward forces, as shown Fig. C, than if they are left in as two vertical bars. It will be seen, from the reciprocal diagram, that the force in *qs* is the same as that in *rt*, which of course must be the case, as they are one and the same bar.

Incomplete and Redundant Framed Structures.—If a jointed structure have not sufficient bars to make it retain its original shape under all conditions of loading, it is termed an "incomplete" structure. Such a structure may, however, be used in practice for one special distribution of loading which never varies, but if the distribution should ever be altered, the structure will change its shape. The determination of the forces acting on the various members can be found by the reciprocal diagram.

But if a structure have more than sufficient bars to make it retain its original shape, it is termed a "redundant" structure. Then the stress on the bars depends entirely upon their relative yielding when loaded, and cannot be obtained from a reciprocal diagram. Such structures are termed "statically indeterminate

structures." Even the most superficial treatment would occupy far too much space. If the reader wishes to follow up the

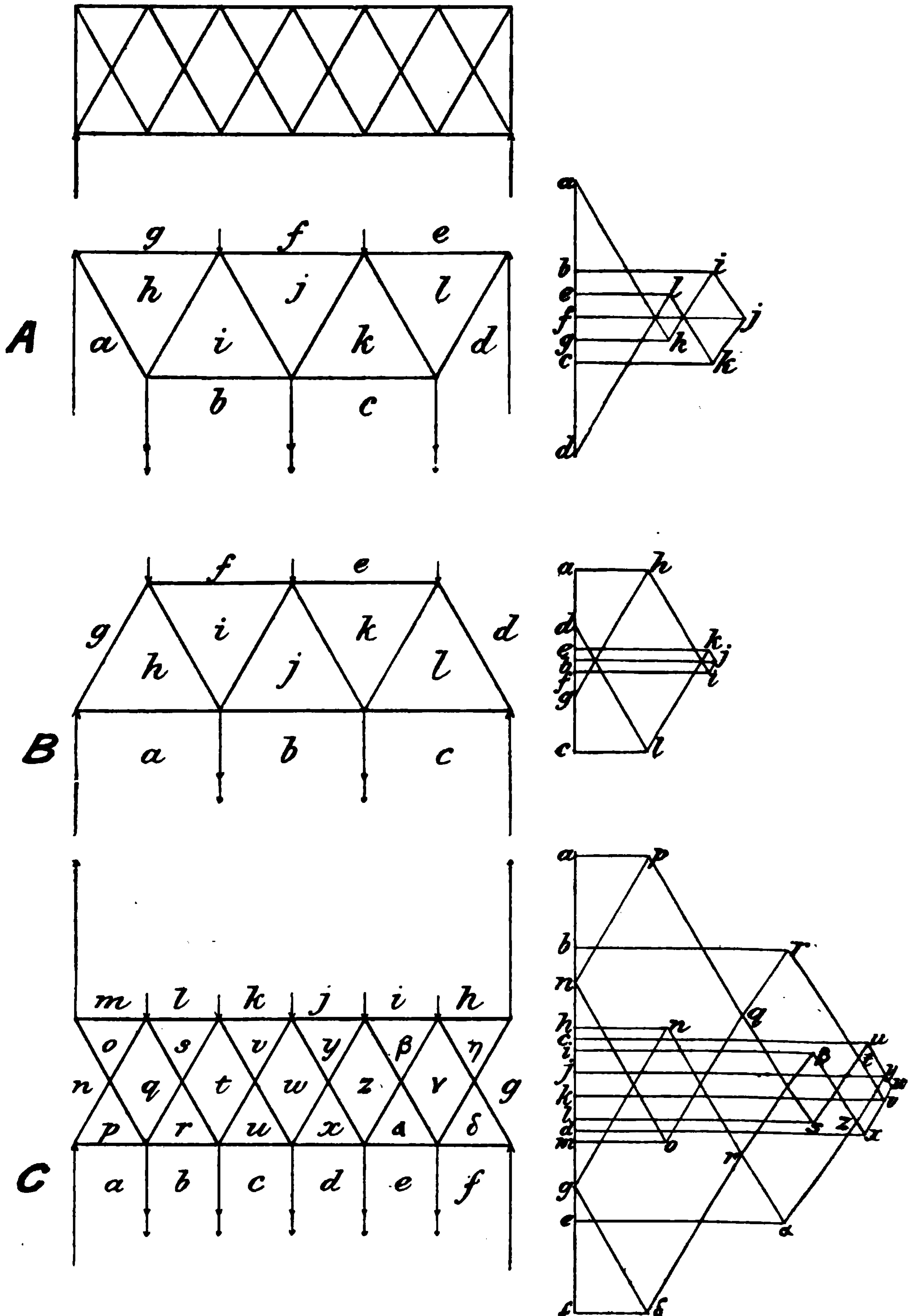


FIG. 498.

subject, he cannot do better than consult an excellent little book on the subject, "Statically Indeterminate Structures," by Martin, published at *Engineering* Office.

Pin Joints.—In all the above cases we have assumed that all the bars are jointed with frictionless pin joints, a condition which, of course, is never obtained in an actual structure. In American bridge practice pin joints are nearly always used, but in Europe the more rigid riveted joint finds favour. When a structure deflects under its load, its shape is slightly altered, and consequently bending stresses are set up in the bars when rigidly jointed. Generally speaking, such stresses are neglected by designers.

Plate Girders.—It is always assumed that the flanges of a rectangular plate girder resist the whole of the bending stresses, and that the web resists the whole of the shear stresses. That such an assumption is not far from the truth is evident from the shear diagram given on p. 319.

In the case of a parabolic plate girder, the flanges take some of the shear, the amount of which is easily determined.

The determination of the bending moment by means of a diagram has already been fully explained. The bending moment at any point divided by the corresponding depth of the girder gives the total stress in the flanges, and this divided by the intensity of the stress gives the net area of one flange. In the rectangular girder the total flange stress will be greatest in the middle, and will diminish towards the abutments, consequently the section of the flanges should correspondingly diminish. This is usually accomplished by keeping the width of the flanges the same throughout, and reducing the thickness by reducing the number of plates. The bending-moment diagram lends itself very readily to the stepping of the plates. Thus suppose it were found that four thicknesses of plate were required to resist the bending stresses in the flanges in the middle of the girder; then, if the bending-moment diagram be divided into four strips of equal thickness, each strip will represent one plate. If these strips be projected up on to the flange as shown, it gives the position where the plates may be stepped.[1]

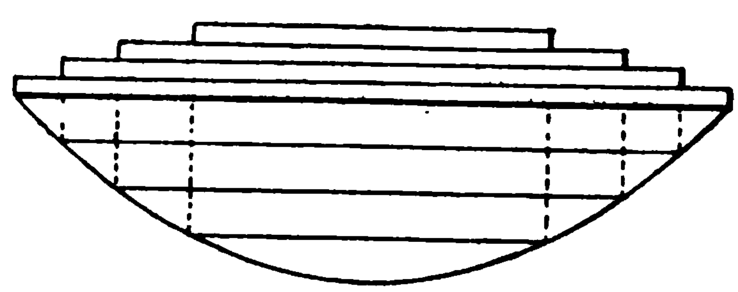


Fig. 499.

[1] It is usual to allow from 6 inches to 12 inches overlap of the plates beyond the points thus obtained.

The shear in the web may be conveniently obtained from the shear diagram (see Chapter X.).

Then if S = shear at any point in tons,

f_s = permissible shear stress, usually not exceeding 2 tons per square inch,

d_w = depth of web in inches,

t = thickness of plate in inches (rarely less than $\frac{3}{8}$ inch),

we have—

$$S = f_s d_w t$$

The depth is usually decided upon when scheming the girder; it is frequently made from $\frac{1}{8}$ to $\frac{1}{12}$ span. The thickness of web is then readily obtained. If on calculation the thickness comes out less than $\frac{3}{8}$ inch, and it has been decided not to use a thinner web, the depth in some cases is decreased accordingly within reasonable limits.

The web is attached to the flanges or booms by means of angle irons arranged thus:

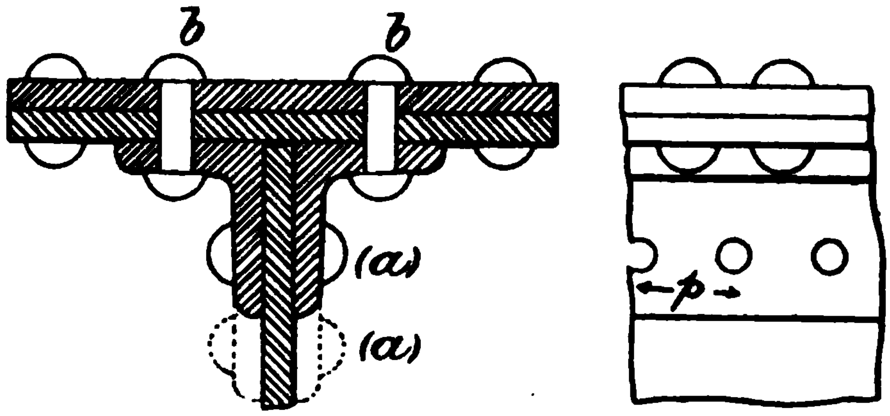


FIG. 500.

The pitch p of the rivets must be such that the bearing and shearing stresses are within the prescribed limits.

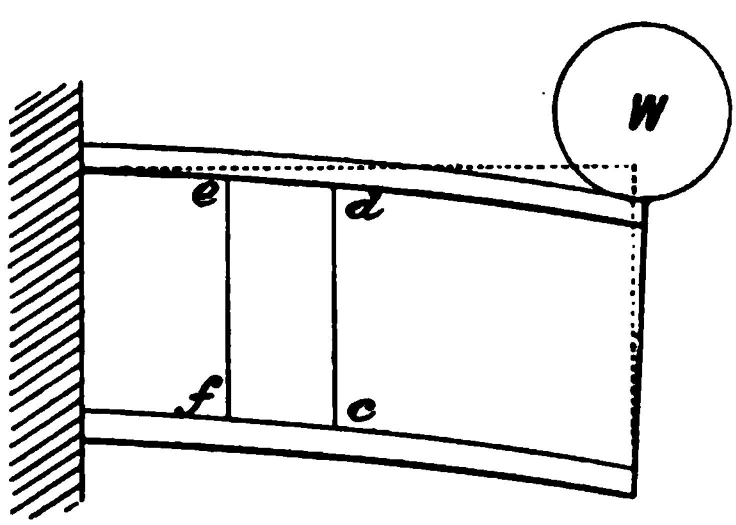


FIG. 501.

On p. 258 we showed that, in the case of any rectangular element subject to shear, the shear stress is equal in two directions at right angles, *i.e.* the shear stress along ef = shear stress along ed, which has to be taken by the rivets a, a, Fig. 500.

But the shear per effective inch run of web plate along *ef* is $f_s t = \frac{S}{d_w}$.

The shearing resistance of each rivet is (in double shear) $\frac{2\pi d^2 f_r}{4} = 1{\cdot}57 d^2 f_r = f_s t(p - d)$.

Whence—

$$p = \frac{1{\cdot}57 d^2}{t} \frac{f_r}{f_s} + d$$

This pitch is, however, often too small to be convenient; then two (zigzag) rows of rivets are used, and p = twice the above value.

The bearing resistance of each rivet is—

$$dtf_b = f_r t(p - d)$$

Let the bearing pressure $f_b = 2f_r$. f_b is usually taken at about 8 tons per square inch. We get—

$$p = 3d \text{ (for a single row of rivets)}$$
$$p = 5d \text{ (for a double row of rivets)}$$

The joint bearing area of the two rivets *b*, *b* attaching the angles to the booms is about twice that of a single rivet (*a*) through the web; hence, as far as bearing pressure is concerned, single rows are sufficient at *b*, *b*. A very common practice is to adopt a pitch of 4 inches, putting two rows in the web at *a*, *a*, and single rows at *b*, *b*.

The pitch of the rivets in the vertical joints of the web (with double cover plates) is the same as in the angles.

The shear diminishes from the abutments to the middle of the span, hence the thickness of the web plates may be diminished accordingly. It often happens, however, that it is more convenient on the whole to keep the web plate of the same thickness throughout. The pitch p of the rivets may then be increased towards the middle. It should be remembered, however, that several changes in the pitch may in the end cost more in manufacture than keeping the pitch constant, and using more rivets.

The rivets should always be arranged in such a manner

that not more than two occur in any one section, in order to reduce the section of the angles as little as possible.

The web and the compression flange under some conditions are liable to buckle if left unstiffened; the usual method of stiffening is to rivet L's or ⊥'s on the web at intervals thus:

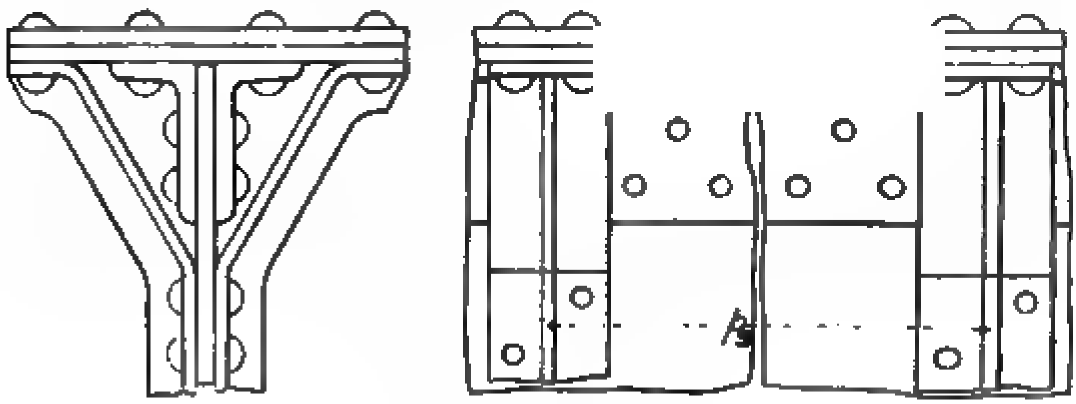


FIG. 502.

We have shown (p. 258), in the case of a square element subjected to shear, that there is a tensile stress acting along one diagonal, and a compressive stress acting along the other, the directions of them being at 45° to the direction of the shear forces. It is considered by some that the compressive stress tends to buckle the web, but others consider that the tensile stress acting along the other diagonal entirely prevents any buckling, and therefore that a web of a plate girder which is subjected to *pure* shear only has no tendency whatever to buckle. When the load is applied to the bottom flange of a plate girder, the author believes that there is no evidence to prove that the web will tend to buckle; but there is very definite evidence to prove that when a concentrated load is placed on the top flange, the web will tend to buckle locally immediately under the load. In such a case a vertical stiffener or stay should be riveted to the web, in order to distribute the load evenly down the web. In the case of a distributed load on the top flange, there is also a tendency to buckle the web, and stiffeners must be provided to prevent buckling. The pitch of them is usually determined by the pitch of the cross-girders, and in other cases the pitch may be made $100t$ near the abutments, increasing to $200t$ at the middle of the span, where t is the thickness of the web. The author considers that the usual method of calculating the pitch of the stiffeners,

from the shear stress is very questionable, and possibly misleading.

Weight of Plate Girders.—For preliminary estimates, the weight of a plate girder may be arrived at thus:

Let w = weight of girder in tons per foot run;

W = total load on the girder, not including its own weight, in tons.

$$\text{Then } w = \frac{W}{500} \text{ roughly}$$

Arched Structures.—We have already shown how to determine the forces acting on the various segments of a suspension-bridge chain. If such a chain were made of suitable form and material to resist compression, it would, when inverted, simply become an arch. The exact profile taken up by a suspension-bridge chain depends entirely upon the distribution of the load, but as the chain is in tension, and, moreover, in stable equilibrium, it immediately and automatically adjusts itself to any altered condition of loading; but if such a chain were inverted and brought into compression, it would be in a state of unstable equilibrium, and the smallest disturbance of the load distribution would cause it to collapse immediately. Hence arched structures must be made of such a section that they will resist change of shape in profile; in other words, they must be capable of resisting bending as well as direct stresses.

Masonry Arches.—In a masonry arch the permissible bending stress is small, in order to ensure that there may be no, or only a small amount of, tensile stress on the joints of the voussoirs, or arch stones. Assuming for the present that there may be no tension, then the resultant line of thrust must lie within the middle third of the voussoir (see p. 382). In order to secure this condition, the form of the arch must be such that under its normal system of loading the line of thrust must pass through or near the middle line of the voussoirs. Then, when under the most trying conditions of live loading, the line of thrust must not pass outside the middle third. This condition can be secured either by increasing the depth of the voussoirs, or by increasing the dead load on the arch in order to reduce the ratio of the live to the dead load. Many writers still insist on the condition that there shall be no tension in the joints of a masonry structure, but every one who has had any experience of such structures is perfectly

well aware that there are very few masonry structures in which the joints do not tend to open, and yet show no signs of instability or unsafeness. There is a limit, of course, to the amount of permissible tension. If the line of thrust pass through the middle third, the maximum intensity of compressive stress on the edge of the voussoir is twice the mean, and if through the middle half, *i.e.* at a distance of one-fourth of the depth from the edge, the intensity of compressive stress is from $2\frac{1}{4}$ to $2\frac{1}{2}$ times the mean according to the adhesion of the mortar.

A masonry arch may fail in several ways, the chief of which are—

1. By the crushing of the voussoirs.
2. By the sliding of one voussoir over the other.
3. By the tilting or rotation of the voussoirs.

The first is avoided by making the voussoirs sufficiently deep, or of sufficient sectional area to keep the compressive stress within that considered safe for the material. The compressive stress f_c is the thrust T divided by the sectional area A of the voussoirs, or—

$$f_c = \frac{T}{A}$$

It is generally considered safe to allow the line of thrust to pass anywhere through the middle half. Hence the maximum intensity of compressive stress is about $2 \cdot 5 f_c$, and we must have—

$$\frac{2 \cdot 5 T}{A} = < \text{the permissible compressive stress}$$

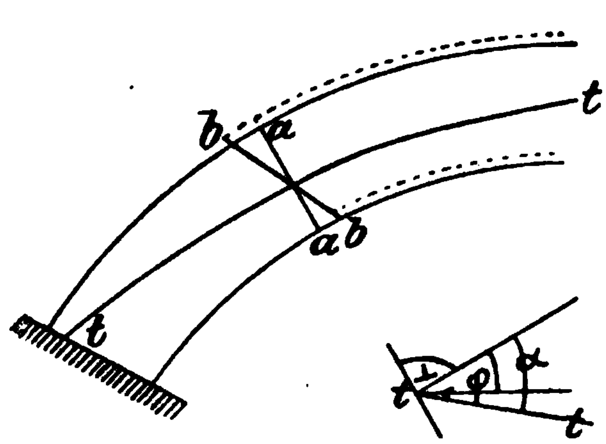


FIG. 503.

The second-mentioned method of failure is avoided by so arranging the joints that the line of thrust never cuts the normal to the joint at an angle greater than the friction angle. Let *tt* be the line of thrust, then sliding will occur in the manner shown by the dotted lines, if the joints be arranged as shown at *bb*—that is, if the angle α exceeds the friction angle ϕ. But if the joint be as shown at *aa*, sliding cannot occur.

The third-mentioned method of failure only occurs when the line of thrust passes right outside the section; the voussoirs then tilt till the line of thrust passes through the pivoting points. An arch can never fail in this way if the line of thrust be kept inside the middle halt.

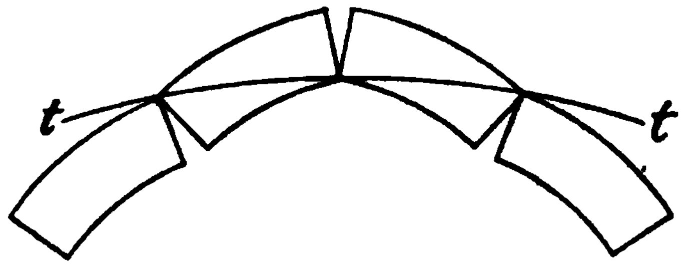


FIG. 504.

The rise of the arch R (Fig. 505) will depend upon local conditions, and the lines of thrust for the various conditions of

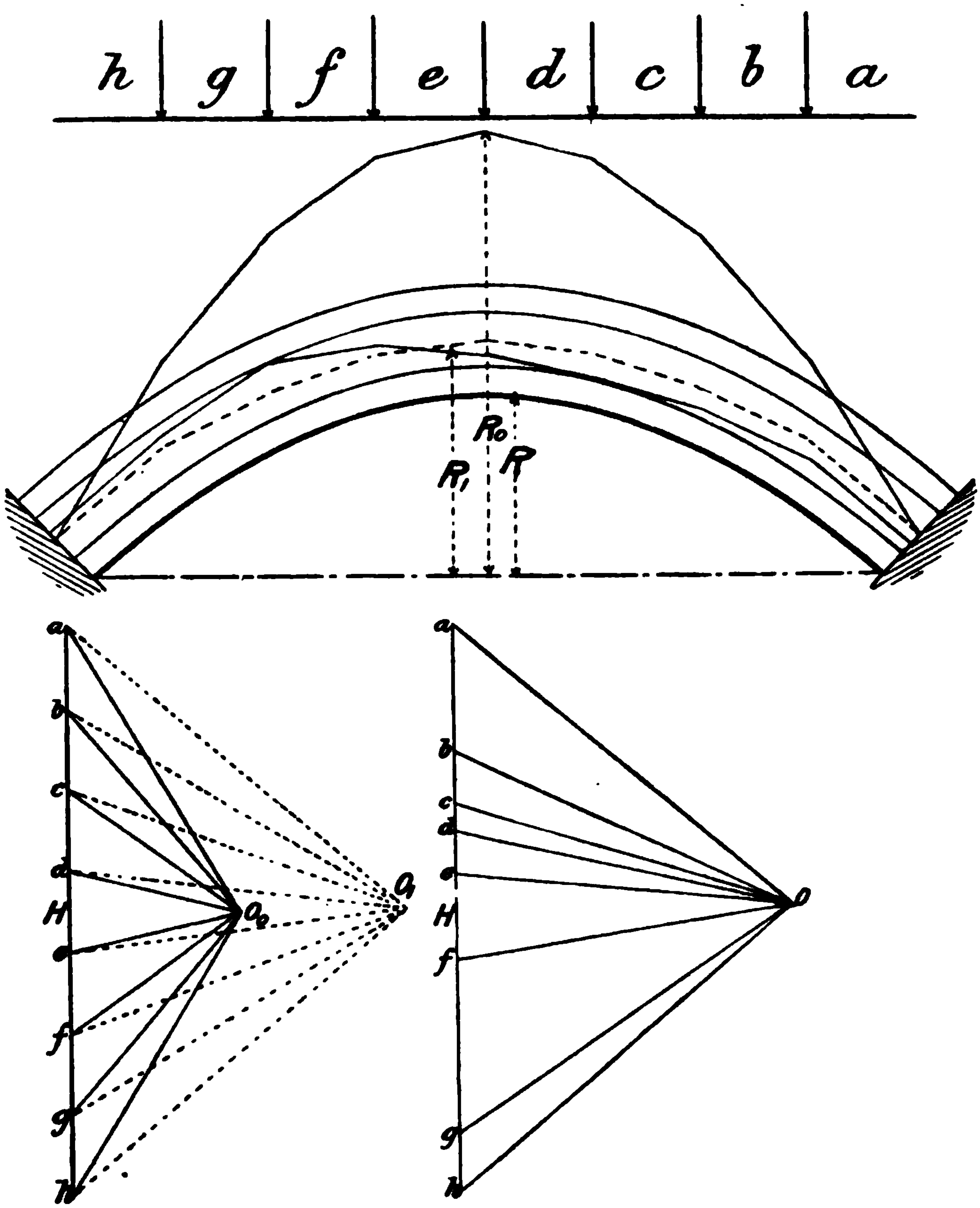


FIG. 505.

loading are constructed in precisely the same manner as the link-and-vector polygon. A line of thrust is first constructed for the distributed load to give the form of the arch, and if the line of thrust come too high or too low to suit the desired rise, it is corrected by altering the polar distance. Thus, suppose the rise of the line of thrust were R_0, and it was required to bring it to R_1. If the original polar distance were O_0H, the new polar distance required to bring the rise to R_1 would be $O_1H = O_0H \times \frac{R_0}{R_1}$.

After the median line of the arch has been constructed, other link polygons, such as the bottom right-hand figure, are drawn in for the arch loaded on one side only, for one-third of the length, on the middle only, and any other ways which are likely to throw the line of thrust away from the median line. After these lines have been put in, envelope curves parallel to the median line are drawn in to enclose these lines of thrust at every point; this gives us the middle half of the voussoirs. The outer lines are then drawn in equidistant from the middle half lines, making the total depth of the voussoirs equal to twice the depth of the envelope curves.

An infinite number of lines of thrust may be drawn in for any given distribution of load. Which of these is the right one? is a question by no means easily answered, and whatever answer may be given, it is to a large extent a matter of opinion. For a full discussion of the question, the reader should refer to I. O. Baker's "Treatise on Masonry" (Wiley and Co., New York); and a paper by H. M. Martin, *I.C.E. Proceedings*, vol. xciii. p. 462.

From an examination of several successful arches, the author considers that if, by altering the polar distance OH, a line of thrust can be flattened or bent so as to fall within the middle half, it may be concluded that such a line of thrust is admissible. One portion or point of it may touch the inner and one the outer middle half lines. As a matter of fact, an exact solution of the masonry arch problem in which the voussoirs rest on plane surfaces is indeterminate, and we can only say that a certain assumption is admissible if we find that arches designed on this assumption are successful.

Arched Ribs.—In the case of iron and steel arches, the line of thrust may pass right outside the section, for in a continuous rib capable of resisting tension as well as compression the rib retains its shape by its resistance to bending. The bending moment varies as the distance of the line of thrust from the centre of gravity of the section of the rib.

The determination of the position of the line of thrust is therefore important.

Arched ribs are often hinged at three points—at the

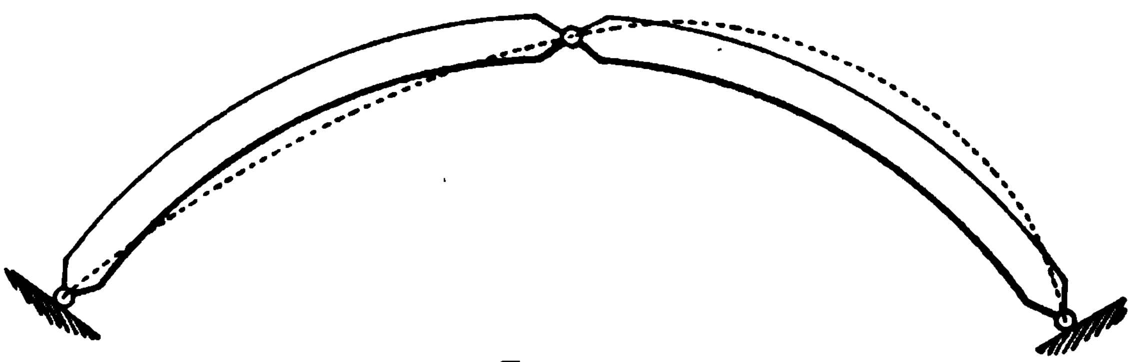


FIG. 506.

springings and at the crown. It is evident in such a case that the line of thrust must pass through the hinges, hence there is no difficulty in finding its exact position. But when the arch is rigidly held at one or both springings, and not hinged at the crown, the position of the line of thrust may be found thus :[1]

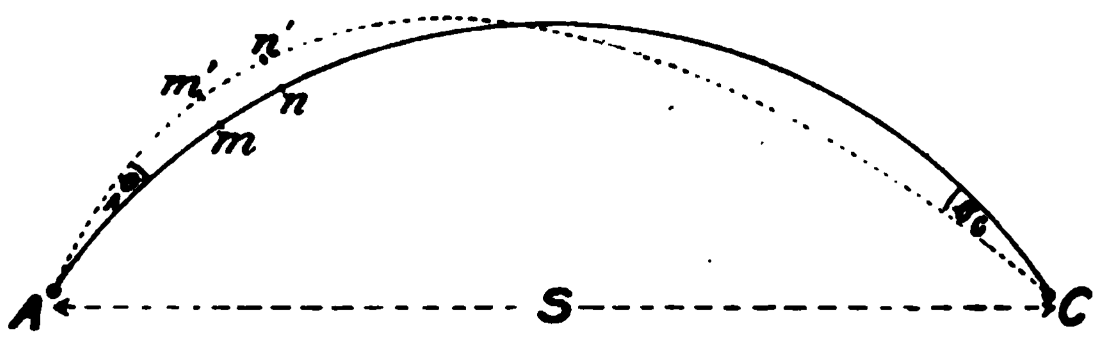


FIG. 507.

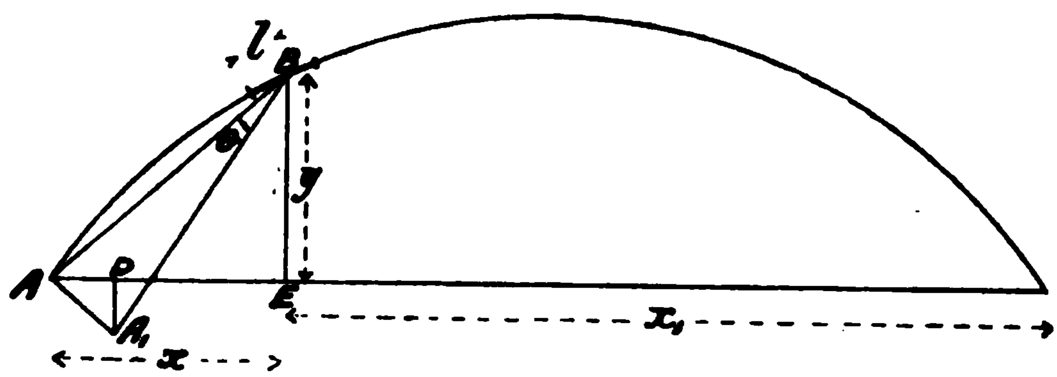


FIG. 508.

Consider a short element of the rib *mn* of length *l*. When the rib is unequally loaded, it is strained so that *mn* takes up the position *m'n'*. Both the slope and the vertical position of the element are altered by the straining of the rib. First consider the effect of the alteration in slope; for this purpose that portion of the rib from B to A may be considered as being pivoted at B. Join BA; then, when the rib is strained, BA

[1] See also a paper by Bell, *I.C.E.*, vol. xxxiii. p. 68.

becomes BA_1. Thus the point A has received a horizontal displacement AD, and a vertical displacement A_1D. The two triangles BAE, AA_1D are similar; hence—

$$\frac{AD}{AA_1} = \frac{y}{AB} \quad AD = \frac{AA_1}{AB} . y = \theta . y \quad . \quad . \quad . \quad \text{(i.)}$$

θ being expressed in circular measure. Likewise—

$$\frac{A_1D}{AA_1} = \frac{x}{AB} \quad A_1D = \frac{AA_1}{AB} . x = \theta . x \quad . \quad . \quad . \quad \text{(ii.)}$$

A similar relation holds for every other small portion of the rib, but as A does not actually move, it follows that some of the horizontal displacements of A are outwards, and some inwards. Hence the algebraic sum of all the horizontal displacements must be zero, or $\Sigma\theta y = 0$ . . . . (iii.)

Now consider the vertical movement of the element. If the rib be pivoted at C, and free at A, when *mn* is moved to *m'n'*, the rib moves through an angle θ_c, and the point A receives a vertical displacement $S . \theta_c$. We have previously seen that A has also a vertical displacement due to the bending of the rib; but as the point A does not actually move vertically, some of the vertical displacements must be upwards, and some downwards. Hence the algebraic sum of all the vertical displacements is also zero, or—

$$\Sigma\theta x + S\theta_c = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(iv.)}$$

By similar reasoning—

$$\Sigma\theta x_1 + S\theta_A = 0$$
$$\text{or } \Sigma(S - x)\,\theta + S\theta_A = 0 \quad . \quad . \quad . \quad . \quad . \quad \text{(v.)}$$

If the arch be rigidly fixed at C—

$$\theta_c = 0$$
$$\text{and } \Sigma\theta x = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(vi.)}$$

If it be fixed at A—

$$\theta_A = 0$$
$$\text{and } \Sigma(S - x)\theta = 0 \text{ from v.}$$

If it be fixed at both ends, we have by addition—

$$\Sigma(S\theta - x\theta + x\theta) = 0$$
$$\Sigma S\theta = 0$$

Then, since S is constant—

$$\Sigma\theta = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(vii.)}$$

Whence for the three conditions of arches we have—

Arch hinged at both ends, $\Sigma y\theta = 0$ from i.
„ fixed „ „ $\Sigma y\theta$, and $\Sigma\theta = 0$ from i. and vii.
„ „ „ one end only, $\Sigma y\theta$, and $\Sigma x\theta = 0$ from i. and vi.

We must now find an expression for θ. Let the full line represent the portion of the unstrained rib, and the dotted line the same when strained.

Let the radius of curvature before straining be ρ_0, and after straining ρ_1.

Then, using the symbols of Chapter XI., we have—

$$M_0 = \frac{EI}{\rho_0}, \text{ and } M_1 = \frac{EI}{\rho_1}$$

Then the bending moment on the rib due to the change of curvature when strained is—

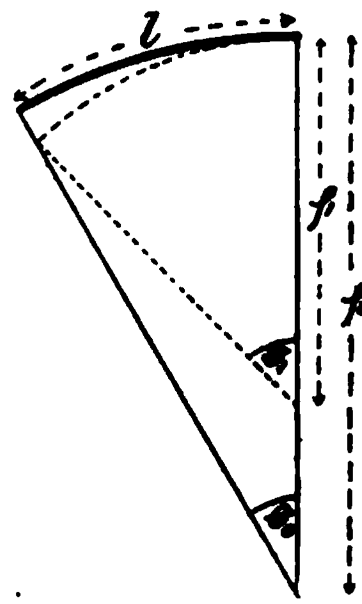


FIG. 509.

$$M = M_1 - M_0 = EI\left(\frac{1}{\rho_1} - \frac{1}{\rho_0}\right) \quad . \quad . \quad \text{(viii.)}$$

But as l remains practically constant before and after the strain, we have—

$$\theta_1 = \frac{l}{\rho_1}, \text{ and } \theta_0 = \frac{l}{\rho_0}$$

$$\text{and } \theta_1 - \theta_0 = \theta = l\left(\frac{1}{\rho_1} - \frac{1}{\rho_0}\right)$$

$$\text{or } l = \frac{\theta}{\frac{1}{\rho_1} - \frac{1}{\rho_0}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(ix.)}$$

Then we have from viii. and ix.—

$$Ml = EI\theta$$

$$\text{and } \theta = \frac{Ml}{EI}$$

If the arched rib be of constant cross-section, $\frac{l}{\mathrm{EI}}$ is constant; but if it be not so, then the length l must be taken so that $\frac{l}{\mathrm{I}}$ is constant.

The bending moment M on the rib is $\mathrm{M} = \mathrm{F} \cdot \overline{ab}$, where the lowest curved line through cb is the line of thrust, and the upper dark line the median line of the rib.

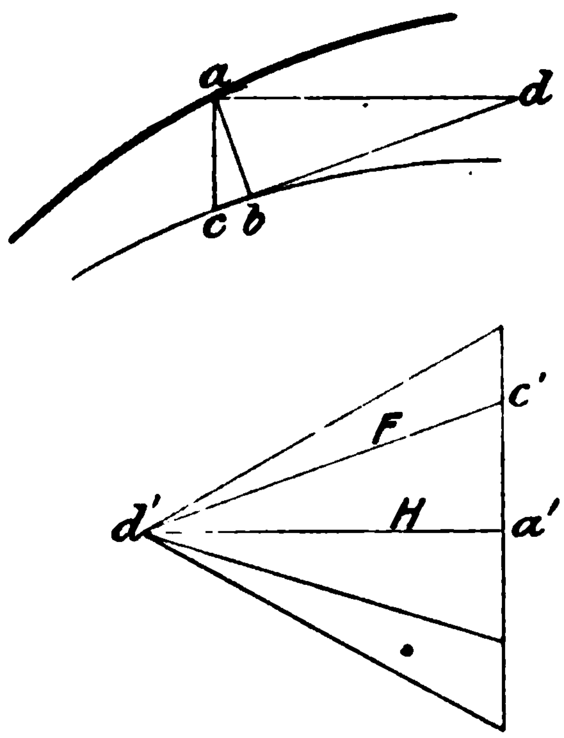


FIG. 510.

Draw ac vertical at c;
dc tangential „
ab normal to dc;
ad horizontal.

Let H be the horizontal thrust on the vector polygon. Then the triangles adc, $a'd'c'$, also adb, cda, are similar; hence—

$$\frac{\mathrm{F}}{\mathrm{H}} = \frac{cd}{ad} \text{ and } \frac{ab}{ad} = \frac{ac}{cd}$$

$$\text{or } \frac{ab}{ac} = \frac{ad}{cd} = \frac{\mathrm{H}}{\mathrm{F}}$$

$$\text{or } \mathrm{F} \cdot \overline{ab} = \mathrm{H} \cdot \overline{ac} = \mathrm{M}$$

but H is constant for any given case.

Let $\overline{ac} = z$.

Hence the expression $\Sigma\theta y = 0$

may be written $\Sigma \mathrm{M} y = 0$, since $\frac{l}{\mathrm{EI}}$ is constant

or $\Sigma z \cdot y = 0$

Then, substituting in a similar manner in the equations above, we have—

Arch hinged at both ends, $\Sigma z \cdot y = 0$
„ fixed „ „ $\Sigma z \cdot y = 0$, and $\Sigma z = 0$
„ „ at one end only, $\Sigma z \cdot y = 0$, and $\Sigma xz = 0$

Thus, after the median line of the arch has been drawn, a line of thrust for uneven loading is constructed; and the median line is divided up into a number of parts of length l,

and perpendiculars dropped from each. The horizontal distances between them will, of course, not be equal; then all the values $z \times y$ must be found, some z's being negative, and

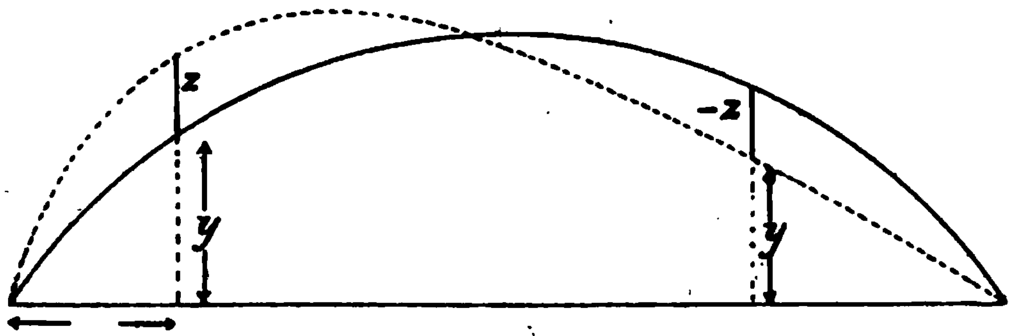


FIG. 511.

some positive, and the sum found. If the sum of the negative values are greater than the positive, the line of thrust must be raised by reducing the polar distance of the vector polygon, and *vice versâ* if the positive are greater than the negative. The line of thrust always passes through the hinged ends. In the case of the arch with fixed ends, the sum of the z's must also

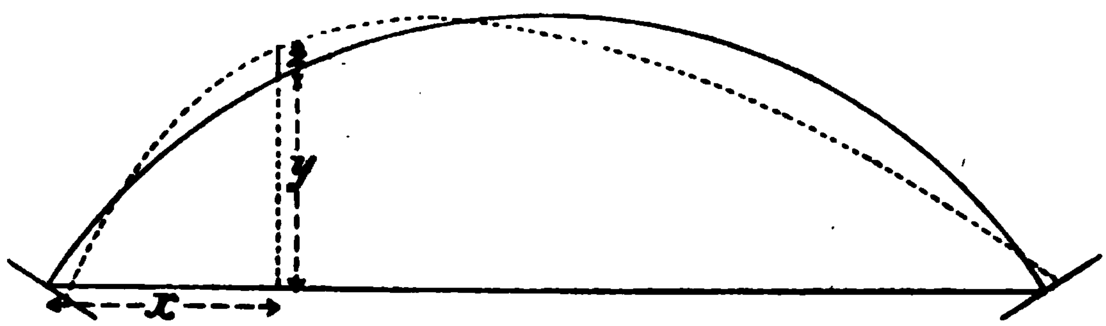


FIG. 512.

be zero; this can be obtained by raising or lowering the line of thrust bodily. When one end only is fixed, the sum of all the quantities $x \, . \, z$ must be zero as well as zy; this is obtained by shifting the line of thrust bodily sideways.

Having fixed on the line of thrust, the stresses in the rib are obtained thus:—

The compressive stress all over the rib at any section is—

$$f_c = \frac{T}{A}$$

Where T is the thrust obtained from the vector polygon, and A is the sectional area of the rib.

The skin stress due to bending is—

$$f = \frac{M}{Z} = \frac{T \, . \, \overline{ab}}{Z} \text{ (see Fig. 510)}$$

$$\text{or } f = \frac{H \, . \, z}{Z}$$

and the maximum stress in the material due to both—

$$f_c + f = \mathrm{T}\left(\frac{1}{\mathrm{A}} + \frac{ab}{\mathrm{Z}}\right)$$

$$\text{or} = \frac{\mathrm{T}}{\mathrm{A}} + \frac{\mathrm{H}\,.\,z}{\mathrm{Z}}$$

Except in the case of very large arches, it is never worth while to spend much time in getting the *exact* position of the worst line of thrust; in many instances its correct position may be detected by eye within very small limits of error.

Effect of Change of Length and Temperature on Arched Ribs.—Long girders are always arranged with expansion rollers at one end to allow for changes in length as the temperature varies. Arched ribs, of course, cannot be so treated —hence, if their length varies due to any cause, the radius of curvature is changed, and bending stresses are thereby set up.

The change of curvature and the stress due to it may be arrived at by the following approximation, assuming the rib to be an arc of a circle:—

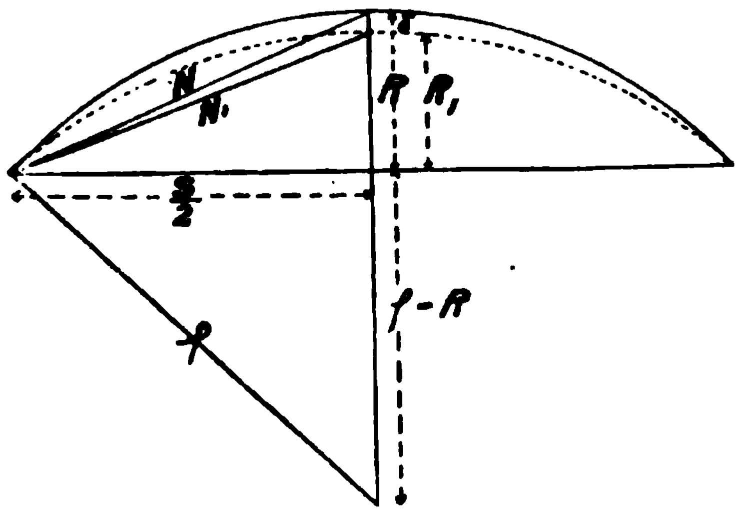


FIG. 513.

$$\text{Let } \mathrm{N}_1 = \mathrm{N}\left(1 - \frac{1}{n}\right)$$

$$\mathrm{N}^2 = \mathrm{R}^2 + \frac{\mathrm{S}^2}{4}$$

$$\mathrm{N}_1^2 = \mathrm{R}_1^2 + \frac{\mathrm{S}^2}{4}$$

$$\mathrm{N}^2 - \mathrm{N}_1^2 = \mathrm{R}^2 - \mathrm{R}_1^2$$

Substituting the value of N_1 and reducing, we get—

$$-\frac{\mathrm{N}^2}{n^2} + \frac{2\mathrm{N}^2}{n} = (\mathrm{R} + \mathrm{R}_1)(\mathrm{R} - \mathrm{R}_1)$$

$$\text{But } \mathrm{R} - \mathrm{R}_1 = \delta$$

$$\text{and } \mathrm{R} + \mathrm{R}_1 = 2\mathrm{R} \text{ (nearly)}$$

The fraction $\frac{1}{n}$ is very small, viz. $\frac{f_c}{\mathrm{E}}$, and is rarely more than $\frac{1}{3000}$; hence the quantity involving its square, viz. $\frac{\mathrm{N}^2}{9000000}$, is negligible.

Then we get—

$$\delta = \frac{N^2}{nR} \quad \text{(x.)}$$

$$\text{But } \left(\frac{S}{2}\right)^2 = N^2 - R^2$$

$$\text{also } \left(\frac{S}{2}\right)^2 = \rho^2 - (\rho - R)^2$$

$$N^2 - R^2 = \rho^2 - (\rho - R)^2$$

from which we get—

$$N^2 = 2\rho R$$

Substituting in x., we have—

$$\delta = \frac{2\rho}{n}$$

$$\text{and } \rho = \frac{N^2}{2R}$$

$$\text{also } \rho_1 = \frac{N_1^2}{2R_1}$$

The bending moment on the rib due to the change of curvature is—

$$M = EI\left(\frac{1}{\rho} - \frac{1}{\rho_1}\right) \text{ from viii.}$$

$$= EI\left(\frac{2R}{N^2} - \frac{2R_1}{N_1^2}\right)$$

and the corresponding skin stress is—

$$f = \frac{M}{Z} = \frac{My}{I} = \frac{Md}{2I}$$

where d is the depth of the rib in inches if R and N are taken in inches.

Then, substituting the value of M, we have—

$$f = \frac{2EId}{2I}\left(\frac{R}{N_2} - \frac{R_1}{N_1^2}\right)$$

N_1^2 may without sensible error (about 1 in 2000) be written N^2; then—

$$f = \frac{Ed}{N^2}(R - R_1)$$

$$f = \frac{Ed\delta}{N_2}$$

The stress at the crown due to the change of curvature on account of the compression of the rib then becomes—

$$f = \frac{2Ed\rho}{nN^2}$$

$$\text{and } \frac{1}{n} = \frac{f_c}{E} \text{ (see p. 280)}$$

where f_c is the compressive stress all over the section of the rib at the crown; hence—

$$f = \frac{2d\rho f_c}{N^2} = \frac{8d\rho}{N^2}$$

taking f_c as a preliminary estimate at about 4 tons per sq. inch; then $\frac{1}{n} = \frac{1}{3500}$ (about). In the case of the change of curvature due to change of temperature, it is usual to take the expansion and contraction on either side of the mean temperature of 60° Fahr. as $\frac{1}{4}$ inch per 100 feet for temperate climates such as England, and twice this amount for tropical climates. Hence for England—

$$\frac{1}{n} = \frac{0 \cdot 25}{1200} = \frac{1}{4800}$$

putting E = 14,000 tons per square inch;

$$\text{and } f = \frac{5 \cdot 1 d\rho}{N^2}$$

Thus in England the stress due to temperature change of curvature is about five-eighths as great as that due to the compression change of curvature.

The value of ρ varies from $1 \cdot 25N$ to $2 \cdot 5N$, and d from $\frac{N}{30}$ to $\frac{N}{20}$.

Hence, due to the compression change of curvature—

$$f = 0 \cdot 6 \text{ to } 0 \cdot 8 \text{ tons per square inch}$$

and due to temperature in England—

$$f = 0 \cdot 4 \text{ to } 0 \cdot 5 \text{ tons per square inch}$$

In the case of ribs fixed rigidly at each end, it can be

shown by a process similar to that given in Chapter XI., of the beam built in at both ends, that the change of curvature stresses at the abutments of a rigidly held arch is nearly twice as great, and the stress at the crown about 50 per cent. greater than the stress in the hinged arch.

An arched rib must, then, be designed to withstand the direct compression, the bending stresses due to the line of thrust passing outside the section, the bending stresses due to the change of curvature, and finally it must be checked to see that it is safe when regarded as a long strut; to prevent side buckling, all the arched ribs in a structure are braced together.

Effect of Sudden Loads on Structures.—If a bar be subjected to a gradually increasing stress, the strain increases in proportion, provided the elastic limit be not passed. The work done in producing the strain is given by the shaded area *abc* in Fig. 514.

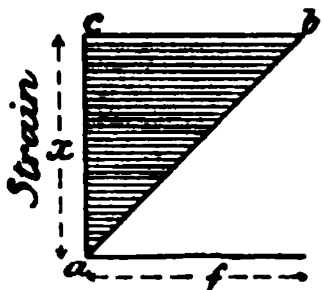


FIG. 514.

Let x = the elastic strain produced;
l = the unstrained length of the bar;
f = the stress over any cross-section;
E = Young's Modulus of Elasticity;
A = sectional area of bar.

$$\text{Then } x = \frac{fl}{E}$$

$$\left.\begin{array}{r}\text{The work done in straining}\\ \text{a bar of section A}\end{array}\right\} = \frac{fAx}{2} = \frac{f^2Al}{2E}$$
$$= \frac{Al}{2} \times \frac{f^2}{E}$$
$$= \tfrac{1}{2} \text{ vol. of bar} \times \text{modulus of elastic resilience}$$

The work done in stretching a bar beyond the elastic limit has been treated in Chapter VIII.

If the stress be produced by a hanging weight (Fig. 515), and the whole load be suddenly applied instead of being gradually increased, then, taking A as 1 square inch to save the constant repetition of the symbol, we have—

$$\left.\begin{array}{r}\text{The work stored in the weight } w_0\\ \text{due to falling through a height } x\end{array}\right\} = w_0x$$
$$= fx$$

But when the weight reaches c (Fig. 515), only a portion of the energy developed during the fall has been expended i· stretching the bar, and the remainder is still stored in t'

falling weight; the bar, therefore, continues to stretch until the kinetic energy of the weight is absorbed by the bar.

The work done in stretching the bar by an amount x is $\frac{fx}{2}$; hence the kinetic energy of the weight, when it reaches c, is the difference between the total work done by the weight and the work done in stretching the bar, or $fx - \frac{fx}{2} = \frac{fx}{2}$.

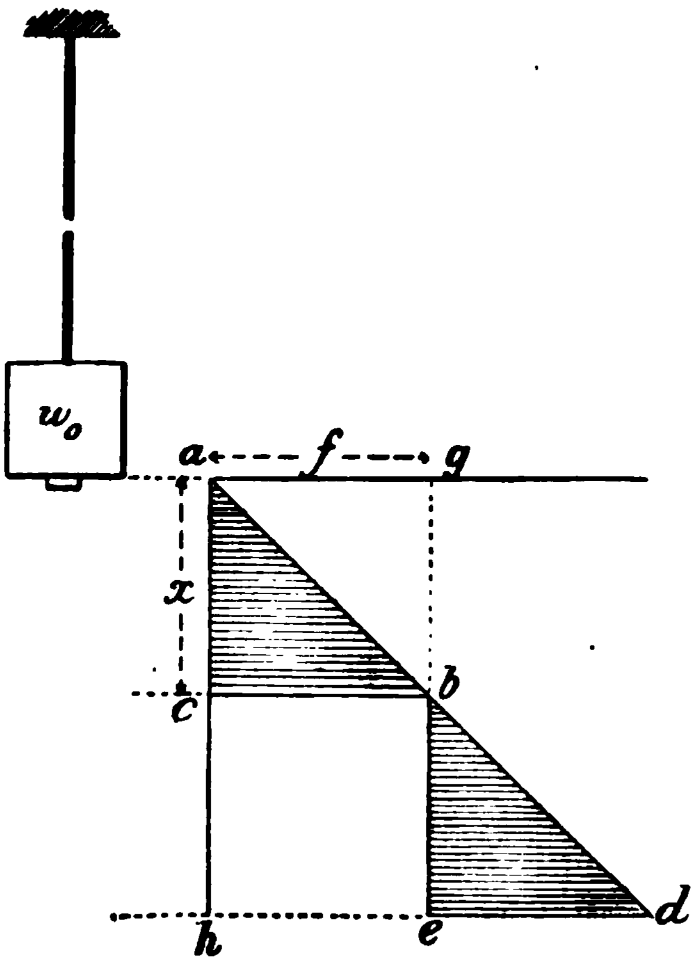


FIG. 515.

The bar will therefore continue to stretch until the work taken up by it is equal to the total work done by gravity on the falling weight, or until the area *ahd* is equal to the area *ageh*. Then, of course, the area *bed* is equal to the area *acb*.

When the weight reaches h, the strain, and therefore the stress, is doubled; thus, when a load is suddenly applied to a bar or a structure, the stress produced is twice as great as if the load were gradually applied.

When the weight reaches h, the tension on the bar is greater than the weight; hence the bar contracts, and in doing so raises the weight back to nearly its original position at a; it then drops again, oscillating up and down until it finally comes to rest at c.

If the bar in question supported a dead weight W, and a further weight w_1 were suddenly applied, the momentary load on the bar would, by the same process of reasoning, be $W + 2w_1$.

If the suddenly applied load acted upwards, tending to compress the bar, the momentary load would be $W - 2w_1$. Whether or not the bar ever came into compression would entirely depend upon the relative values of W and w_1.

The special case when $w_1 = 2W$ is of interest; the momentary load is then—

$$W - 2(2W) = -3W$$

the negative sign simply indicating that the stress has been

reversed. This is the case of a revolving loaded axle. Consider it as stationary for the moment, and overhanging as shown—

The upper skin of the axle in case (i.) is in tension, due to W. In order to relieve it of this stress, *i.e.* to straighten the axle, a force −W must be applied, as in (ii.); and, further, to bring the upper skin into compression of the same intensity, as in (i.), a force −2W must be applied, as shown in (iii.). Now, when an axle revolves, every portion of the skin is alternately brought into tension and compression; hence we may regard a revolving axle as being under the same system of loading as in (iii.), or that the momentary load is three times the steady load.

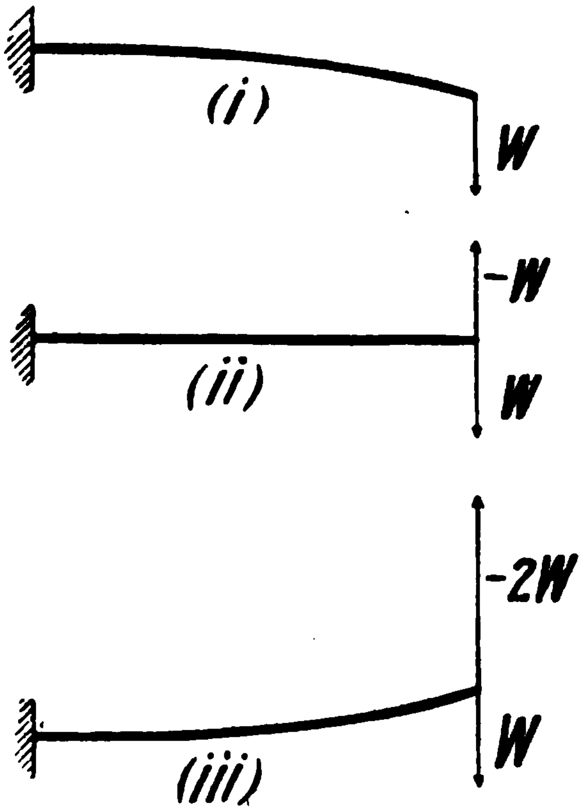


FIG. 516.

Experiments on the Repetition of Stress.—In 1870 Wöhler published[1] the results of some extremely interesting experiments on the effects of repeated stresses on materials; since then Spangenburg, Bauschinger, and Baker have done similar work. In these experiments various materials were subjected to tensile, compressive, and torsional stresses, which were wholly or partially removed, and in some instances were even reversed from tension to compression in the same bar. The intensity of the stresses were at first large, almost approaching the static breaking strength of the material. Such stresses caused fracture after a small number of repetitions, but as the intensity of the stress was reduced, the number of repetitions before fracture rapidly increased, and after a time a low limit of stress was reached, at which it appeared that the bar was capable of withstanding an infinite number of repetitions. This stress appears to depend upon the ultimate strength of the material and the amount of fluctuation of the stress, often termed the "range of stress," to which the material is subjected:—

In a general way, the results of all the experimenters are fairly concordant, to what extent will be shown shortly.

The following examples taken from Wöhler will serve to illustrate the sort of results obtained:—

[1] An English translation will be found in *Engineering*, vol. ii.

MATERIAL.

Krupp's Axle Steel.

Tensile strength, varying from 42 to 49 tons per sq. inch.

Tensile stress applied in tons per square inch from	to	Number of repetitions before fracture.	Nominal bending stress in tons per square inch from	to	Number of repetitions before fracture.
0	38·20	18,741	0	26·25	1,762,300
0	33·40	46,286	0	25·07	1,031,200
0	28·65	170,170	0	24·83	1,477,400
0	26·14	123,770	0	23·87	5,234,200
0	23·87	473,766	0	23·87	40,600,000 (unbroken)
0	22·92	13,600,000 (unbroken)			

Nominal bending stress in a revolving axle from	to	Number of repetitions before fracture.
20·1	−20·1	55,100
17·2	−17·2	127,775
16·3	−16·3	797,525
15·3	−15·3	642,675
,,	,,	1,665,580
,,	,,	3,114,160
14·3	−14·3	4,163,375
,,	,,	45,050,640

MATERIAL.

Krupp's Spring Steel.

Tensile strength, 57·5 tons per sq. inch.

Tensile stress applied in tons per square inch from	to	Number of repetitions before fracture.	Tensile stress applied in tons per square inch from	to	Number of repetitions before fracture.
47·75	7·92	62,000	38·20	4·77	99,700
,,	15·92	149,800	,,	9·55	176,300
,,	23·87	400,050	,,	14·33	619,600
,,	27·83	376,700	,,	,,	2,135,670
,,	31·52	19,673,000 (unbroken)	,,	19·10	35,800,000 (unbroken)
42·95	9·55	81,200	33·41	4·77	286,100
,,	14·33	1,562,000	,,	9·55	701,800
,,	19·10	225,300	,,	11·94	36,600,000 (unbroken)
,,	23·87	1,238,900			
,,	,,	300,900			
,,	28·65	33,600,000 (unbroken)			

Various theories have been advanced to account for the results obtained by Wöhler, all more or less unsatisfactory, but there are one or two empirical formulas which fairly well represent the results.

Of these, probably Gerber's parabolic equation with Unwin's constants[1] best fits the results; it is, however, not easy to remember. The "dynamic theory" equation, however, fairly fits the results, and is very easy of application, and is, moreover, simple to remember; but whether the assumptions of the theory are justifiable or not is quite an open question, which we

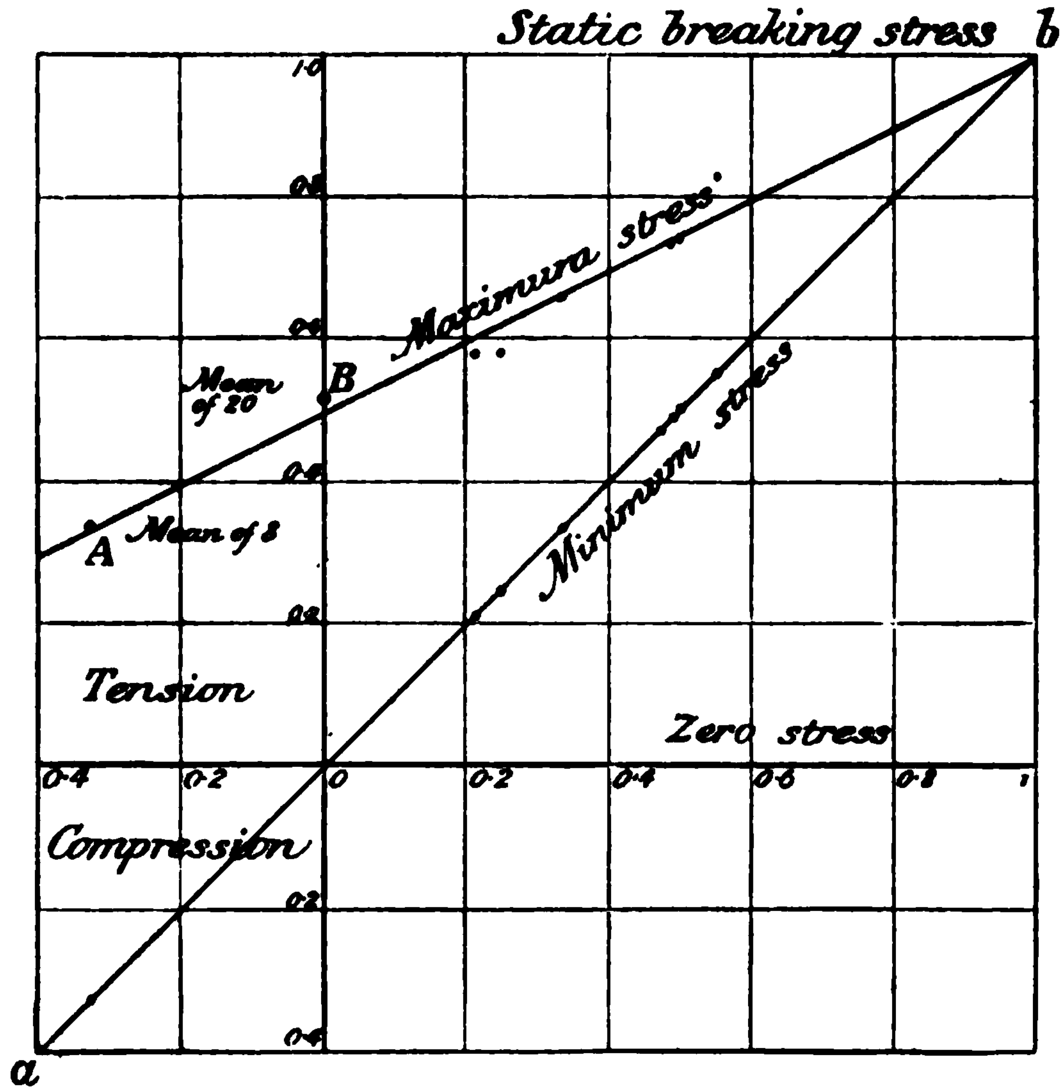


FIG. 517.

shall not discuss. We adopt it simply because it is the easiest to use, and, for all practical purposes, represents Wöhler's, Spangenberg's, and Bauschinger's results. The "dynamic theory" assumes that the varying loads applied by Wöhler and others were equivalent to suddenly applied loads, and consequently a piece of material will not break under repeated loadings unless the "momentary" stress, due to sudden applications, does not exceed the statical breaking strength of the material. In Fig. 517 we show, by means of a diagram, the results of all the

[1] "El.ments of Machine Design," p. 25.

published experiments reduced to a common scale. In every case represented the material stood over four million repetitions before fracture. The horizontal scale is immaterial; the vertical scale shows the ratio of the applied stress to the static breaking stress. The minimum stress on the material is plotted on the line *aob*, and the corresponding maximum stress, which may be repeated over four million times, is shown by the small circles above. If the dynamic theory were perfectly true, all the points would lie on the line marked "maximum stress;" for then the minimum stress (taken as being due to a dead load) *plus* twice the range of stress (*i.e.* maximum stress − minimum stress) taken as being due to a live load should together be equal to the statical breaking strength of the material.

The results of tests of revolving axles are shown in group A; the dynamic theory demands that they should be represented by a point situated 0·33 from the zero stress axis.

Likewise, when the stress varies from 0 to a maximum, the results are shown at B; by the dynamic theory, they should be represented by a point situated 0·5 from the zero axis. For all other cases the upper points should lie on the maximum stress line. Whether they do lie reasonably near this line must be judged from the diagram. When one considers the many accidental occurrences that may upset such experiments as these, one can hardly wonder at the points not lying regularly on the mean line.

To sum up, then. When designing a member which will be subjected to both a steady load W and a fluctuating load w_1, we must so proportion its section that it shall be capable of safely withstanding a dead load W_0, where—

$$W_0 = W \mp 2w_1$$

the *plus* sign when both loads act in the same way, and the *minus* when they tend to produce opposite effects.

For a fuller discussion of this question the reader is referred to Unwin's "Testing of Materials;" Fidler's "Practical Treatise on Bridge Construction;" Johnson's "Materials of Construction."

CHAPTER XVI.

HYDRAULICS.

IN Chapter VIII. we stated that a body which resists a change of form when under the action of a distorting stress is termed a *solid* body, and if the body returns to its original form after the removal of the stress, the body is said to be an *elastic solid* (*e.g.* wrought iron, steel, etc., under small stresses); but if it retains the distorted form it assumed when under stress, it is said to be a *plastic solid* (*e.g.* putty, clay, etc.). If, on the other hand, the body does not resist a change of form when under the action of a distorting stress, it is said to be a *fluid* body; if the change of form takes place immediately it comes under the action of the distorting stress, the body is said to be a *perfect fluid* (*e.g.* alcohol, ether, water, etc., are very nearly so); if, however, the change of form takes place gradually after it has come under the action of the distorting stress, the body is said to be a *viscous fluid* (*e.g.* tar, treacle, etc.). The viscosity is measured by the rate of change of form under a given distorting stress.

In nearly all that follows in this chapter, we shall assume that water is a perfect fluid; in some instances, however, we shall have to carefully consider some points depending upon its viscosity.

Weight of Water.—The weight of water for all practical purposes is taken at 62·5 lbs. per cubic foot, or 0·036 lb. per cubic inch. It varies slightly with the temperature, as shown in the table on the following page, which is for pure distilled water.

The volume corresponding to any temperature can be found very closely by the following empirical formula:—

$$\left.\begin{array}{l}\text{Volume at absolute temperature T, taking}\\ \quad\text{the volume at } 39{\cdot}2 \text{ Fahr. or } 500^\circ\\ \quad\text{absolute as } 1\end{array}\right\} = \frac{T^2 + 250{,}000}{1000T}$$

Pressure due to a Given Head.—If a cube of water of

1 foot side be imagined to be composed of a series of vertical columns, each of 1 square inch section, and 1 foot high, each will weigh $\frac{62\cdot5}{144} = 0\cdot434$ lb. Hence a column of water 1 foot high produces a pressure of 0·434 lb. per square inch.

Temp. Fahr.	32°.		39·2°.	50°.	100°.	150°.	200°.	212°.
	Ice.	Water.						
Weight per cubic foot in lbs.	57·2	62·417	62·425	62·409	62·00	61·20	60·14	59·84
Volume of a given weight taking water at 39·2° Fahr. as 1	1·091	1·0001	1·0000	1·0002	1·007	1·020	1·038	1·043

The height of the column of water above the point in question is termed the *head.*

Let h = the head of water in feet above any surface ;
p = the pressure in pounds per square inch on that surface ;
w = the weight of a column of water 1 foot high and 1 square inch section
= 0·434 lb.

$$\text{Then } p = wh, \text{ or } 0\cdot434h$$
$$\text{or } h = \frac{p}{w} = 2\cdot305p, \text{ or say } 2\cdot31p$$

Thus a head of 2·31 feet of water produces a pressure of 1 lb. per square inch.

Taking the pressure due to the atmosphere as 14·7 lbs. per square inch, we have the head of water corresponding to the pressure of the atmosphere—

$$14\cdot7 \times 2\cdot31 = 34 \text{ feet (nearly)}$$

This pressure is the same in all directions, and is entirely independent of the shape of the containing vessel. Thus in Fig. 518—

The pressure over any unit area of surface at $a = p_a = 0{\cdot}434h_a$
" " " " $b = p_b = 0{\cdot}434h_b$

and so on.

The horizontal width of the triangular diagram at the side shows the pressure per square inch at any depth below the surface. Thus, if the height of the triangle be made to a scale of 1 inch to the foot, and the width of the base $0{\cdot}434h$, the width of the triangle measured in inches will give the pressure in pounds per square inch at any point, at the same depth below the surface.

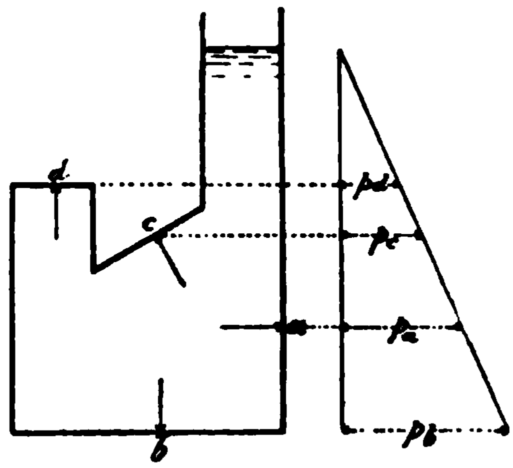


FIG. 518.

Compressibility of Water.—The popular notion that water is incompressible is erroneous; the alteration of volume under such pressures as are usually used is, however, very small. Experiments show that the alteration in volume is proportional to the pressure, hence the relation between the change of volume when under pressure may be expressed in the same form as we used for Young's modulus on p. 280.

Let v = the diminution of volume under any given pressure p in pounds per square inch (corresponding to x on p. 280);
V = the original volume (corresponding to l on p. 280);
K = the modulus of elasticity of volume of water;
p = the pressure in pounds per square inch.

$$\text{Then } \frac{v}{V} = \frac{p}{K}, \text{ or } K = \frac{pV}{v}$$

K = from 320,000 to 300 000 lbs. per square inch.

Thus water is reduced in bulk or increased in density by 1 per cent. when under a pressure of 3000 lbs. per square inch. This is quite apart from the stretch of the containing vessel.

Total Pressure on an Immersed Surface.—If, for any purpose, we require the total normal pressure acting on an immersed surface, we must find the mean pressure acting on the surface, and multiply it by the area of the surface. We shall show that the mean pressure acting on a surface is the pressure due to the head of water above the centre of gravity of the surface.

Let Fig. 519 represent an immersed surface. Let it be divided up into a large number of narrow strips of length l_1, l_2, etc., and of width b each at a depth h_1, h_2, etc., respectively from the surface. Then the total pressure on each strip is p_1l_1b, p_2l_2b, etc., where p_1, p_2, etc., are the pressures corresponding to h_1, h_2, etc.

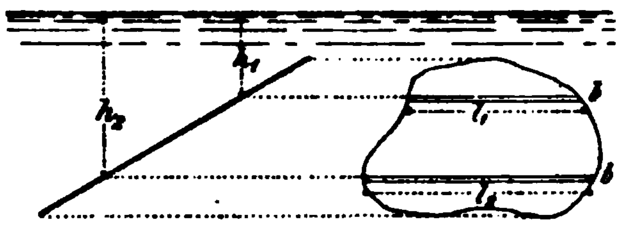


FIG. 519.

But $p = wh$, and $l_1b = a_1$, $l_2b = a_2$, etc.

The total pressure on each strip $= wa_1h_1$, wa_2h_2, etc.

Total pressure on whole surface $= \text{P}_0 = w\,(a_1h_1 + a_2h_2 + \text{etc.})$

But the sum of all the areas a_1, a_2, etc., make up the whole area of the surface A, and by the principle of the centre of gravity (p. 58) we have—

$$a_1h_1 + a_2h_2 +, \text{etc.}, = \text{AH}_0$$

where H_0 is the depth of the centre of gravity of the immersed surface from the surface of the water, or—

$$\text{P}_0 = w\text{AH}_0$$

Thus the total pressure in pounds on the immersed surface is the area of the surface in square units × the pressure in pounds per square unit due to the head of water above the centre of gravity of the surface.

Centre of Pressure.—The centre of pressure of a plane immersed surface is the point in the surface through which the resultant of all the pressure on the surface acts.

It can be found thus:

Let $\text{H}_c =$ the depth of the centre of pressure of the immersed surface from the surface of the water;

$\text{P}_1 =$ the total pressure on any small strip at a depth h_1 from the surface.

$$\text{P}_1 = p_1a_1 = wa_1h_1$$

$$\text{then } \text{P}_1h_1 + \text{P}_2h_2 +, \text{etc.}, = \text{P}_0\text{H}_c$$

Substituting the values for P, we have—

$$wa_1h_1^2 + wa_2h_2^2 +, \text{etc.}, = w\text{H}_0\text{AH}_c$$

$$\text{hence } \text{H}_c = \frac{a_1h_1^2 + a_2h_2^2 +, \text{etc.}}{\text{H}_0\text{A}}$$

but in Chapter III., p. 76, we showed that $a_1h_1^2 + a_2h_2^2 +$, etc., = the moment of inertia I_0 of the surface about an axis lying in its own plane situated on the surface of the water. Then, adhering to the same notation, which we now repeat—

I_0 = the moment of inertia of any surface or body about a given axis;
I = the moment of inertia about a parallel axis passing through the c. of g. of the surface;
A = the area of the surface;
R_0 = the perpendicular distance between the two axes;
K = the radius of gyration;

we have from the same source—

$$I_0 = I + AR_0^2, \text{ or } I + AH_0^2$$

$$\text{also } H_c = \frac{I + AH_0^2}{AH_0} = \frac{AK^2 + AH_0^2}{AH_0} = \frac{K^2 + H_0^2}{H_0}$$

The lateral position of the centre of pressure is found by taking a vertical plane through the centre of gravity of the surface; where this plane cuts the horizontal drawn at a depth H_c below the surface is the centre of pressure.

The depth of the centre of pressure from the surface of the water is given for a few cases in the following table:—

Surface.	K^2.	H_0.	H_c.
Rectangle of depth d with upper edge at surface of water	$\frac{d^2}{12}$	$\frac{d}{2}$	$\frac{2}{3}d$
Circle of diameter d with circumference touching surface of water	$\frac{d^2}{16}$	$\frac{d}{2}$	$\frac{5}{8}d$
Triangle of height d with apex at surface of water	$\frac{d^2}{18}$	$\frac{2d}{3}$	$\frac{3}{4}d$

The methods of finding K^2 and H_0 have been fully described in Chapter III.

Graphical Method for finding the Centre of Pressure.—In some cases of irregularly shaped surfaces the algebraic method given above is not easy of application, but the following graphic method is extremely simple.

In the figure shown draw a series of lines across, not necessarily equidistant; project them on to a base-line drawn parallel to the surface of the water. In the figure shown only one line, *aa*, is projected on to the base-line in *bb*. Join *bb* to any convenient point *o* on the surface line of the water, which cuts off a line *d'd'*; then we have, by similar triangles—

Fig. 520.

$$\frac{aa}{d'a'} = \frac{bb}{d'd'} = \frac{h_b}{h_a}$$

or the width of the shaded figure *d'd'* at any depth h_a below the surface is proportional to the total pressure on a very narrow strip *aa* of the surface; hence the shaded figure may be regarded as an equivalent surface on which the pressure is uniform; hence the c. of g. of the shaded figure is the centre of pressure of the original figure.

It will be seen that precisely the same idea is involved here as in the modulus figures of beams given in Chapter IX.

The total normal pressure on the surface is the shaded area A, multiplied by the pressure due to the head at the base-line, or—

$$\text{Total normal pressure} = w\text{A}h_b$$

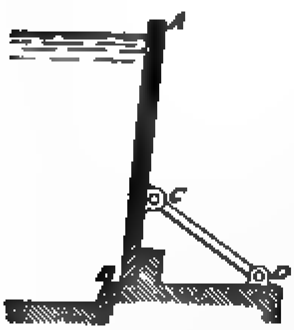


Fig. 521.

Practical Application of the Centre of Pressure.—A good illustration of a practical application of the use of the centre of pressure is shown in Fig. 521, which represents a self-acting movable flood dam. The dam AB, usually of timber, is pivoted to a back stay, CD, the point C being placed at a distance $= \frac{2}{3}$AB from the top; hence, when the level of the water is below A the centre of pressure falls below C, and the dam is stable; if, however, the water

flows over A, the centre of pressure rises above C, and the dam tips over. Thus as soon as a flood occurs the dam automatically tips over and prevents the water rising much above its normal. Each section, of course, has to be replaced by those in attendance when the flood has abated.

Velocity of Flow due to a Given Head.—Let the tank shown in the figure be provided with an orifice in the bottom as shown, through which water flows with a velocity V feet per second. Let the water in the tank be kept level by a supply-pipe as shown, and suppose the tank to be very large compared with the quantity passing the orifice per second, and that the water is sensibly at rest and free from eddies.

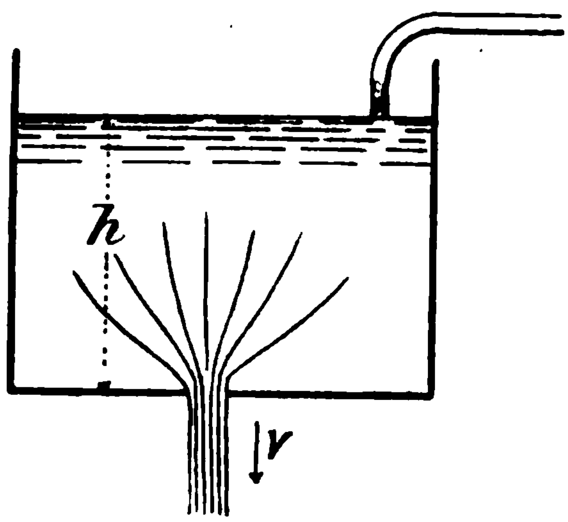


FIG. 522.

Let A = area of orifice in square feet;
A_c = the contracted area of the jet;
Q_0 = quantity of water passing through the orifice in cubic feet per second (neglecting contraction);
V = velocity of flow in feet per second;
W = weight of water passing in pounds per second;
h = head in feet above the orifice.

Then $Q_0 = A_cV$

Work done per second by W lbs. of water falling through h feet $= Wh$ foot-lbs.

Kinetic energy of the water on leaving the orifice $= \dfrac{WV^2}{2g}$

But these two quantities must be equal, or—

$$Wh = \frac{WV^2}{2g}, \text{ and } h = \frac{V^2}{2g}$$
$$\text{and } V = \sqrt{2gh}$$

that is, the velocity of flow is equal to the velocity acquired by a body in falling through a height of h feet.

Contraction and Friction of a Stream passing through an Orifice.—The actual velocity with which water flows through an orifice is less than that due to the head,

mainly on account of the friction of the stream on the sides of the orifice; and, moreover, the stream contracts after it leaves the orifice, the reason for which will be seen from the figure. If each side of the orifice be regarded as a ledge over which a stream of water is flowing, it is evident that the path taken by the water will be the resultant of its horizontal and vertical movements, and therefore it does not fall vertically as indicated by the dotted lines, which it would have to do if the area of the stream were equal to the area of the orifice. Both the friction and the contraction can be measured experimentally, but they are usually combined in one coefficient of discharge K, which is found experimentally.

FIG. 523.

$$K = \frac{\text{actual discharge}}{\text{discharge if no friction or contraction}}$$

then the actual discharge of an orifice is—

$$Q = KQ_0, \text{ and } W = KW_0$$

Values of K will be given for the various orifices as we proceed.

Plain Orifice.—The edges should be chamfered off as shown; if not, the water dribbles down the sides and makes the coefficient variable. The sharper the edges the smaller is the coefficient, but it rarely gets below 0·61, and sometimes reaches 0·64. As a mean value K = 0·62.[1]

FIG. 524.

$$Q = 0{\cdot}62A\sqrt{2gh}$$

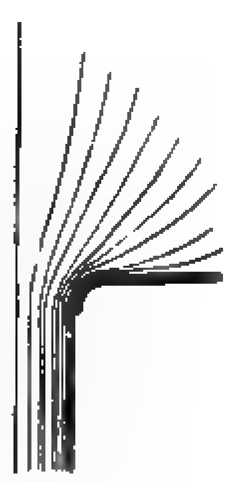


FIG. 525.

Rounded Orifice.—If the orifice be rounded to the same

[1] This coefficient has been obtained rationally by Lord Rayleigh for a narrow slit orifice; the author has recently succeeded in obtaining a similar value for a circular orifice in a thin plate.

form as a contracted jet, the contraction can be entirely avoided, but the friction is rather greater than in the plain orifice. The head h must be measured from the bottom of the orifice, and the diameter at the same place. The coefficient varies slightly with the curvature from 0·94 to 0·97. As a mean value—

$$Q = 0{\cdot}95\ A \sqrt{2gh}$$

[1] **Pipe Orifice.**—The length of the pipe should be not less than three times the diameter. The jet contracts after leaving the square corner, as in the sharp-edged orifice; it expands again lower down, and fills the tube. It is possible to get a clear jet right through, but a very slight disturbance will make it run as shown. In the case of the clear stream, the value of K is approximately the same as in the plain orifice. When the pipe runs full, there is a sudden change of velocity from the contracted to the full part of the jet, with a consequent loss of energy and velocity of discharge.

FIG. 526.

Let the velocity at $b = V_b$; and the head $= h_b$
the velocity at $a = V_a$; „ „ $= h_a$

Then the loss of head $= \dfrac{(V_b - V_a)^2}{2g}$ (see p. 482)

$$\text{and } h_a = \frac{V_a^2}{2g} + \frac{(V_b - V_a)^2}{2g}$$

The velocities at the sections a and b will be inversely as the respective areas. If c be the coefficient of contraction at b, we have $V_b = \dfrac{V_a}{c}$; inserting this value in the expression given above, we get—

$$V_a = \frac{1}{\sqrt{1 + \left(\frac{1}{c} - 1\right)^2}} \sqrt{2gh_a}$$

[1] The beginner will do well to leave the next four paragraphs until he has mastered pp. 475–478, also p. 482.

Putting $c = 0.64$, as in the plain orifice, we have—

$$V_a = 0.87\sqrt{2gh_a}$$

or the coefficient of velocity = 0·87. The coefficient of discharge found by experiment is K = 0·82, the difference being due to friction.

$$Q = 0.82A\sqrt{2gh_a}$$

The pressure at a is atmospheric, but at b it is less (see p. 476). If it discharged into the atmosphere, the discharge would be—

$$Q = 0.62A\sqrt{2gh}$$

but in this case it is—

$$Q = 0.82A\sqrt{2gh}$$

or $\dfrac{0.82}{0.62} = 1.32$ times greater; hence we may write—

$$Q = 0.62 \times 1.32A\sqrt{2gh}$$

$$\text{or } Q = 0.62A\sqrt{2g \times 1.75h}$$

hence the effective head is $1.75h$, that is to say, there is a partial vacuum at $b = 0.75h$, which is easily shown to be the case by a vacuum gauge attached to the nozzle as shown.[1]

The head h must be measured from the bottom of the pipe orifice.

Re-entrant Orifice or Borda's Mouthpiece.—If a plain orifice in the bottom of a tank be closed by a cover or valve on the upper side, the total pressure on the bottom of the tank will be P, where P is the weight of water in the tank; but if the orifice be opened, the pressure P will be reduced by an amount P_0, equal to (i.) the downward pressure on the valve, viz. whA; and (ii.) by a further amount P_f, due to the flow of water over the surface of the tank all round the orifice (see p. 476). Then we have—

Fig. 527.

$$P_0 = whA + P_f$$

[1] Experiments in the author's laboratory sometimes give higher value than this.

In the case of Borda's mouthpiece, the orifice is so far removed from the side of the tank that the velocity of flow over the surface is practically zero, hence no such reduction of pressure occurs.

Let the section of the jet be a, and the area of the orifice A.

$$\left.\begin{array}{l}\text{Then the total pressure due to the column}\\ \quad\text{of water over the orifice}\end{array}\right\} = whA$$

$$= \frac{wAV^2}{2g}$$

$$\text{the volume of water flowing per second} = aV$$

$$\text{the mass of water flowing per second} = \frac{waV}{g}$$

$$\text{the momentum of the water flowing per second} = \frac{waV^2}{g}$$

The water before entering the mouthpiece was sensibly at rest, hence this expression gives us the change of momentum per second.

$$\left.\begin{array}{l}\text{Change of momentum}\\ \quad\text{per second}\end{array}\right\} = \text{impulse per second, or pressure}$$

$$\frac{waV^2}{g} = \frac{wAV^2}{2g}$$

$$\text{whence } a = 0{\cdot}5A$$

$$\text{or } K = 0{\cdot}5$$

$$\text{experiments give } K = 0{\cdot}52 \text{ to } 0{\cdot}55$$

$$\text{or say } K = 0{\cdot}53$$

$$\text{then } Q = 0{\cdot}53A\sqrt{2gh}$$

Diverging Mouthpiece.—This form of mouthpiece is of great interest in that the discharge of a pipe can be greatly increased by adding a nozzle of this form to the outlet end, because the velocity of flow in the throat a is greater than the velocity due to the head of water h above it. The pressure at b is atmospheric;[1] hence the pressure at a is less than atmospheric (see p. 476); thus the water is discharging into a partial vacuum. If a water-gauge be attached at a, and

[1] This reasoning will not hold if the mouthpiece discharges into a vacuum.

the vacuum measured, the velocity of flow at a will be found to be due to the head of water above it *plus* the vacuum head.

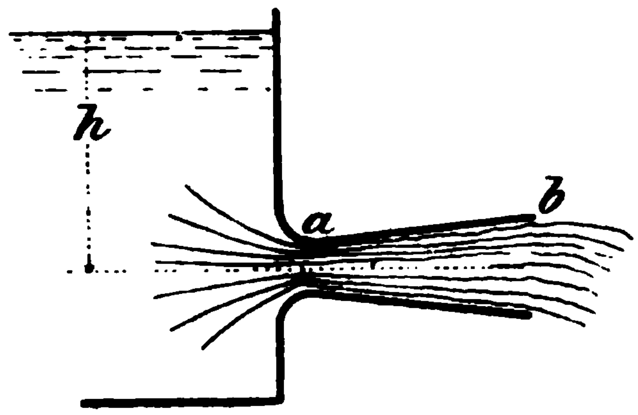


FIG. 528.

We shall shortly show that the energy of any steadily flowing stream of water in a pipe in which the diameter varies gradually is constant at all sections, neglecting friction. The energy per pound of the water at the level of the mouthpiece, reckoning from absolute zero of pressure, is $h + \frac{p}{w}$, $\frac{p}{w}$ being the head equivalent to the pressure of the atmosphere. Hence, neglecting friction, if the water have a velocity V_a, and a pressure p_a, or a head $\frac{p_a}{w}$ at the throat a, it is obvious that the energy has been derived from the head above it. A part of the energy exists as energy of motion, $\frac{V_a^2}{2g}$, and a part as head still unconverted into energy of motion, viz. $\frac{p_a}{w}$; likewise at b the water has energy of motion $\frac{V_b^2}{2g}$, and head energy $\frac{p_b}{w}$. Then, since there is assumed to be no loss by friction, these must all be equal, or—

$$h + \frac{p}{w} = \frac{V_a^2}{2g} + \frac{p_a}{w} = \frac{V_b^2}{2g} + \frac{p_b}{w}$$

But the pressure at b is atmospheric, therefore—

$$\frac{p}{w} = \frac{p_b}{w}, \text{ or } h = \frac{V_b^2}{2g}$$

$$\text{or } V_b = \sqrt{2gh}$$

and the discharge—

$$Q = KV_bA_b = KA_b\sqrt{2gh}$$

In the case above, the mouthpiece is horizontal, but if it be placed vertically with b below, the proof given above still holds;

the h must then be measured from b, *i.e.* the bottom of the mouthpiece, provided the conditions mentioned below are fulfilled.

Thus we see that the discharge depends upon the area at b, and is independent of the area at a; there is, however, a limit to this, for if the pressure at a be negative, the stream will not be continuous.

From the above, we have—

$$\frac{p_a}{w} = \frac{V_b^2 - V_a^2}{2g} + \frac{p_b}{w}$$

If $\frac{p_a}{w}$ becomes zero, the stream breaks up, or when—

$$\frac{V_a^2 - V_b^2}{2g} = \frac{p_b}{w} = 34 \text{ feet}$$

$$\text{But } \frac{V_a}{V_b} = \frac{A_b}{A_a} = n, \text{ or } V_a = nV_b$$

$$\text{hence } \frac{n^2V_b^2 - V_b^2}{2g} = \frac{V_b^2}{2g}(n^2 - 1) = 34$$

$$\text{or } h_b(n^2 - 1) = 34$$

In order that the stream may be continuous, $h_b\,(n^2 - 1)$ should be less than 34 feet, and the maximum discharge will occur when the term to the left is slightly less than 34 feet.

The following experiments demonstrate the accuracy of the statement made above, that the discharge is due to the head of water + the vacuum head. The experiments were made by Mr. Brownlee, and are given in the *Proceedings* of the Shipbuilders of Scotland for 1875–6.

The experiments were arranged in such a manner that, in effect, the water flowed from a tank A through a diverging mouthpiece into a tank B, a vacuum gauge being attached at the throat t.

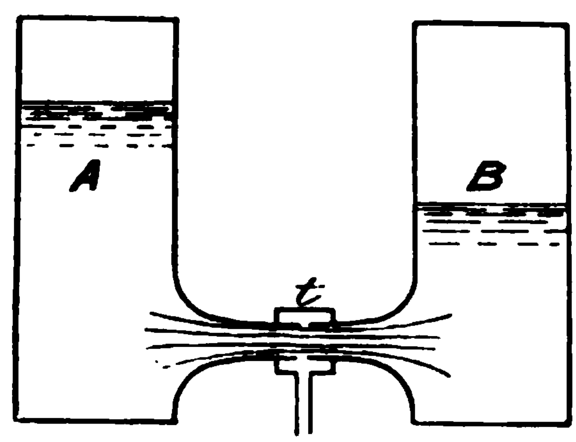


FIG. 529.

The close agreement between the experimental and the calculated values as given in the last two columns, is a clear proof of the accuracy of the theory given above.

Head of water in tank A. Feet. H_a.	Head of water in tank B. Feet. H_b.	Vacuum at throat in feet of water. H_v.	Velocity of flow at throat. Feet per second.	
			By experiment.	$\sqrt{2g(H_a + H_v)}$
69·24	58·85	None	65·97	66·78
69·24	50·78	33·5	80·97	81·34
69·24	None	33·5	81·43	81·34
12·50	8·50	11·3	37·90	38·84
12·50	5·00	33·5	53·98	54·43
12·50	1·50	33·5	54·60	54·43
8·00	None	33·5	51·67	51·70
2·00	None	8·2	24·74	25·63
0·25	None	0·52	6·66	7·04

Jet Pump or Hydraulic Injector.—If the height of the column of water in the vacuum gauge at t (Fig. 529), be less than that due to the vacuum produced, the water will be sucked in and carried on with the jet. Several inventors have endeavoured to utilize an arrangement of this kind for saving water in hydraulic machinery when working below their full power. The high-pressure water enters by the pipe A; when passing through the nozzles on its way to the machine cylinder, it sucks in a supply of water from the exhaust sump viâ B, and the greater volume of the combined stream at a lower pressure passes on to the cylinder. All the water thus sucked in is a direct source of gain, but the efficiency of the apparatus as usually constructed is very low, about 30 per cent. The author and Mr. R. H. Thorpe, of New York, made a long series of experiments on jet pumps, and succeeded in designing one which gave an efficiency of 72 per cent.

FIG. 530.

An ordinary jet pump is shown in Fig. 530. The main trouble that occurs with such a form of pump is that the water churns round and round the suction spaces of the nozzles instead of going straight through. Each suction space between the nozzles should be in a separate chamber provided with a back-pressure valve, and the spaces should gradually increase in area as the high-pressure water proceeds—that is to say, the

first suction space should be very small, and the next rather larger, and so on.

Rectangular Notch.—An orifice in a vertical plane with an open top is termed a notch, or sometimes a weir. The only two forms of notches commonly used are the rectangular and the triangular.

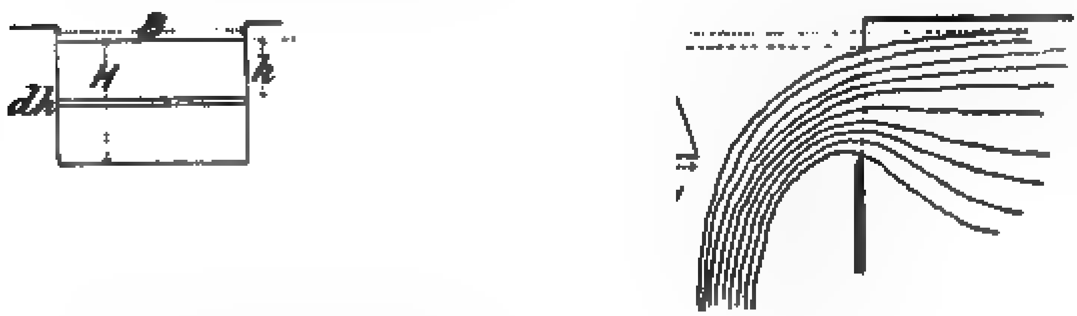


FIG. 531.

From the figure, it will be observed that the head of water immediately over the crest is less than the head measured further back, which is, however, the true head H.

In calculating the quantity of water Q that flows over such a notch, we proceed thus—

The area of any elementary strip as shown $= \mathrm{B} \cdot dh$

$$\left.\begin{array}{c}\text{quantity of water passing strip per}\\ \text{second, neglecting contraction}\end{array}\right\} = \mathrm{V} \cdot \mathrm{B} \cdot dh$$

where V = velocity of flow in feet per second,

$$= \sqrt{2gh}, \text{ or } \sqrt{2g}h^{\frac{1}{2}}$$

$$\left.\begin{array}{l}\text{hence the quantity of water passing strip}\\ \text{per second, neglecting contraction}\end{array}\right\} = \sqrt{2g}\mathrm{B}h^{\frac{1}{2}}dh$$

$$\left.\begin{array}{l}\text{the whole quantity of water Q passing}\\ \text{over the notch in cubic feet per}\\ \text{second, neglecting contraction}\end{array}\right\} = \sqrt{2g}\mathrm{B}\int_{h=0}^{h=\mathrm{H}} h^{\frac{1}{2}}dh$$

$$\mathrm{Q} = \sqrt{2g}\mathrm{B}\frac{\mathrm{H}^{\frac{3}{2}}}{\frac{3}{2}}$$

$$\mathrm{Q} = \sqrt{2g}\,\mathrm{B}\tfrac{2}{3}\mathrm{H}\sqrt{\mathrm{H}}$$

$$\mathrm{Q} = \tfrac{2}{3}\mathrm{BH}\sqrt{2g\mathrm{H}}$$

$$\left.\begin{array}{c}\text{introducing a coefficient to}\\ \text{allow for contraction}\end{array}\right\}\mathrm{Q} = \mathrm{K}\tfrac{2}{3}\mathrm{BH}\sqrt{2g\mathrm{H}}$$

where B and H are both measured in feet; where K has values varying from 0·59 to 0·64 depending largely on the proportions of the section of the stream, *i.e.* the ratio of the

depth to the width, and on the relative size of the notch and the section of the stream above it. In the absence of precise data it is usual to take $K = 0{\cdot}62$. The above value for Q may also be arrived at thus, the velocity of flow at any strip at a depth h from the surface is $\sqrt{2gh}$. Hence, if we construct a diagram of velocities we shall get the parabolic figure shown a the side, the area of which is $\frac{2}{3}H\sqrt{2gH}$ (see p. 30), and the mean width, *i.e.* the mean velocity of flow, is—

$$\frac{\frac{2}{3}H\sqrt{2gH}}{H} = \frac{2}{3}\sqrt{2gH}$$

and the total flow over the notch = area × mean velocity of flow

$$= BH\tfrac{2}{3}\sqrt{2gH}$$

the same result as we arrived at above.

Triangular Notch.—In order to avoid the uncertainty of the value of K, Professor James Thompson proposed the use of V notches; the form of the section of the stream then always remains constant however the head may vary. Experiments show that K for such a notch is very nearly constant. Hence, in the absence of precise data, it may be used with much greater confidence than the rectangular notch. The quantity of water that passes is arrived at thus:

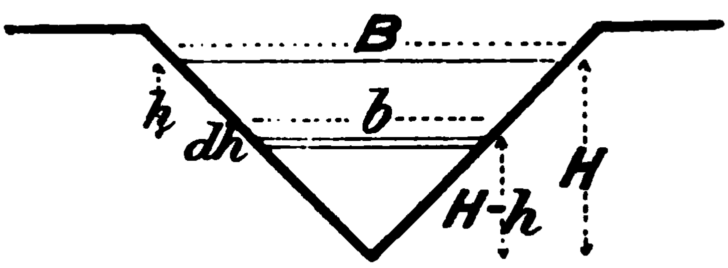


FIG. 532.

$$\text{Area of elementary strip} = b \,.\, dh$$

$$\text{But } \frac{b}{B} = \frac{H - h}{H}, \text{ and } b = \frac{B(H - h)}{H}$$

$$\text{area of elementary strip} = \frac{B(H - h)}{H} \,.\, dh$$

$$\text{velocity of water passing strip} = V = \sqrt{2gh} = \sqrt{2g}h^{\frac{1}{2}}$$

$$\left.\begin{array}{l}\text{quantity of water passing}\\ \text{strip per second, neglect-}\\ \text{ing contraction}\end{array}\right\} = \frac{B(H - h)}{H}\sqrt{2g}\,h^{\frac{1}{2}} \,.\, dh$$

$$\left.\begin{array}{l}\text{whole quantity of water Q}\\ \text{passing over the notch in}\\ \text{cubic feet per second,}\\ \text{neglecting contraction}\end{array}\right\} = \frac{B}{H}\sqrt{2g}\int_{h=0}^{h=H}(H - h)h^{\frac{1}{2}} \,.\, dh$$

$$Q = \frac{B}{H}\sqrt{2g}\int_{h=0}^{h=H}(Hh^{\frac{1}{2}} - h^{\frac{3}{2}})dh$$

$$Q = \frac{B}{H}\sqrt{2g}\left[\frac{Hh^{\frac{3}{2}}}{\frac{3}{2}} - \frac{h^{\frac{5}{2}}}{\frac{5}{2}}\right]_{h=0}^{h=H}$$

$$Q = \frac{B}{H}\sqrt{2g}\left(\tfrac{2}{3}H^{\frac{5}{2}} - \tfrac{2}{5}H^{\frac{5}{2}}\right)$$

$$Q = \frac{B}{H}\sqrt{2g}\tfrac{4}{15}H^{\frac{5}{2}}$$

$$Q = \tfrac{4}{15}BH\sqrt{2gH}$$

Introducing a coefficient for the contraction of the stream—

$$Q = K\tfrac{4}{15}BH\sqrt{2gH}$$

where K = 0·59 to 0·60.

Rectangular Orifice in a Vertical Plane.—When the vertical height of the orifice is small compared with the depth of water above it, the discharge is commonly taken to be the same as that of an orifice in a horizontal plane, the head being H, *i.e.* the head to the centre of the orifice. When, however, the vertical height of the orifice is not small compared with the

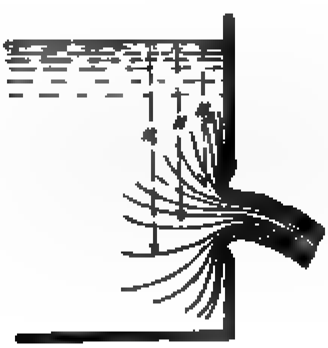


FIG. 533. FIG. 534.

depth, the discharge is obtained by precisely the same reasoning as in the two last cases; it is—

$$Q = K\tfrac{2}{3}B\sqrt{2g}(H_1^{\frac{3}{2}} - H_2^{\frac{3}{2}})$$

K, however, is a very uncertain quantity; it varies with the shape of the orifice and its depth below the surface.

Drowned Orifice.—When there is a head of water on both sides of an orifice, the discharge is not free; the calculation of the flow is, however, a very simple matter. The head

producing flow at any section xy (Fig. 534) is $H_1 - H_2 = H$; likewise, if any other section be taken, the head producing flow is also H. Hence the velocity of flow $V = \sqrt{2gH}$, and the quantity discharged—

$$Q = KA\sqrt{2gH}$$

K varies somewhat, but is usually taken 0·62 as a mean value.

Flow under a Constant Head.—It is often found necessary to keep a perfectly constant head in a tank when making careful measurements of the flow of liquids, but it is often very difficult to accomplish by keeping the supply exactly equal to the delivery. It can, however, be easily managed with the device shown in the figure. It consists of a closed tank fitted with an orifice, also a gland and sliding pipe open to the atmosphere. The vessel is filled, or nearly so, with the fluid, and the sliding pipe adjusted to give the required flow. The flow is due to the head H, and the negative pressure p above the surface of the water, for as the water sinks a partial vacuum is formed in the upper part of the vessel, and air bubbles through. Hence the pressure p is always due to the head h, and the effective head producing flow through the orifice is $H - h$, which is independent of the height of water in the vessel, and is constant provided the water does not sink below the bottom of the pipe. The quantity of water delivered is—

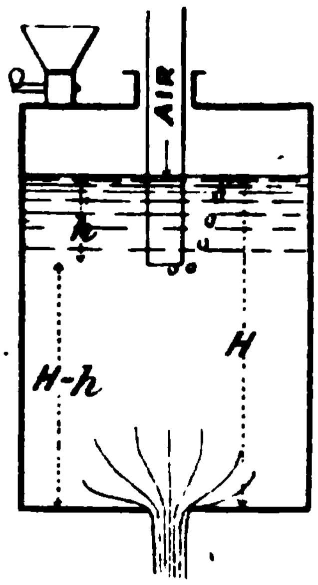


FIG. 535.

$$Q = KA\sqrt{2g(H - h)}$$

where K has the values given above for different orifices.

Velocity of Approach.—If the water approaching a notch or weir have a velocity V_a, the quantity of water passing will be correspondingly greater, but the exact amount will depend upon whether the velocity of the stream is uniform at every part of the cross-section, or whether it varies from point to point as in the section over the crest of a weir or notch.

If the velocity be uniform, the increase of flow due to this will be KAV_a, or $KA\sqrt{2gh_a}$, where h_a is the head corresponding to the velocity V_a. Then, on this assumption—

$$Q = K\tfrac{2}{3}BH\sqrt{2gH} + KBH\sqrt{2gh_a}$$
$$Q = KBH\sqrt{2g}\left(\tfrac{2}{3}\sqrt{H} + \sqrt{h_a}\right)$$

But if the velocity vary as in a weir or notch, the mean velocity will be $\frac{2}{3}V_a$, and we get—

$$Q = K\tfrac{2}{3}BH\sqrt{2g}\,(\sqrt{H} + \sqrt{h_a})$$

It is impossible to say which is correct for any given case in the absence of experimental data. Precautions should always be taken to diminish the velocity of approach as much as possible.

Flow-through Pipes of Variable Section.—For the present we shall only deal with pipes running full, in which the section varies very gradually from point to point. If the variation be abrupt, an entirely different action takes place. This particular case we shall deal with later on. The main point that we have to concern ourselves with at present is to show that the energy of the water at any section of the pipe is constant—neglecting friction.

If W lbs. of water be raised from a given datum to a receiver at a certain height h feet above, the work done in raising the water is Wh foot-lbs., or h foot-lbs. per pound of water. By lowering the water to the datum, Wh foot-lbs. of work will be done. Hence, when the water is in the raised position its energy is termed its energy of position, or—

$$\text{The energy of position} = Wh \text{ foot-lbs.}$$

If the water were allowed to fall freely, *i.e.* doing no work in its descent, it would attain a velocity V feet per second, where $V = \sqrt{2gh}$, or $h = \frac{V^2}{2g}$. Then, since no energy is destroyed in the fall, we have $Wh = \frac{WV^2}{2g}$ foot-lbs. of energy stored in the falling water when it reaches the datum, or $\frac{V^2}{2g}$ foot-lbs. per pound of water. This energy, which is due to its velocity, is termed its kinetic energy, or energy of motion; or—

$$\text{The energy of motion} = \frac{WV^2}{2g}$$

If the water in the receiver descends by a pipe to the datum level—for convenience we will take the pipe as one square inch area—the pressure p at the foot of the pipe will be wh lbs. per square inch. This pressure is capable of overcoming a resistance through a distance l feet, and thereby doing pl foot-

lbs. of work; then, as no energy is destroyed in passing along the pipe, we have $pl = Wh = \frac{Wp}{w}$ foot-lbs. of work done by the water under pressure, or $\frac{p}{w}$ foot-lbs. per pound of water. This is known as its pressure-energy, or—

$$\text{The pressure-energy} = \frac{Wp}{w}$$

Thus the energy of a given quantity of water may exist exclusively in either of the above forms, or partially in one form and partially in another, or in any combination of the three.

$$\left.\begin{array}{r}\text{Total energy per}\\ \text{pound of water}\end{array}\right\} = \left\{\begin{array}{l}\text{energy of}\\ \text{position}\end{array}\right\} + \left\{\begin{array}{l}\text{energy of}\\ \text{motion}\end{array}\right\} + \left\{\begin{array}{l}\text{pressure-}\\ \text{energy}\end{array}\right.$$

$$= h + \frac{V^2}{2g} + \frac{p}{w}$$

This may, perhaps, be more clearly seen by referring to the figure.

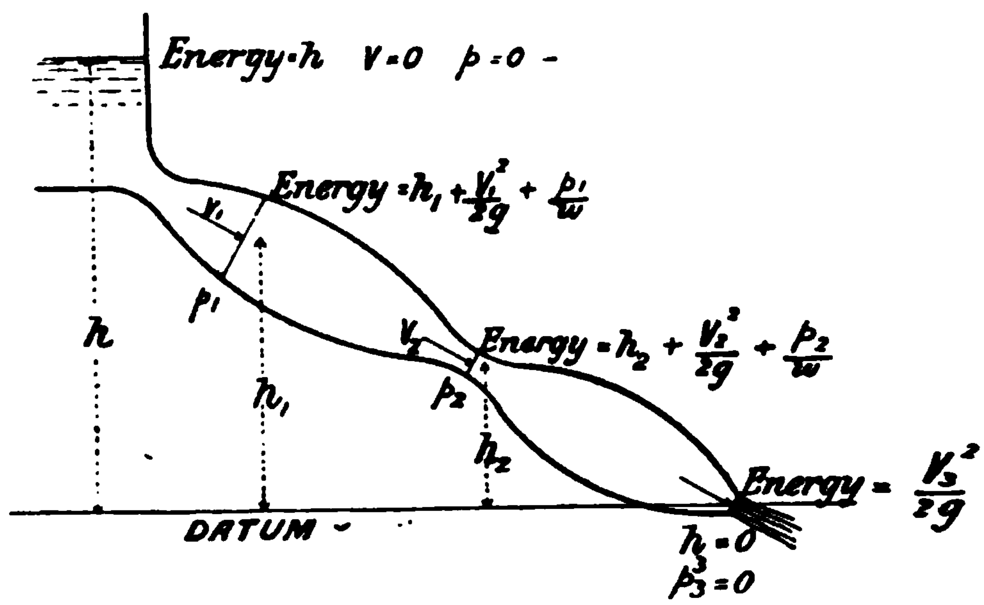


FIG. 536.

Then, as no energy of the water is destroyed on passing through the pipe, the total energy at each section must be the same, or—

$$h_1 + \frac{V_1^2}{2g} + \frac{p_1}{w} = h_2 + \frac{V_2^2}{2g} + \frac{p_2}{w} = \text{constant}$$

The quantity of water passing any given section of the pipe in a given time is the same, or—

$$Q_1 = Q_2$$
$$\text{or } A_1V_1 = A_2V_2$$
$$\frac{V_1}{V_2} = \frac{A_2}{A_1}$$

or the velocity of the water varies inversely as the sectional area—

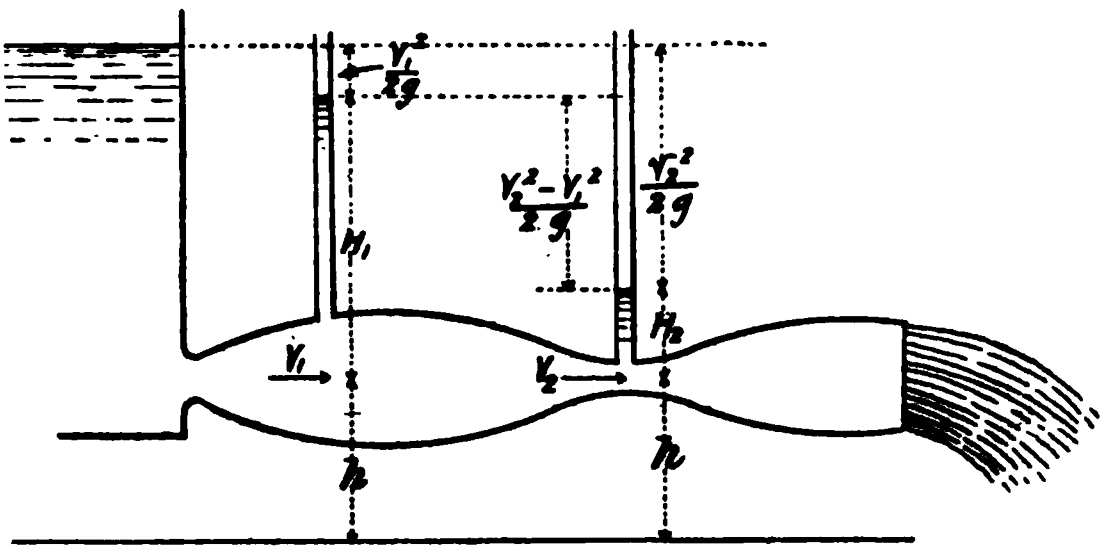


FIG. 537.

Some interesting points in this connection were given by the late Mr. Froude at the British Association in 1875.

Let vertical pipes be inserted in the main pipe as shown; then the height H, to which the water will rise in each, will be proportional to the pressure, or—

$$H_1 = \frac{p_1}{w}, \text{ and } H_2 = \frac{p_2}{w}$$

and the total heights of the water-columns above datum—

$$\frac{p_1}{w} + h, \text{ and } \frac{p_2}{w} + h$$

and the differences of the heights—

$$\frac{p_1}{w} + h - \frac{p_2}{w} - h = \frac{V_2^2}{2g} - \frac{V_1^2}{2g}$$
$$H_1 - H_2 = \frac{V_2^2 - V_1^2}{2g}$$

from the equation given above.

Thus we see that, when water is steadily running through

a full pipe of variable section, the pressure is greatest at the greatest section, and least at the least section.

In addition to many other experiments that can be made to prove that such is the case, one has been devised by Professor Osborne Reynolds that beautifully illustrates this point. Take a piece of glass tube, say $\frac{3}{4}$ inch bore drawn down to a fine waist in the middle of, say, $\frac{1}{20}$ inch diameter; then, when water is forced through it at a high velocity, the pressure is so reduced at the waist that the water boils and hisses loudly. The pressure is atmospheric at the outlet, but very much less at the waist. The hissing in water-injectors and partially opened valves is also due to this cause.

Venturi Water-meter.—An interesting application of this principle is the Venturi water-meter. The water is forced through a very easy waist in a pipe, and the pressure measured at the smallest and largest section; then, if the difference of the heads corresponding to the two pressures be H_0—

$$\frac{V_2^2 - V_1^2}{2g} = H_0, \text{ or } V_2^2 - V_1^2 = 2gH_0$$

If the areas of the sections of the pipes be A_1, A_2, we have, from above—

$$V_1 = \frac{A_2V_2}{A_1}$$

Let $A_1 = nA_2$; then—

$$V_1 = \frac{V_2}{n}$$

$$\text{hence } V_2^2 - \left(\frac{V_2}{n}\right)^2 = 2gH$$

$$\text{and } V_2 = \sqrt{\frac{2gH_0}{1 - \frac{1}{n^2}}}$$

$$Q = A_2V_2 = A_2\sqrt{\frac{2g}{1 - \frac{1}{n^2}}}\sqrt{H_0}$$

$$Q = C\sqrt{H_0},$$

$$\text{where } C = A_2\sqrt{\frac{2g}{1 - \frac{1}{n^2}}}$$

which is constant for any given case.

This meter has been used on a large scale in the United States, and is accurate to within ½ per cent. (see *Engineering*, August 14, 1896).

Radiating Currents and Free Vortex Motion.[1]—Let the figure represent the section of two circular plates at a small distance apart, and let water flow up the vertical pipe and

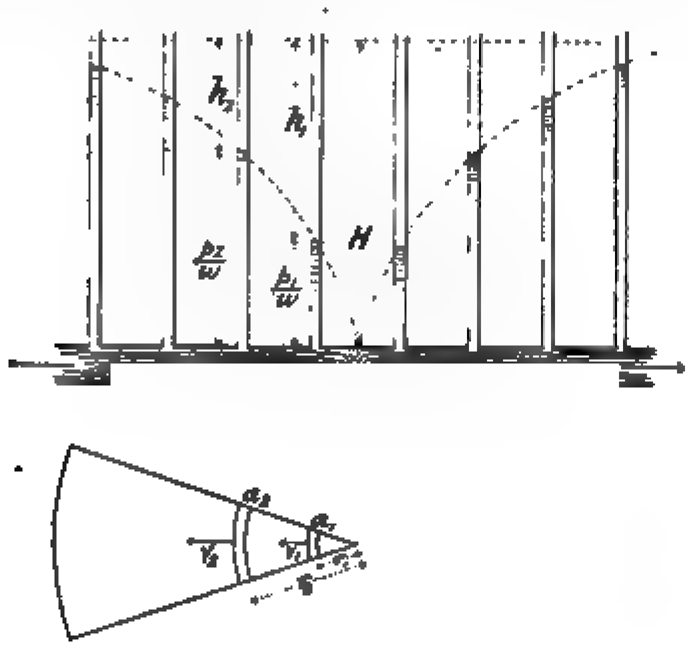


Fig. 538.

escape round the circumference of the plates. Take any small portion of the plates as shown; the strips represent portions of rings of water moving towards the outside. Let their areas be a_1, a_2; then, since the flow is constant, we have—

$$v_1a_1 = V_2a_2, \text{ or } \frac{v_2}{v_1} = \frac{a_1}{a_2} = \frac{r_1}{r_2}$$

$$\text{hence } v_2 = v_1\frac{r_1}{r_2}$$

or the velocity varies inversely as the radius. The plates being horizontal, the energy of position remains constant; therefore—

$$\frac{v_1^2}{2g} + \frac{p_1}{w} = \frac{v_2^2}{2g} + \frac{p_2}{w} = \text{H}$$

Substituting the value of v_2 found above, we have—

$$\frac{v_1^2}{2g} + \frac{p_1}{w} - \frac{v_1^2r_1^2}{2g \cdot r_2^2} = \frac{p_2}{w}$$

[1] See Professor Unwin's article, "Hydromechanics," in "Encyclopædia Britannica."

Then, substituting $\frac{v_1^2}{2g} + \frac{p_1}{w} = \text{H}$ from above, and putting $\left(\frac{v_1^2}{2g}\right)\frac{r_1^2}{r_2^2} = h_1\frac{r_1^2}{r_2^2} = h_2$, we have—

$$\text{H} - h_2 = \frac{p_2}{w}$$

Then, starting with a value for h_1, the h_2 for other positions is readily calculated and set down from the line above.

If a large number of radial segments were taken, they would form a complete cylinder of water, in which the water enters at the centre and escapes radially outwards. The distribution of pressure will be the same as in the radial segments, and the form of the water will be a solid of revolution formed by spinning the dotted line of pressures, known as Barlow's curve, round the axis.

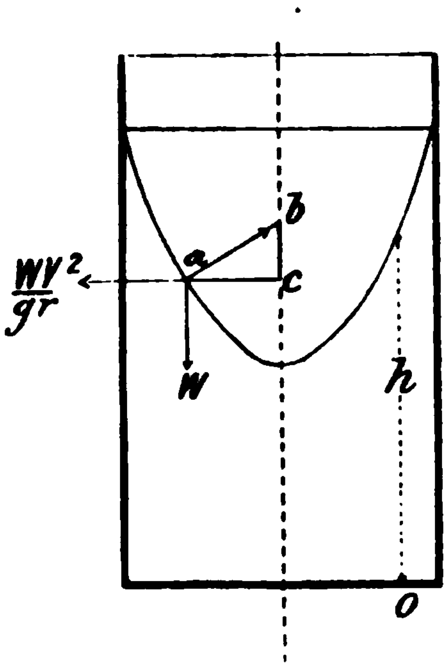


FIG. 539.

The case in which this kind of vortex is most commonly met with is when water flows in radially to a central hole, and then escapes.

Forced Vortex.—If water be forced to revolve in and with a revolving vessel, the form taken up by the surface is readily found thus :

Let the vessel be rotating n times per second.

Any particle of water is acted upon by the following forces—

(i.) The weight W acting vertically downwards.

(ii.) The centrifugal force $\frac{\text{WV}^2}{gr}$ acting horizontally, where V is its velocity in feet per second, and r its radius in feet.

(iii.) The fluid pressure, which is equal to the resultant of i. and ii.

From the figure, we have—

$$\frac{\frac{\text{WV}^2}{gr}}{\text{W}} = \frac{ac}{bc}$$

which may be written—

$$\frac{W 2^2 \pi^2 r^2 n^2}{W g r} = \frac{ac}{bc}$$

But $\frac{2^2\pi^2}{g}$ is constant, say C;

$$\text{Then } Crn^2 = \frac{ac}{bc}$$

$$\text{But } ac = r$$

$$\text{therefore } Cn^2 = \frac{1}{bc}$$

And for any given number of revolutions per second n^2 does not vary; therefore *bc*, the subnormal, is constant, and the curve is therefore a parabola. If an orifice were made in the bottom of the vessel at *o*, the discharge would be due to the head *h*.

Loss of Energy due to Abrupt Change of Direction.—If a stream of water flow down an inclined surface AB with a velocity V_1 feet per second, when it reaches B the direction of flow is suddenly changed from AB to BC, and the layers of water overtop one another, thus causing a breaking-up of the stream, and an eddying action which rapidly dissipates the energy of the stream by the frictional resistance of the particles of the water; this is sometimes termed the loss by shock. The velocity V_2 with which the water flows after passing the corner is given by the diagram of velocities ABD, from which we see that the component V_3, normal to BC, is wasted in eddying, and the energy wasted per pound of water is $\frac{V_3^2}{2g} = \frac{V_1^2 \sin^2\theta}{2g}$.

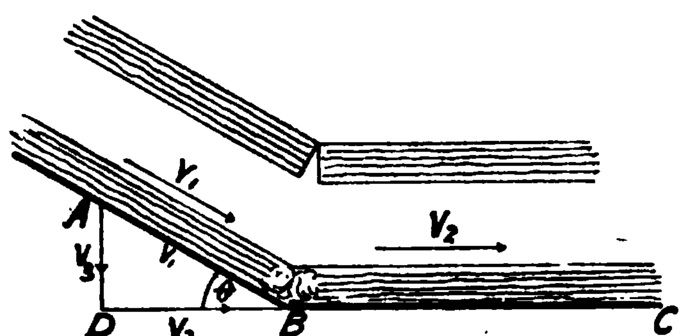


FIG. 540.

As the angle ABD increases the loss of energy increases, and when it becomes a right angle the whole of the energy is wasted by shock (Fig. 541).

If the surface be a smooth curve (Fig. 542) in which there is no abrupt change of direction, there will be no loss due to shock; hence the smooth easy curves that are adopted for the vanes of motors, etc.

If the surface against which the water strikes (normally) is

moving in the same direction as the jet with a velocity $\frac{V_1}{n}$, then the striking velocity will be—

$$V_1 - \frac{V_1}{n} = V_0$$

and the loss of energy per pound of water will be—

$$\frac{V_0}{2g} = \frac{\left(V_1 - \frac{V_1}{n}\right)^2}{2g} = \frac{V_1^2}{2g}\left(1 - \frac{1}{n}\right)^2$$

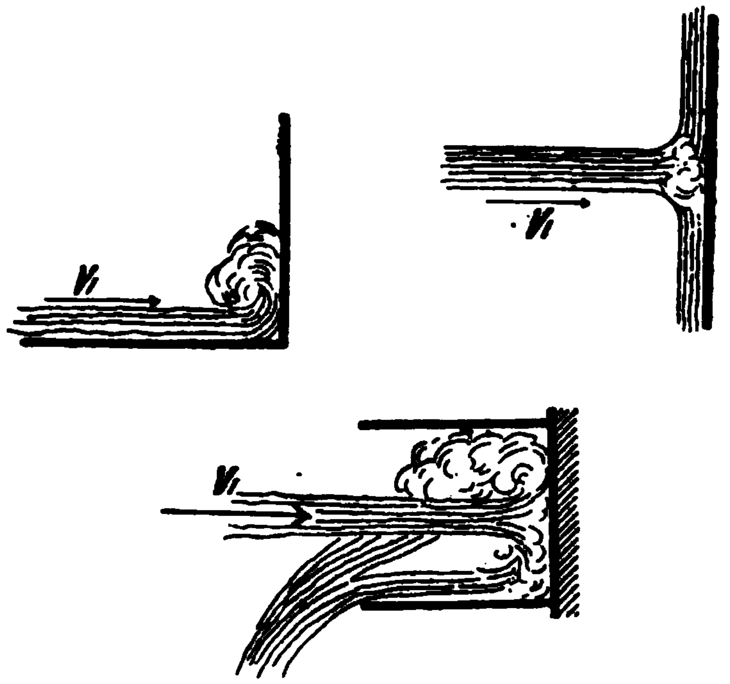


FIG. 541.

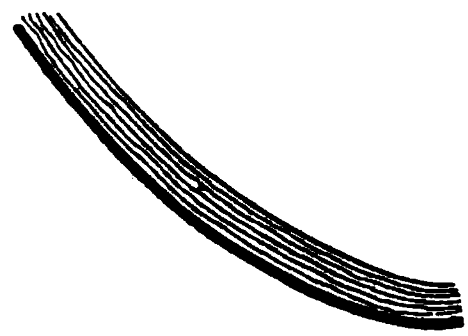


FIG. 542.

When $n = 1$, no striking takes place, and consequently no loss of energy; when $n = \infty$, *i.e.* when the surface is stationary, the loss is $\frac{V_1^2}{2g}$, *i.e.* the whole energy of the jet is dissipated.

Loss of Energy due to Abrupt Change of Section.—When water flows along a pipe in which there is an abrupt change of section, as shown, we may regard it as a jet of water moving with a velocity V_1 striking against a surface (in this case a body of water) moving in the same direction, but with a velocity $\frac{V_1}{n}$; hence the loss of energy per pound of water is precisely the same as in the last paragraph, viz. $\frac{\left(V_1 - \frac{V_1^2}{n}\right)}{2g}$. The

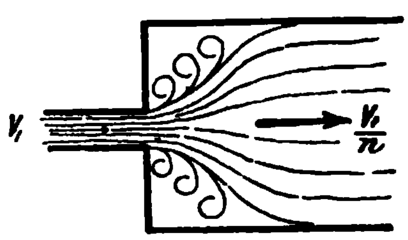


FIG. 543.

energy lost in this case is in eddying in the corners of the large section, as shown. As the water in the large section is moving $\frac{1}{n}$ as fast as in the small section, the area of the large section is n times the area of the small section. Then the loss of energy per pound of water, or the loss of head when a pipe suddenly enlarges n times, is—

$$\frac{V_1^2\left(1 - \frac{1}{n}\right)^2}{2g}$$

Or if we refer to the velocity in the large section as V_1, we have the velocity in the small section nV_1, and the loss of head—

$$\frac{(nV_1 - V_1)^2}{2g} = \frac{V_1^2}{2g}(n - 1)^2$$

When the water flows in the opposite direction, *i.e.* from the large to the small section, the loss of head is due to the abrupt change of velocity from the contracted to the full section of the small stream. The contracted section in pipes under pressure is, according to some experiments made in the author's laboratory, from 0·62 to 0·66; hence, n = from 1·61 to 1·5; then, the loss of head $= \frac{0{\cdot}3V_1^2}{2g}$.

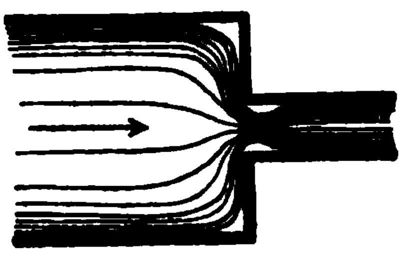


FIG. 544.

Let p_1 = the pressure in the small part, where the velocity is V_1;

p_2 = the pressure in the large part, where the velocity is $\frac{V_1}{n}$.

Then, from p. 476, we have—

$$\frac{p_1}{w} + \frac{V_1^2}{2g} = \frac{p_2}{w} + \frac{\left(\frac{V_1}{n}\right)^2}{2g}$$

when there is a gradual change of section; but when there is an abrupt change of section, this becomes—

$$\frac{p_1}{w} + \frac{V_1^2}{2g} = \frac{p_2}{w} + \frac{\left(\frac{V_1}{n}\right)^2}{2g} + \frac{0{\cdot}3V_1^2}{2g}$$

Then the difference in pressure in the two sections due to the abrupt change in section is, by simple reduction—

$$p_2 - p_1 = \frac{wV_1^2}{2g}\left(0\cdot87 - \frac{1}{n^2}\right)$$

We shall require to make use of this in one or two special cases.

Surface Friction.—When a body immersed in water is caused to move, or when water flows over a body, a certain resistance to motion is experienced; this resistance is termed the surface or fluid friction between the body and the water.

The experimental methods recently devised by Prof. Hele-Shaw enable the effect of a surface in contact with which water is flowing to be clearly seen, and also the manner in which obstacles affect the form of the stream-lines. See *Trans. Naval Architects*, 1897–8.

At very low velocities, only a thin film of the water actually in contact with the body appears to be affected, a mere skimming action; but as the velocity is increased, the moving body appears to carry more or less of the water with it, and to cause local eddying for some distance from the body. Experiments made by Professor Osborne Reynolds clearly demonstrate the difference between the two kinds of resistances—the surface resistance and the eddying resistance. Water is caused to flow through the glass pipe AB at a given velocity; a bent glass tube and funnel C is fixed in such a manner that a fine stream of deeply coloured dye is ejected. When the water flows through at a low velocity, the stream of dye runs right through like an unbroken thread; but as soon as the velocity is increased beyond a certain point, the thread breaks up and passes through in sinuous fashion, thus demonstrating that the water is not flowing through as a steady stream, as it did at the lower velocities.

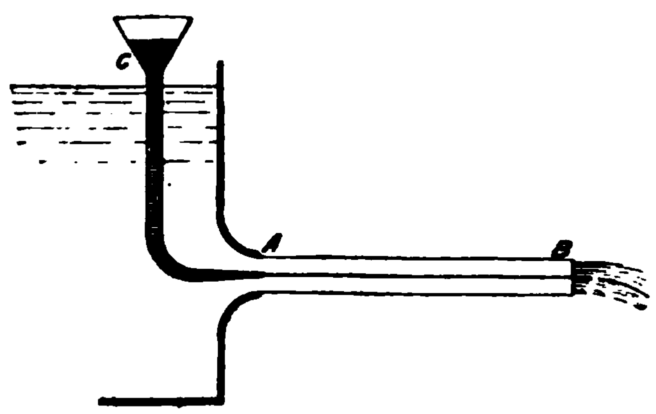


FIG. 545.

Professor Reynolds found that the change from steady to unsteady flow occurred when DV > 0·02 for a temperature of 60° Fahr., where D = diameter of pipe in feet, V = velocity of flow in feet per second.

For further details of this investigation, the reader is referred to the original papers in the *Philosophical Transactions* for

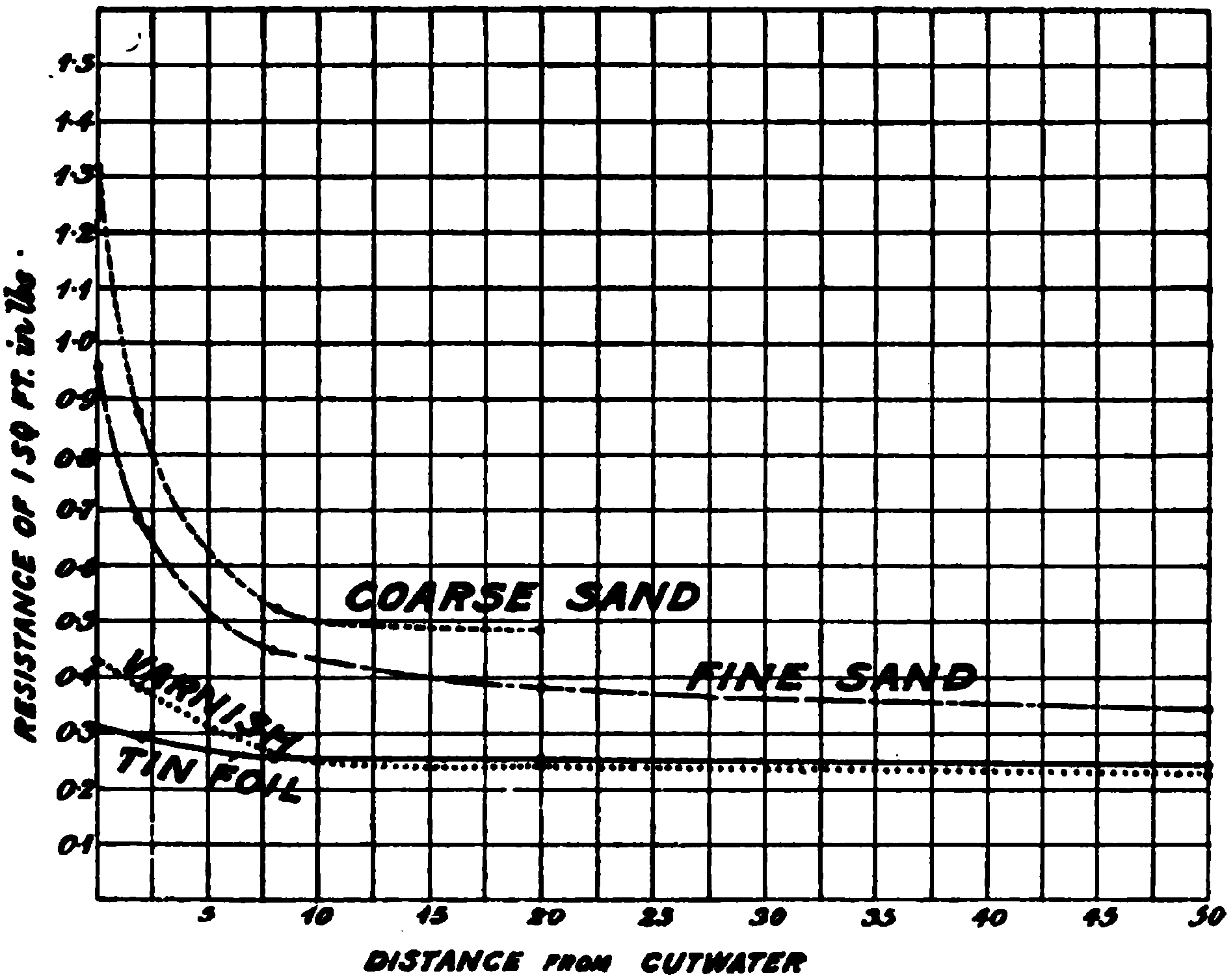


FIG. 546.

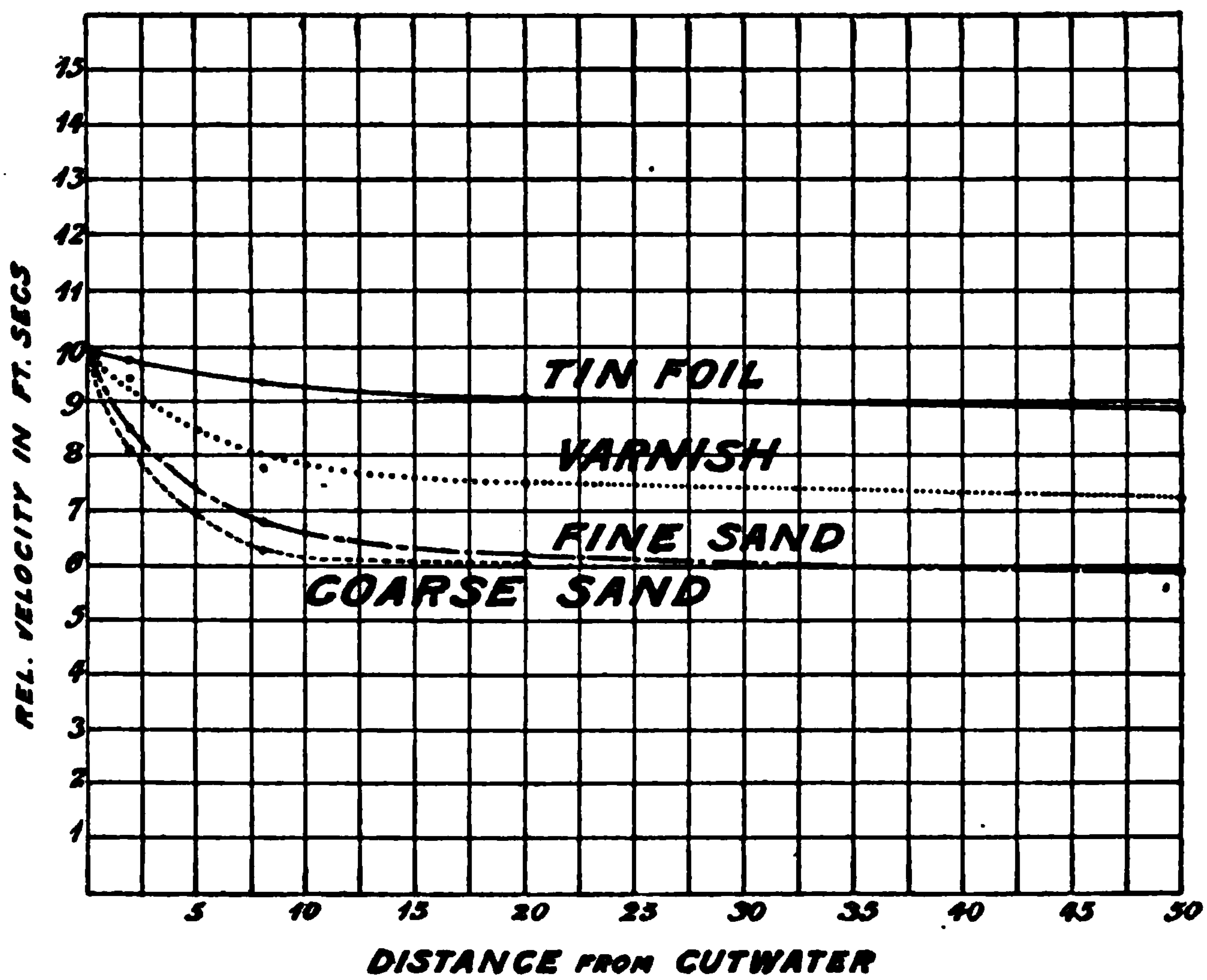


FIG. 547.

1883, p. 943; and for 1896, p. 167. Also to Turner and Brightmore's "Waterworks Engineering," p. 67.

Experiments by Mr. Froude at Torquay (see *Brit. Ass. Proceedings*, 1874) demonstrate very clearly that, when a thin long plank is towed edgewise through the water, the surrounding water at the rear end is carried along with the plank. This action is more marked in the case of planks covered with rough materials than with smooth materials.

The curves in the accompanying diagram illustrate this point very clearly. They were not given in this form by Mr. Froude; they have been deduced by the author from the published figures referred to above.

These curves very clearly show that the rougher the surface the greater is the tendency for the moving surface to carry the water along with it, which is quite what one would expect; they also show that the carrying along of the water is done almost entirely by the first 8 or 10 feet, and after that the relative velocities of the two remain practically constant.

Professor Unwin and others have also experimented on the friction of discs revolving in water, and have obtained results very closely in accord with those obtained by Mr. Froude.

The general results of experiments on the flow of water show that the following statements are approximately correct:—

(i.) The friction varies directly as the extent of the wetted surface.

(ii.) The friction varies directly as the roughness of the surface.

(iii.) The friction varies directly as the square of the velocity.

(iv.) The friction is independent of the pressure.

For fluids other than water, we should have to add—

(v.) The friction varies directly as the density and viscosity of the fluid.

Hence, if S = the wetted surface in square feet;

f = a coefficient depending on the roughness of the surface;

V = velocity of flow relatively to the surface in feet per second;

R = frictional resistance;

Then, neglecting any variation of the friction with the temperature, we have—

$$R = SfV^2$$

Reducing this to a form suitable for application to pipes, we have, for any length of pipe L feet, the pressure P_1 in pounds per square foot at one end greater than the pressure P_2 at the other end, on account of the friction of the water. Then, if A be the area of the pipe in square feet, we have—

$$R = (P_1 - P_2)A$$

Then, putting $P_1 = h_1 W_w$ and $P_2 = h_2 W_w$, we have—

$$R = W_w A(h_1 - h_2) = W_w A h$$

where h is the loss of head due to friction on any length of pipe L; then—

$$W_w A h = S f V^2$$

$$\text{or } \frac{W_w \pi D^2 h}{4} = L \pi D f V^2$$

$$\text{hence } h = \frac{4f}{W_w} \cdot \frac{LV^2}{D}$$

The coefficient f has to be obtained by experiment, according to Anderson and Hawkesley.

A mean value is about 0·0065; this gives—

$$\frac{4f}{W_w} = \frac{1}{2400}$$

The above formula then becomes—

$$h = \frac{LV^2}{2400D}$$

The value of f, however, varies very largely for different surfaces, and the resistance does not always vary as the square of the velocity, nor simply inversely as D; hence this formula must only be taken as a rough approximation.

The energy of motion of 1 lb. of water moving with a velocity V feet per second is $\frac{V^2}{2g}$; hence the whole energy of motion of the water is dissipated in friction when—

$$\frac{V^2}{2g} = \frac{LV^2}{2400D}$$

Putting in the numerical value for g, we get $L = 37D$.

This value 37, of course, depends on the roughness of the pipe. We shall find this method of regarding frictional resistances exceedingly convenient when dealing with the resistances of T's, elbows, etc., in pipes.

Still adhering to the rough formula given above, we can calculate the discharge of any pipe thus:

$$\left.\begin{array}{c}\text{The quantity discharged in}\\ \text{cubic feet per second}\end{array}\right\} = Q = AV = \frac{\pi D^2 V}{4}$$

From the formula given above, we have—

$$V = \sqrt{\frac{2400Dh}{L}}$$

Substituting this value, we have—

$$Q = 38{\cdot}5D^{\frac{5}{2}}\sqrt{\frac{h}{L}}$$

Or, more conveniently—

$$Q = 38{\cdot}5D^2\sqrt{\frac{Dh}{L}}$$

Thrupp's Formula for the Flow of Water.—All formulas for the flow of water are, or should be, constructed to fit experiments, and that which fits the widest range of experiments is of course the most reliable. Several investigators in recent years have collected together the results of published experiments, and have adjusted the older formulas or have constructed new ones to better accord with experiments. There is very little to choose between the best of recent formulas, but on the whole the author believes *that* this formula best fits the widest range of experiments; others are equally as good for smaller ranges. It is a modification of Hagen's formula, and was published in a paper read before the Society of Engineers in 1887.

Let V = velocity of flow in feet per second;
R = hydraulic mean radius in feet, *i.e.* the area of the stream divided by the wetted perimeter, and is $\frac{D}{4}$ for circular and square pipes:
L = length of pipe in feet;
h = loss of head due to friction in feet;

S = cosecant of angle of slope $= \frac{L}{h}$;

Q = quantity of water flowing in cubic feet per second.

$$\text{Then } V = \frac{R^x}{C\sqrt[n]{S}}$$

where x, C, n are coefficients depending on the nature of the surface of the pipe or channel.

For small values of R, more accurate results will be obtained by substituting for the index x the value $x + y\sqrt{\frac{z - R}{R}}$.

In this formula the effect of a change of temperature is not taken into account; that of Professor Osborne Reynolds is the only one that professes to do so. The friction varies, roughly, inversely as the absolute temperature of the water.

Surface.	n.	C.	x.	y.	z.
Wrought-iron pipes ...	1·80	0·004787	0·65	0·018	0·07
Riveted sheet-iron pipes ...	1·825	0·005674	0·677	—	—
New cast-iron pipes ...	1·85	0·005347	0·67	—	—
	2·00	0·006752	0·63	—	—
Lead pipes	1·75	0·005224	0·62	—	—
Pure cement rendering ...	1·74	0·004000	0·67	—	—
	1·95	0·006429	0·61	—	—
Brickwork (smooth) ...	2·00	0·007746	0·61	—	—
,, (rough) ...	2·00	0·008845	0·625	0·01224	0·50
Unplaned plank	2·00	0·008451	0·615	0·03349	0·50
Small gravel in cement ...	2·00	0·01181	0·66	0·03938	0·60
Large ,, ,, ...	2·00	0·01415	0·705	0·07590	1·00
Hammer-dressed masonry ...	2·00	0·01117	0·66	0·07825	1·00
Earth (no vegetation) ...	2·00	0 01536	0·72	—	—
Rough stony earth	2·00	0·02144	0·78	—	—

If we take x as 0·62, and $n = 2$, $C = 0·0067$, we get—

$$Q = \frac{D^{2·62}}{9·44C\sqrt[n]{S}}$$

which reduces to—

$$Q = 17·6D^{2·62}\sqrt{\frac{h}{L}}$$

This expression should be compared with the approximate one on p. 488.

Resistance of Knees, Bends, etc.—We have already shown that if the direction of a stream of water be abruptly changed through a right angle, the whole of its energy of motion is destroyed, a similar action occurs in a right-angled knee or elbow in a pipe, hence its resistance is at least equivalent to the friction in a length of pipe about 37 diameters long. In addition to this loss, the water overshoots the corner, as shown in Fig. 548, and causes a sudden contraction and enlargement of section with a further loss of head. The losses in sockets, sudden enlargements, etc., can be readily calculated; others have been obtained by experiment, and their values are given in the following table. When calculating the friction of systems of piping, the equivalent lengths as given should be added, and the friction calculated as though it were a length of straight pipe.

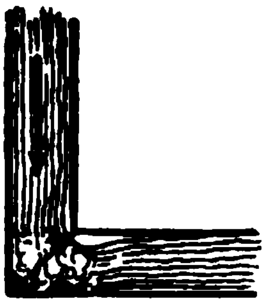


FIG. 548.

Nature of resistance.	Equivalent length of straight pipe expressed in diameters, on the basis of L = 36D
Right-angled knee or elbow (experiments)	30–40 in plain pipe 50–60 with screwed elbow
Right-angled bends, exclusive of resistance of sockets at ends, radius of bend, = 4 diameters	3–5
Ditto including sockets (experiments) ...	22–30
Sockets (screwed) calculated from the sudden enlargement and contraction (average sizes)	24
Ditto by experiment	20–28
Sudden enlargement to a square-ended pipe, where $n = \frac{\text{large area}}{\text{small area}}$	$36\left(1 - \frac{1}{n}\right)^2$
Sudden contraction	12 approx.
Mushroom valves (one set of experiments)	120–150
Plug cock, handle 15° } Unwin	27
,, turned 30° } Unwin	200
,, through 45° } Unwin	1100
Sluice and slide valves $n = \frac{\text{port area}}{\text{area of opening}}$	$100(n - 1)^2$

Velocity of Water in Pipes.—Water is allowed to flow

at about the velocities given below for the various purposes named :—

Pressure pipes for hydraulic purposes for long mains	3 to 4 feet per sec.
Ditto for short lengths[1]	Up to 25 ,,
Ditto through valve passages[1]	Up to 50 ,,
Pumping mains	3 to 5 ,,
Waterworks mains	2 to 3 ,,

[1] Such velocities are unfortunately common, but they should be avoided if possible.

CHAPTER XVII.

HYDRAULIC MOTORS AND MACHINES.

THE work done by raising water from a given datum to a receiver at a higher level is recoverable by utilizing it in one of three distinct types of motor.

1. Gravity machines, in which the weight of the water is utilized.

2. Pressure machines, in which the pressure of the water is utilized.

3. Velocity machines, in which the velocity of the water is utilized.

Gravity Machines.—In this type of machine the weight-energy of the water is utilized by causing the water to flow into the receivers of the machine at the higher level, then to descend with the receivers in either a straight or curved path to the lower level at which it is discharged. If W lbs. of water have descended through a height H feet, the work done = WH foot-lbs. Only a part, however, of this will be utilized by the motor, for reasons which we will now consider.

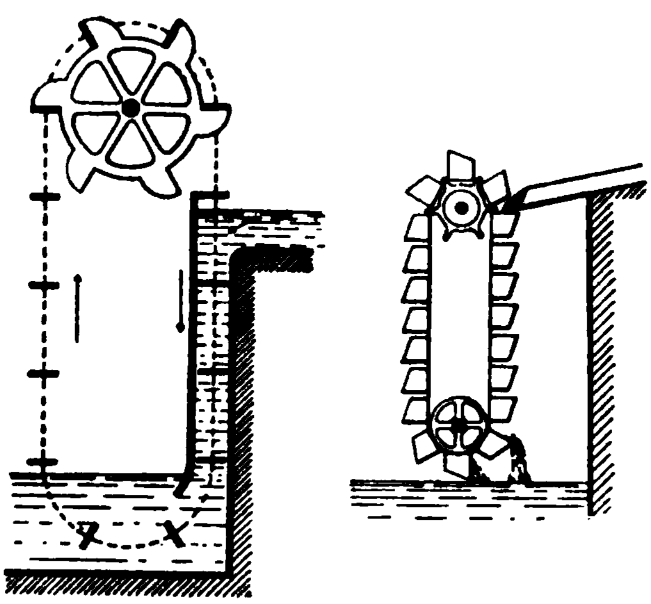


FIG. 549.

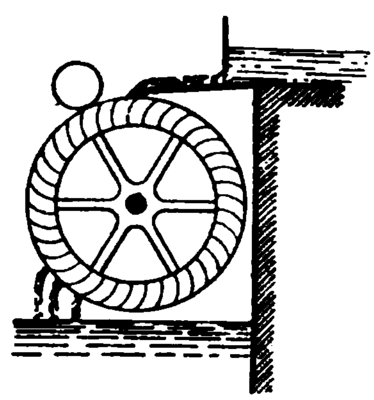


FIG. 550.

The illustrations, Figs. 549, 550, show various methods of

utilizing the weight-energy of water. Those shown in Fig. 549 are very rarely used, but they serve well to illustrate the principle involved. The ordinary overshot wheel shown in Fig. 550 will perhaps be the most instructive example to investigate as regards efficiency.

Although we have termed all of these machines *gravity* machines, they are not purely such, for they all derive a small portion of their power from the water striking the buckets on entry. Later on we shall show that, for motors which utilize the velocity of the water, the maximum efficiency occurs when the velocity of the jet is twice the velocity of the buckets or vanes.

In the case of an overshot water-wheel, it is necessary to keep down the linear velocity of the buckets, otherwise the centrifugal force acting on the water will cause much of it to be wasted by spilling over the buckets. If we decide that the inclination of the surface of the water in the buckets to the horizontal shall not exceed 1 in 8, we get the peripheral velocity of the wheel $V_w = 2\sqrt{R}$, where R is the radius of the wheel in feet.

Take, for example, a wheel required for a fall of 15 feet. The diameter of the wheel may be taken as a first approximation as 12 feet. Then the velocity of the rim should not exceed $2\sqrt{6}$ = say 5 feet per second. Then the velocity of the water issuing from the sluice should be 10 feet per second; the head h required to produce this velocity will be $h = \frac{V^2}{2g}$, or, introducing a coefficient to allow for the friction in the sluice, we may write it $h = \frac{1\cdot 1 V^2}{2g} = 1\cdot 6$ foot. One-half of this head, we shall show later, is lost by shock. The depth of the shroud is usually from 0·75 to 1 foot; the distance from the middle of the stream to the c. of g. of the water in the bucket may be taken at about 1 foot, which is also a source of loss.

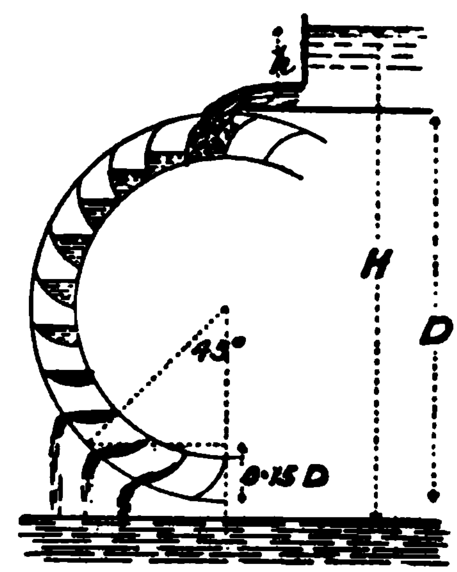


FIG. 551.

The next source of waste is due to the water leaving the wheel before it reaches the bottom. The exact position at which it leaves varies with the form of buckets adopted, but for our present purpose it may be taken that the mean discharge occurs at an angle of 45° as

shown. Then by measurement from the diagram, or by a simple calculation, we see that this loss is $0 \cdot 15D$. A clearance of about $0 \cdot 5$ foot is usually allowed between the wheel and the tail water. We can now find the diameter of the wheel, remembering that $H = 15$ feet, and taking the height from the surface of the water to the wheel as 2 feet. This together with the $0 \cdot 5$ foot clearance at the bottom gives us $D = 12 \cdot 5$ feet.

Thus the losses with this wheel are—

Half the sluice head	=	0·8 foot
Drop from centre of stream to buckets	=	1·0 „
Water leaving wheel too early, 0·15 × 12·5 feet	=	1·9 „
Clearance at bottom	=	0·5 „
		4·2 feet

$$\text{Hydraulic efficiency of wheel} = \frac{15 - 4 \cdot 2}{15} = 72 \text{ per cent.}$$

The mechanical efficiency of the axle and one toothed wheel will be about 90 per cent., thus giving a total efficiency of the wheel of 65 per cent.

With greater falls this efficiency can be raised to 80 per cent.

The above calculations do not profess to be a complete treatment of the overshot wheel, but they fairly indicate the sort of losses such wheels are liable to.

Pressure Machines.—In these machines the water at the

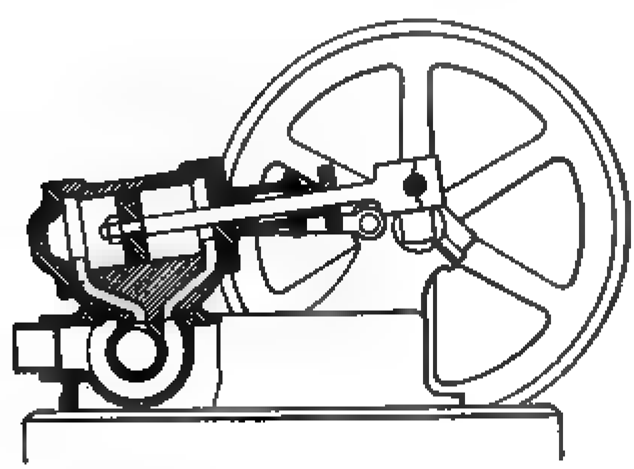


Fig. 552.

higher level descends by a pipe to the lower level, it passes to a closed vessel or a cylinder, and acts

piston in precisely the same manner as in a steam-engine. The work done is the same as before, viz. WH foot-lbs. for

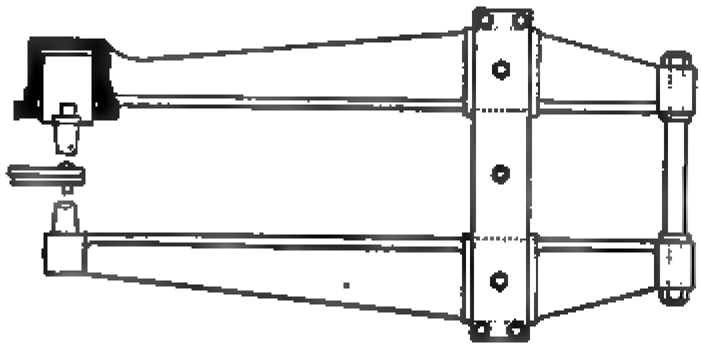


Fig. 553.

the pressure at the lower level is W_wH lbs. square foot; and the weight of water used per square foot of piston $= W_w L = W$,

(*a*) (*b*)

Fig. 554.

where L is the distance moved through by the piston in feet. Then the work done by the pressure water $= W_w LH = WH$ foot-lbs. Several examples of pressure machines are shown in Figs. 552, 553, 554, *a* and *b*. Fig. 552 is an oscillating cylinder pressure motor used largely on the continent. Fig. 553 is an ordinary hydraulic pressure riveter. Fig. 554, *a*, is a passenger lift, with a wire-rope multiplying arrangement. Fig. 554, *b*, is an ordinary ram lift. For details the reader is referred

to special books on hydraulic machines, such as Blaine[1] a
Robinson.[2]

The chief sources of loss in efficiency in these motors are—

1. Friction of the water in the mains and passages.
2. Losses by shock through abrupt changes in velocity of water.
3. Friction of mechanism.
4. Waste of water due to the same quantity being used when running under light loads as when running with the full load.

The friction and shock losses may be reduced to a minimum by careful attention to the design of the ports and passages; re-entrant angles, abrupt changes of section of ports and passages, high velocities of flow, and other sources of loss given in the chapter on hydraulics should be carefully avoided.

By far the most serious loss in most motors of this type is that mentioned in No. 4 above. Many very ingenious devices have been tried with the object of overcoming this loss.

Amongst the most promising of those tried are devices for automatically regulating the length of the stroke in proportion to the resistance overcome by the motor. Perhaps the best known of these devices is that of the Hastie engine, a full description of which will be found in Professor Unwin's article on Hydromechanics in the "Encyclopædia Britannica."

In an experiment on this engine, the following results were obtained :—

Weight in pounds lifted 22 feet	chain only	427	633	745	857	969	1081	1193
Water used in gallons at 80 lbs. per square inch	7·5	10	14	16	17	20	21	22
Efficiency per cent. (actual)	—	51	54	56	60	58	61	65
Efficiency per cent. if stroke were of fixed length	—	23	34	40	46	53	59	65

The efficiency in lines 3 and 4 has been deduced from the other figures by the author, on the assumption that the motor was working full stroke at the highest load given.

The great increase in the efficiency at low loads due to the compensating gear is very clear.

Cranes and elevators are often fitted with two cylinders of different sizes, or one cylinder and a differential piston. When lightly loaded, the smaller cylinder is used, and the larger one

[1] "Hydraulic Machinery" (Spon).
[2] "Hydraulic Power and Machinery" (Griffin).

only for full loads. The valves for changing over the conditions are usually worked by hand, but it is very often found that the man in charge does not take advantage of the smaller cylinder. In order to place it beyond his control, the extremely ingenious device shown in Fig. 555 is sometimes used.

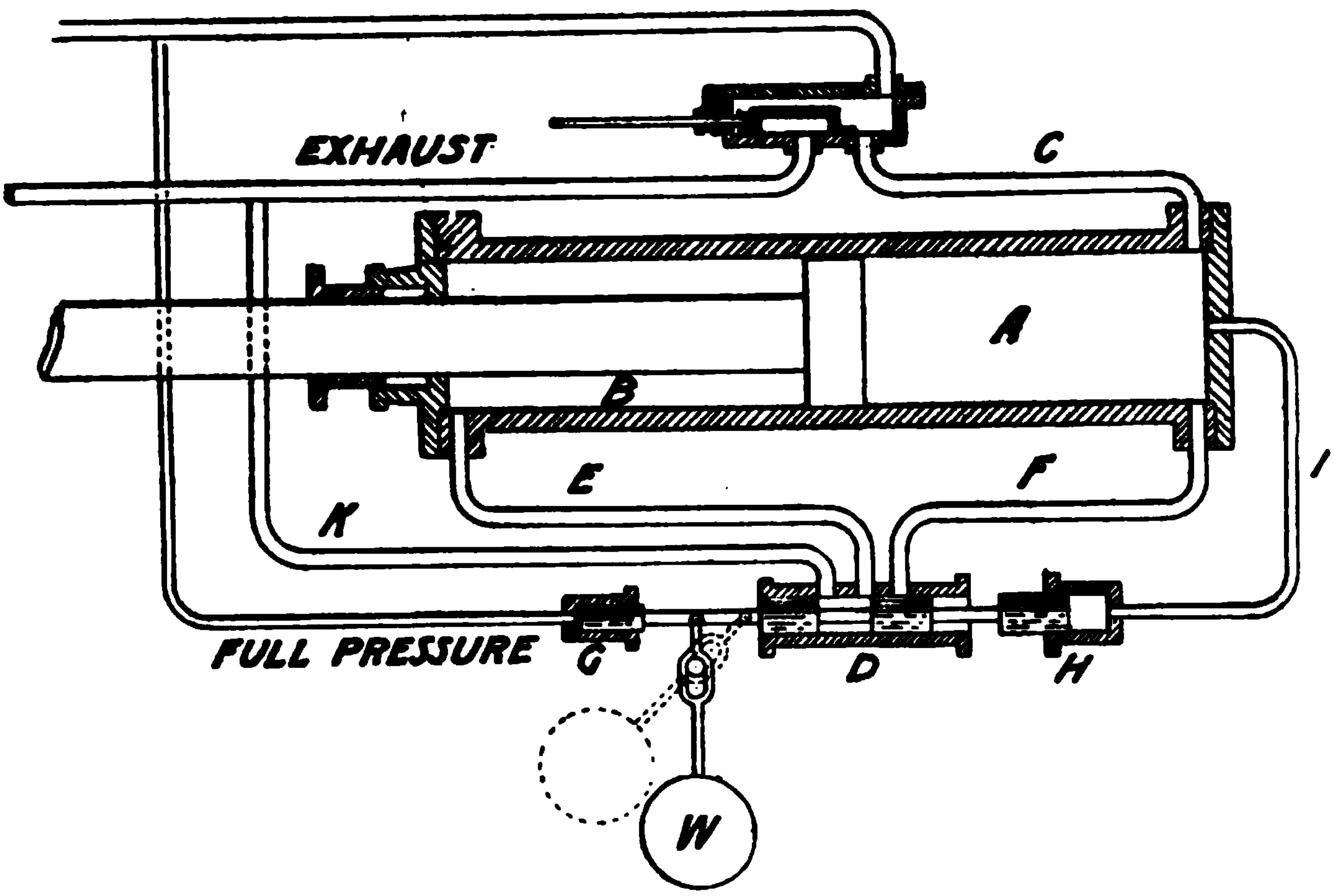


FIG. 555.

The author is indebted to Mr. R. H. Thorp, of New York, the inventor, for the drawings and particulars from which the following account is taken. The working cylinder is shown at AB. When working at full power, the valve D is in the position shown in full lines, which allows the water from B to escape freely by means of the exhaust pipes E and K; then the quantity of water used is given by the volume A. But when working at half-power, the valve D is in the position shown in dotted lines; the water in B then returns *viâ* the pipe E, the valve D, and the pipe F to the A side of the piston. Under such conditions it will be seen that the quantity of high-pressure water used is the volume A *minus* the volume B, which is usually one-half of the former quantity. The position of the valve D, which determines the conditions of full or half power is generally controlled by hand. The action of the automatic device shown depends upon the fact that the pressure of the water in the cylinder is proportional to the load lifted, for if the pressure were in excess of that required to steadily raise a light

load, the piston would be accelerated, and the pressure would be reduced, due to the high velocity in the ports. In general, the man in charge of the crane throttles the water at the inlet valve in order to prevent any such acceleration. In Mr. Thorp's arrangement, the valve D is worked automatically.

In the position shown, the crane is working at full power: but if the crane be only lightly loaded, the piston will be accelerated and the pressure of the water will be reduced by friction in passing through the pipe C, until the total pressure on the plunger H will be less than the total full water-pressure on the plunger G, with the result that the valve D will be forced over to the right, thus establishing communication between B and A, through the pipes E and F, and thereby putting the crane at half-power. As soon as the pressure is raised in A, the valve D returns to its full-power position, due to the area of H being greater than that of G, and to the pendulum weight W.

It very rarely happens that a natural supply of high-pressure water can be obtained, consequently a power-driven pump has to be resorted to as a means of raising the water to a sufficiently high pressure. In certain simple operations the water may be used direct from the pump, but nearly always some method of storing the power is necessary. If a tank could be conveniently placed at a sufficient height, the pump might be arranged to deliver into it, from whence the hydraulic installation would draw its supply of high-pressure water. In the absence of such a convenience, which, however, is seldom met with, a hydraulic accumulator (Fig. 556) is used. It consists essentially of a vertical cylinder, provided with a long-stroke plunger, which is weighted to give the required pressure, usually from 700 to 1000 lbs. per square inch. With such a means of storing energy, a very large amount of power—far in excess

FIG. 556.

of that of the pump—may be obtained for short periods. In fact, this is one of the greatest points in favour of hydraulic methods of transmitting power. The levers shown at the side are for the purpose of automatically stopping and starting the pumps when the accumulator weights get to the top or bottom of the stroke.

Energy stored in an Accumulator.—

If s = the stroke of the accumulator in feet;
d = the diameter of the ram in inches;
p = the pressure in pounds per square inch.

Then the work stored in foot-lbs. $= 0{\cdot}785d^2ps$

Work stored per cubic foot of water in foot-lbs. $= 144p$

Work stored per gallon of water $= \dfrac{144p}{6{\cdot}25} = 23{\cdot}04p$

Number of gallons required per minute at the pressure p per horse-power $= \dfrac{33{,}000}{23{\cdot}04p} = \dfrac{1432}{p}$

Number of cubic feet required per minute at the pressure p $= \dfrac{33{,}000}{144p} = \dfrac{229{\cdot}2}{p}$

Effects of Inertia of Water in Pressure Systems.—In nearly all pressure motors and machines, the inertia of the water seriously modifies the pressures actually obtained in the cylinders and mains. For this reason such machines have to be run at comparatively low piston speeds, seldom exceeding 100 feet per minute. In the case of free piston machines, such as hydraulic riveters, the pressure on the rivet due to this cause is frequently twice as great as would be given by the steady accumulator pressure.

In the case of a water-pressure motor, the water in the mains moves along with the piston, and may be regarded as a part of the reciprocating parts. The pressure set up in the pipes, due to bringing it to rest, may be arrived at in the same manner as the "Inertia pressure," discussed in Chapter VI.

Let w = weight of a column of water 1 square inch in section, whose length L in feet is that of the main along which the water is flowing to the motor $= 0{\cdot}434L$;

m = the ratio $\dfrac{\text{area of plunger or piston}}{\text{area of section of water main}}$

p = the pressure in pounds per square inch set up in the pipe, due to bringing the water to rest at the end of the stroke (with no air-vessel);

N = the number of revolutions per minute of the motor;

R = the radius of the crank in feet.

Then, remembering that the pressure varies directly as the velocity of the moving masses, we have, from pp. 145, 147—

$$p = 0{\cdot}00034m(0{\cdot}434\text{L})\text{RN}^2\left(1 \pm \frac{1}{n}\right)$$

$$p = 0{\cdot}00015m\text{LRN}^2\left(1 \pm \frac{1}{n}\right)$$

Relief valves are frequently placed on long lines of piping, in order to relieve any dangerous pressure that may be set up by this cause.

Pressure due to Shock.—If water flow along a long pipe with a velocity V feet per second, and a valve at the outlet end be suddenly closed, the kinetic energy of the water will be expended in compressing the water and in stretching the walls of the pipe. The latter is so small a quantity as compared with the former that we shall neglect it. If the water were a material of an unyielding character, the whole of it would be instantly brought to rest, and the pressure set up would be infinitely great. Water, however, yields considerably under pressure. Hence, even after the valve is closed, water continues to enter at the inlet end with undiminished velocity for a period of t seconds, until the whole of the water in the pipe is compressed, thus producing a momentary pressure greater than the static pressure of the water. The compressed water than expands, and a return-wave is set up, causing the water-pressure to fall below the static pressure.

Let K = the modulus of elasticity of bulk of water = 300,000 lbs. per square inch (see p. 271);

x = the amount the column of water is shortened, due to the compressive stress or pressure due to shock, in feet;

f = the compressive stress or pressure in pounds per square inch due to shock;

w = the weight of a unit column of water, *i.e.* 1 sq. inch section, 1 foot long, = 0·434 lb.;

L = the length of the column of flowing water in feet.

Immediately after the valve is suddenly closed, the water continues to enter the pipe with its former velocity V, but, as the water cannot escape, it is compressed first at the valve, and a compression wave is started, which runs back along the pipe until the whole column of water is compressed. The section *ab* takes up the new position *a′b′*, or the column has been shortened by the amount x_1, where—

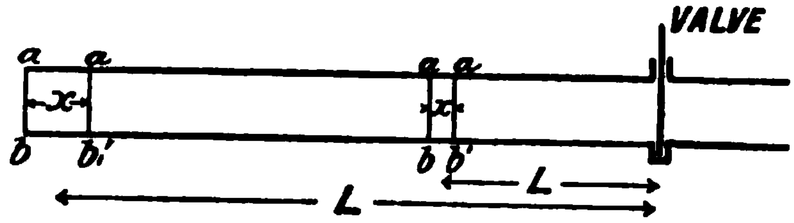


FIG. 557.

$$x_1 = \frac{fL_1}{K}, \text{ and } x = \frac{fL}{K}$$

$$\text{or } f = K\frac{x_1}{L_1}, \text{ or } K\frac{x}{L}$$

But x is proportional to L, hence f is constant as the wave travels along, and is independent of the length of the column or the length of the pipe.

The quantity of water that entered the pipe during the compression process is the volume between the two sections *ab* and *a′b′*; but, as the water continued to enter for a period of t seconds, the velocity with which the column moves along is—

$$V = \frac{x}{t}$$

and the mass of the moving column per square inch of section—

$$M = \frac{Lw}{g}$$

The impulse per square inch of section during the time t, or the pressure due to this mass of water being brought to rest, is equal to the change of momentum during the time t; or—

$$ft = MV = \frac{Lw}{g} \times \frac{x}{t}$$

Substituting the value of x, we have—

$$f = \frac{Lwx}{gt^2} = \frac{L^2wf}{Kgt^2}$$

from whence we have—

$$\frac{L}{t} = \sqrt{\frac{gK}{w}}, \text{ and } L = t\sqrt{\frac{gK}{w}}$$

The quantity $\frac{L}{t}$ is the velocity with which the compressio wave traverses the pipe, or the velocity of pulsation. Insertin numerical values for the symbols under the root, we get th velocity of pulsation 4720 feet per second, *i.e.* the velocity o sound in water. When $t = 1$, L = the distance the com pression wave travels per second.

Returning to the first value of f, we have—

$$f = \frac{Kx}{L} = \frac{Kx}{t\sqrt{\frac{gK}{w}}} = \frac{KV}{4720}$$

inserting the value of K—

$$f = 63{\cdot}5V$$

When the valve is closed uniformly in a given time, the manner in which the pressure varies at each instant can be readily obtained by constructing (i.) a velocity time curve; (ii.) a retardation or pressure curve, as explained on p. 133. Also see Appendix.

Maximum Power transmitted by a Water-Main.—We showed on p. 488 that the quantity of water that can be passed through a pipe with a given loss of head is—

$$Q = 38{\cdot}5D^{\frac{5}{2}}\sqrt{\frac{h}{L}}$$

Each cubic foot of water falling per second through a height of one foot gives—

$$\frac{62{\cdot}5 \times 60}{33{,}000} = 0{\cdot}1135 \text{ horse-power}$$

$$\text{hence H.P.} = 0{\cdot}1135QH$$

where H is the fall in feet, and Q the quantity of water in cubic feet per second.

Then, if h be the loss of head due to friction, the horse-power delivered at the far end of the main L feet away is—

$$\overline{\text{H.P.}} = 0{\cdot}1135Q(H - h)$$

Substituting the value of Q from above, we have—

$$\overline{\text{H.P.}} = 0{\cdot}1135 \times 38{\cdot}5D^{\frac{5}{2}}\sqrt{\frac{h}{L}}(H - h)$$

Let $h = n$H. Then, by substitution and reduction, we get the power delivered at the far end—

$$\overline{\text{H.P.}} = 4{\cdot}37\sqrt{\frac{nH^3D^5}{L}}(1 - n)$$

$$\text{or } L = \frac{19{\cdot}1nH^3D^5(1 - n)}{\overline{\text{H.P.}}^2}$$

These equations give us the horse-power that can be transmitted with any given fraction of the head lost in friction; also the permissible length of main for any given loss when transmitting a certain amount of power.

The power that can be transmitted through a pipe depends on (i.) the quantity of water that can be passed; (ii.) the effective head, *i.e.* the total head *minus* the friction head.

$$\text{Power transmitted } P = Q(H - h) \times \text{a constant}$$

$$\text{or } P = AV\left(H - \frac{LV^2}{2400D}\right) \times \text{a constant}$$

But all the quantities in this expression are constant in any given case except V, hence—

$$P = V(1 - V^2) \times \text{a constant}$$

$$\text{or } P = (V - V^3) \times \text{a constant}$$

Then, omitting the constant, which does not affect us, we have—

$$\frac{dP}{dV} = 1 - 3V^2 = 0$$

when the power is a maximum,

$$\text{or } V^2 = \tfrac{1}{3}$$

The loss of head is proportional to V^2, and the power to V,

during the interval, this increase of energy must be equal to the work expended in shoving the body, or—

$$\left.\begin{array}{l}\text{The work done in}\\ \text{shoving the body}\end{array}\right\} = \left\{\begin{array}{c}\text{impulse or}\\ \text{shove}\end{array}\right\} \times \left\{\begin{array}{c}\text{distance through which}\\ \text{it is exerted}\end{array}\right.$$

$$= \text{increase in kinetic energy}$$

$$\left.\begin{array}{l}\text{The kinetic energy}\\ \text{before the shove}\end{array}\right\} = \frac{MV^2}{2}$$

$$\left.\begin{array}{l}\text{The kinetic energy}\\ \text{after the shove}\end{array}\right\} = \frac{MV_1^2}{2}$$

$$\left.\begin{array}{l}\text{Increase in kinetic}\\ \text{energy}\end{array}\right\} = \frac{M}{2}(V_1^2 - V^2)$$

The distance through which the impulse is exerted is—

$$\frac{V_1 + V}{2} \cdot t$$

$$\text{hence } \frac{M}{2}(V_1^2 - V^2) = Pt\frac{V_1 + V}{2}$$

$$\text{or } Pt = M(V_1 - V)$$

or impulse in time t = change of momentum in time t

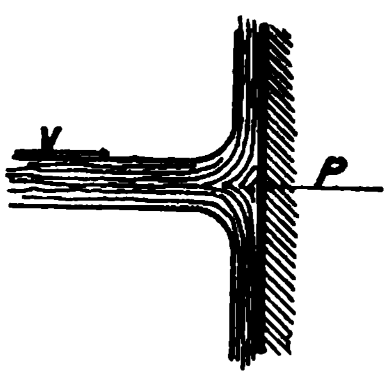


FIG. 559.

Let a jet of water moving with a velocity V feet per second impinge on a plate, as shown. After impinging, its velocity in its original direction is zero, hence its change of velocity on striking is V, and therefore—

$$Pt = MV$$

$$\text{or } P = \frac{M}{t}V$$

But $\frac{M}{t}$ is the mass of water delivered per second.

Let W = weight of water *delivered per second.*

$$\text{Then } \frac{W}{g} = \frac{M}{t}$$

$$\text{and } P = \frac{WV}{g}$$

It should be noticed that the pressure due to an impinging jet is just twice as great as the pressure due to the head of

water corresponding to the same velocity. This can be shown thus:

$$h = \frac{V^2}{2g}$$

$$p = wh = \frac{wV^2}{2g}$$

where w = the weight of a unit column of water.

We have $W = wV$. Substituting this value of wV—

$$p = \frac{WV}{2g}$$

The impinging jet corresponds to a dynamic load, and a column of water to a steady load (see Chapter XV.).

In this connection it is interesting to note that, in the case of a sea-wave, the pressure due to a wave of oscillation is approximately equal to that of a head of water of the same height as the wave, and, in the case of a wave of translation, to twice that amount.

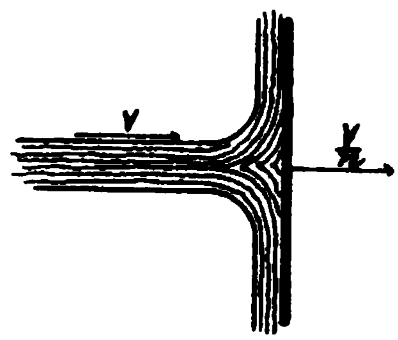


FIG. 560.

Pressure on a Moving Surface due to an Impinging Jet.—Let the plate shown in the Fig. 560 be one of a series on which the jet impinges at very short intervals. The reason for making this stipulation will be seen shortly.

Let the weight of water delivered per second be W lbs. as before; then, if the plates succeed one another very rapidly as in many types of water-wheels, the quantity impinging on the plates will also be sensibly equal to W. The impinging velocity is $V - \frac{V}{n}$, or $V\left(1 - \frac{1}{n}\right)$; hence the pressure in pounds' weight on the plates is—

$$P = \frac{WV\left(1 - \frac{1}{n}\right)}{g}$$

And the work done per second on the plates in foot-lbs.—

$$P\frac{V}{n} = \frac{WV^2\left(1 - \frac{1}{n}\right)}{ng} \quad \ldots \ldots \quad \text{(i.)}$$

and the energy of the jet is—

$$\frac{WV^2}{2g} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(ii.)}$$

$$\text{hence the efficiency of the jet} = \frac{\text{i.}}{\text{ii.}} = \frac{2}{n}\left(1 - \frac{1}{n}\right)$$

The value of n for maximum efficiency can be obtained by plotting or by differentiation.[1] It will be found that $n = 2$. The efficiency is then 50 per cent., which is the highest that can be obtained with a jet impinging on flat vanes. A common example of a motor working in this manner is the ordinary undershot water-wheel, but due to leakage past the floats, axle friction, etc. The efficiency is rarely over 30 per cent.

If the jet had been impinging on only one plate instead of a large number, the quantity of water that reached the plate per second would only have been $W\left(1 - \frac{1}{n}\right)$; then, substituting this value for W in the equation above, it will be seen that the efficiency of the jet $= \frac{2}{n}\left(1 - \frac{1}{n}\right)^2$, and the maximum efficiency occurs when $n = 3$, and is equal to about 30 per cent.

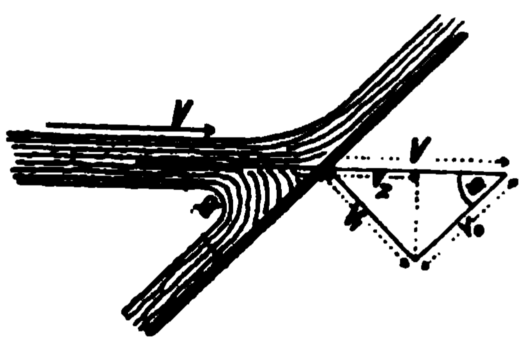


FIG. 561.

Pressure on an Oblique Surface due to an Impinging Jet.—The jet impinges obliquely at an angle θ to the plate, and splits up into two streams. The velocity V may be resolved into V_1 normal and V_0 parallel to the plate. After impinging, the water has no velocity normal to the plate, therefore the normal pressure—

$$P = \frac{WV_1}{g} = \frac{WV \sin \theta}{g}$$

---

[1] Efficiency $= \eta = \frac{2}{n}\left(1 - \frac{1}{n}\right)$

$$\eta = \frac{2}{n} - \frac{2}{n^2} = 2n^{-1} - 2n^{-2}$$

$$\frac{d\eta}{dn} = 2n^{-2} - 4n^{-3} = 0, \text{ when } \eta \text{ is a maximum}$$

$$\text{or } 2n^{-2} = 4n^{-3}$$

$$\frac{2}{n^2} = \frac{4}{n^3}, \text{ whence } n = 2$$

Pressure on a Moving Oblique Surface due to an Impinging Jet.—If the plate be one of a series moving in a direction *normal* to itself with a velocity $\frac{V_1}{n}$, the jet impinges with a velocity normal to the plate of—

$$V_1\left(1 - \frac{1}{n}\right)$$

and the normal pressure—

$$P = \frac{WV_1\left(1 - \frac{1}{n}\right)}{g}$$

Work done per second on the plate in foot-lbs.—

$$\frac{PV_1}{n} = \frac{WV_1^2\left(1 - \frac{1}{n}\right)}{ng}$$

Then, substituting the value V sin θ for V_1, we have—

$$\text{Efficiency of the jet} = \frac{2 \sin^2 \theta\left(1 - \frac{1}{n}\right)}{n}$$

Similarly, if the plates be moving in the same direction as the jet with a velocity $\frac{V_2}{n}$, we have—

$$\left.\begin{array}{l}\text{The work done per second on} \\ \text{the plate in foot-lbs.}\end{array}\right\} = \frac{WV_2^2\left(1 - \frac{1}{n}\right)}{ng}$$

Substituting the value V $\sin^2 \theta$ for V_2, we have—

$$\text{Efficiency of the jet} = \frac{2 \sin^4 \theta\left(1 - \frac{1}{n}\right)}{n}$$

Pressure on a Smooth Curved Surface due to an Impinging Jet.—We will first consider the case in which the surface is stationary and the water slides on it without shock; how to secure this latter condition we will consider shortly.

We show three forms of surface, to all of which the following reasoning applies.

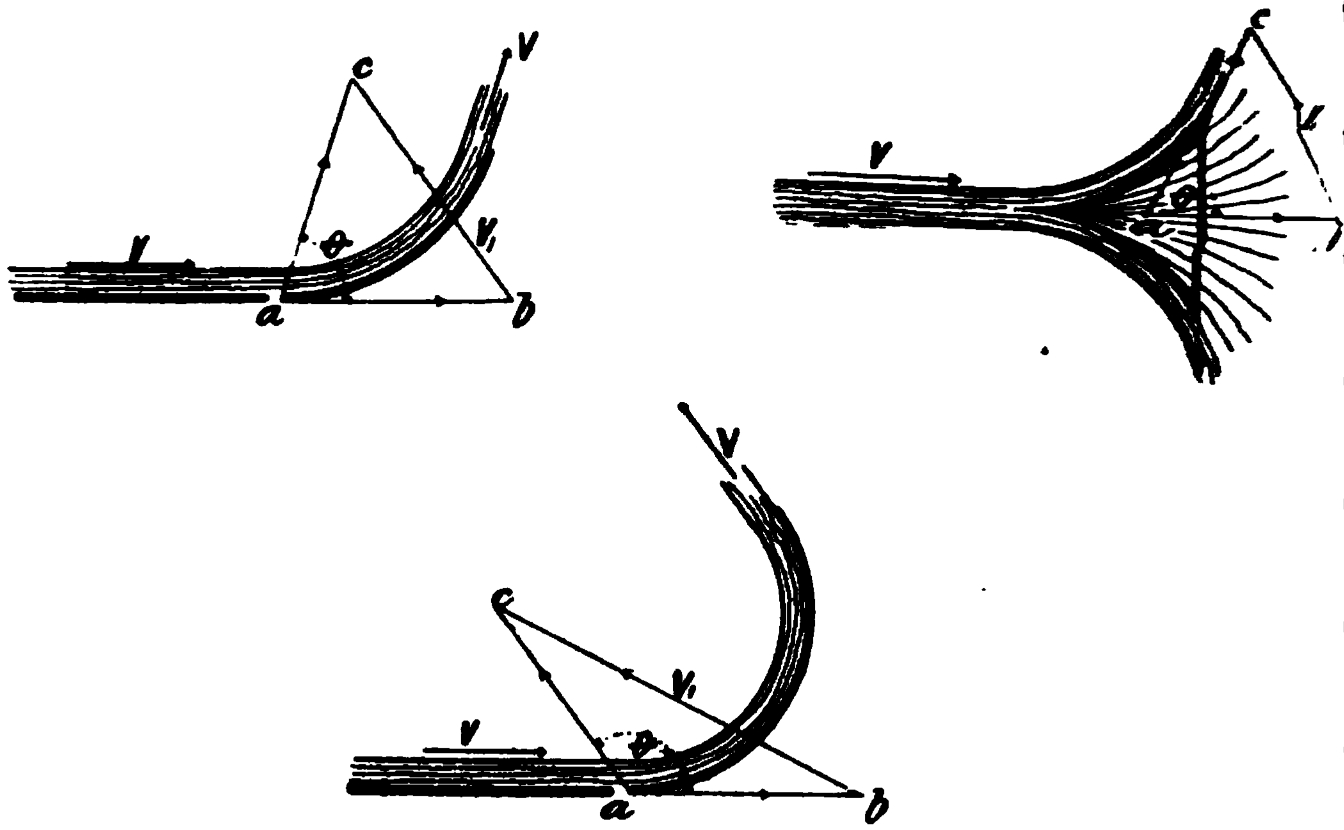


FIG. 562.

Draw *ab* to represent the initial velocity V of the jet in magnitude and direction; then, neglecting friction, the final velocity of the water on leaving the surface will be V, and its direction will be tangential to the last tip of the surface. Draw *ac* parallel to the final direction and equal to *ab*, then *bc* represents the change of velocity V_1; hence the resultant pressure on the surface in *the direction of cb* is—

$$P = \frac{WV_1}{g}$$

Then, reproducing the diagram of velocities above, we have—

$$y = V \sin \theta$$
$$x = V \cos \theta$$
$$V_1^2 = (V - x)^2 + y^2$$

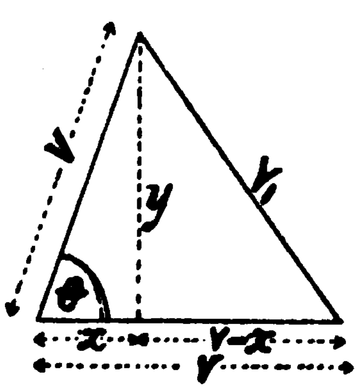


FIG. 563.

Then, substituting the values of x and y and reducing, we have—

$$V_1 = V\sqrt{2(1 - \cos \theta)}$$

We, however, generally require the pressure in a direction parallel to the jet. From Fig. 563 we see that the component parallel to the jet is

$V - x = V(1 - \cos\theta)$. Thus in all the three cases given above we have the pressure parallel to the jet—

$$P_0 = \frac{WV(1 - \cos\theta)}{g}$$

Pressure on a Smooth-curved Moving Surface due to an Impinging Jet.—If the curved surface were one of a moving series, the same construction as given above would hold, *ab* and *ac* still being drawn to represent the initial and final velocities of the water relative to the ground, and, of course, equal to their respective absolute velocities.

Let *ad* represent the velocity of the moving surface, then *db* represents the velocity of the jet relative to the vane. The relative velocity remains the same all along the vane, hence if we make *gh* tangential to the last lip of the vane and equal to *db*, it will represent the final velocity of the jet relative to the vane. Draw *hi* parallel and equal to *ad*, then *gi* represents the final absolute velocity of the jet on leaving the vane. Draw *ac* equal and parallel to *gi*. By similar triangles, it is easily shown that *dc* is equal and parallel to *gh*, and is also equal to *db*; then, since *ab* represents the initial velocity, and *ac* the final velocity of the jet relative to the ground, *bc*, as before, represents the change of velocity V_1. Hence the pressure on the moving vane *in the direction cb* is—

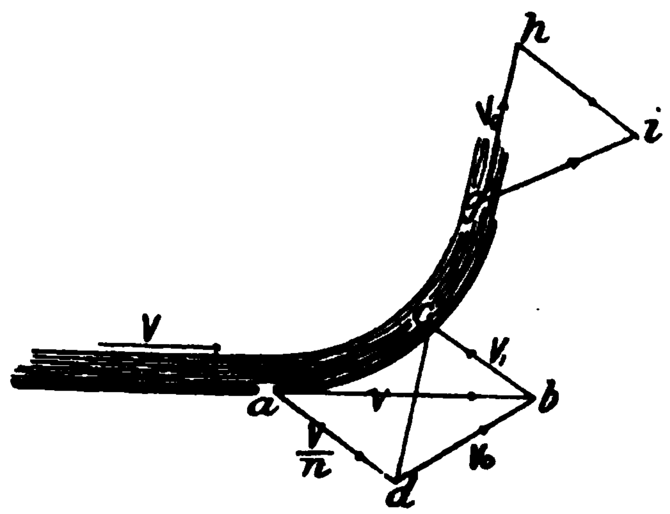


FIG. 564.

$$P = \frac{WV_1}{g}$$

If the vane move in a direction parallel to the jet, the construction then becomes that shown in Fig. 565; then the pressure *in the direction cb* is—

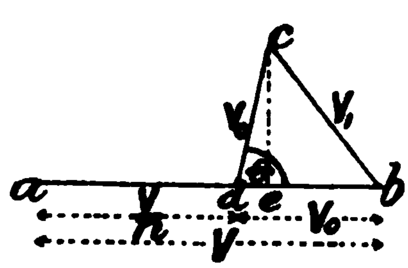


FIG. 565.

$$P = \frac{WV_1}{g}$$

But the component of V_1 in the direction of the jet is *eb*—

$$\text{or } V_0 - de = V_0 - V_0 \cos\theta$$

$$\text{But } V_0 = V - \frac{V}{n}$$

$$\text{hence } eb = V - \frac{V}{n} - V\cos\theta + \frac{V\cos\theta}{n}$$

$$eb = V\left(1 - \frac{1}{n}\right)(1 - \cos\theta)$$

and the pressure on the moving vane in the direction of the jet when the vane is moving parallel to the jet is—

$$P = \frac{WV}{g}\left(1 - \frac{1}{n}\right)(1 - \cos\theta)$$

a result that might have been obtained by inspection from the value of P given on p. 509. This result applies to all the cases shown in Fig. 562.

$$\left.\begin{array}{l}\text{The work done per second}\\ \text{by jet}\end{array}\right\} \frac{PV}{n} = \frac{WV^2}{ng}\left(1 - \frac{1}{n}\right)(1 - \cos\theta)$$

$$\text{efficiency} = \frac{2}{n}\left(1 - \frac{1}{n}\right)(1 - \cos\theta)$$

Pelton Wheel Vanes (Figs. 566, 567).—This is but a special case of the vanes treated in the last paragraph. The

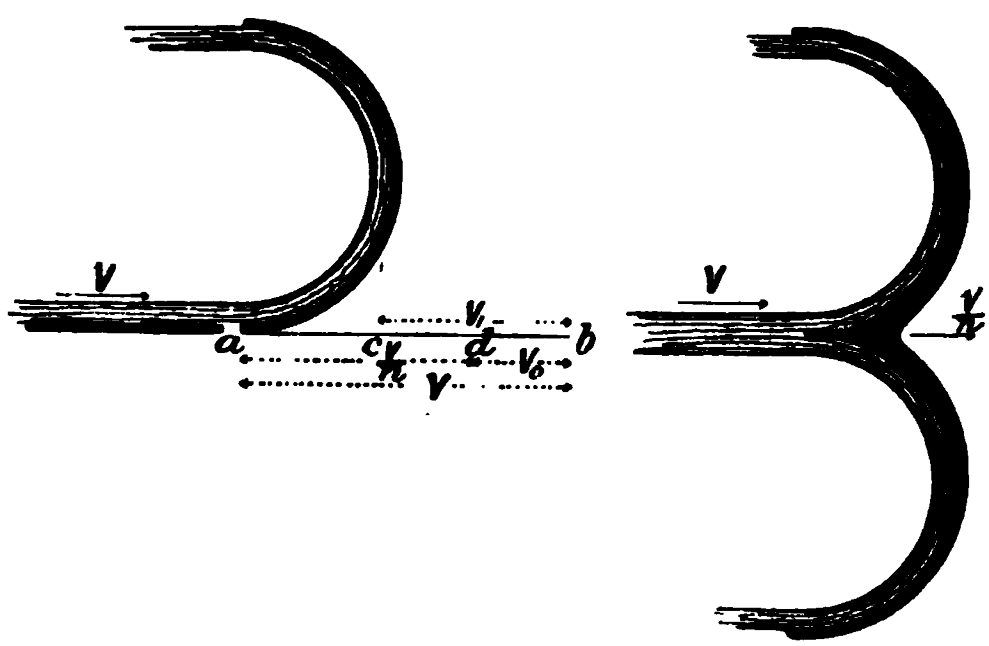


FIG. 566.

last tip of the vane is parallel to the jet, and the diagram of velocities becomes a straight line as shown. The letters

correspond to those in Fig. 564: *ad* is the velocity of the vane $\frac{V}{n}$, *db* the relative velocity of the water and the vane V_0.

$$\text{Then } V_0 = V - \frac{V}{n} = V\left(1 - \frac{1}{n}\right)$$

Fig. 567.[1]

But V_0 remains unchanged as the water traverses the vane; thus the total change of velocity is $bc = 2V_0 = 2V\left(1 - \frac{1}{n}\right)$, and the pressure—

$$P = \frac{W \cdot 2V}{g}\left(1 - \frac{1}{n}\right)$$

which could have been obtained from the expression for P in the last paragraph by putting in the value for $\cos\theta$, where $\theta = 180°$, and $\cos 180 = -1$. We have given both the special and the general treatment in order to make it as clear as possible.

It follows quite simply from the above, or from p. 508, that the efficiency $= \frac{4}{n}\left(1 - \frac{1}{n}\right)$, which has a maximum value when $n = 2$; then the efficiency is 100 per cent., a result which is never reached in practice.

Poncelot Water-wheel Vanes.—In this wheel a thin stream of water having a velocity V feet per sec. glides up curved

[1] Reproduced by the kind permission of Messrs. Gilbert Gilkes and Co., Kendal.

vanes ha

The wate

to it, anc

at its up

falls, att

the nega
which it

wheel wi

with a ve

on leavir

hence, wl
all the e
all the lo
efficiency
cluding t

Form of Vane to prevent Shock.—In order that the water may glide gently on to the vanes of any motor, the tangent to the entering tip of the vanes must be in the same direction as the path of the water relative to the tip of the wheel; thus, in the figure, if *ab* represents the velocity of the entering stream, *ad* the velocity of the vane, then *db* represents the relative path of the water, and the entering tip must be parallel to it. The stream then gently glides on the vane without shock.

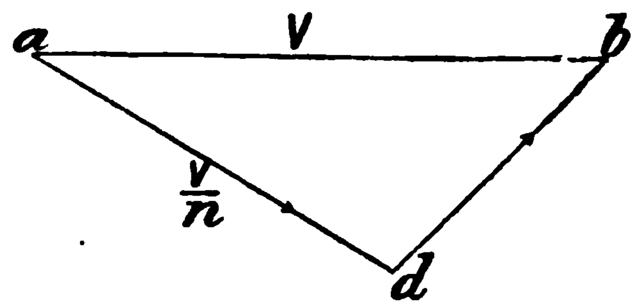


FIG. 569.

Turbines.—Turbines may be conveniently divided into two classes: (1) Those in which the whole energy of the water is converted into energy of motion in the form of free jets or streams which are delivered on to suitably shaped vanes in order to reduce the absolute velocity of the water on leaving to zero or nearly so. Such a turbine wheel receives its impulse from the direct action of impinging jets or streams; and is known as an "impulse" turbine. When the admission only takes place over a small portion of the circumference, it is known as a "partial admission" turbine. The jets of water proceeding from the guide-blades are perfectly free, and after impinging on the wheel-vanes the water at once escapes into the air above the tail-race.

(2) Those in which some of the energy is converted into pressure energy, and some into energy of motion. The water is therefore under pressure in both the guide-blades and in the wheel passages, consequently they must always be full, and there must always be a pressure in the clearance space between the wheel and the guides, which is not the case in impulse turbines. Such are known as "reaction" turbines, because the wheel derives its impulse from the reaction of the water as it leaves the wheel passages. There is often some little difficulty in realizing the pressure effects in reaction turbines. Probably the best way of making it clear is to refer for one moment to the simple reaction wheel shown in Fig. 570, in which water runs into the central chamber and is discharged at opposite sides by two curved horizontal pipes as shown; the reaction of the jets on the horizontal pipes causes the whole to revolve. Now, instead of allowing the central chamber to revolve with the horizontal pipes, we may fix the central chamber, as in Fig. 571, and allow the arms only to revolve; we shall get a

crude form of a reaction turbine. It will be clear that a water-tight joint must be made between the arms and the chamber,

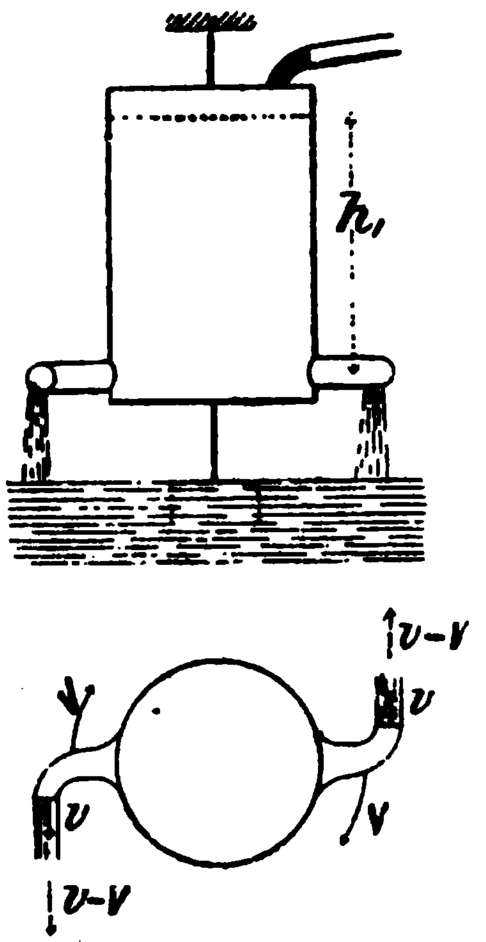


Fig. 570.

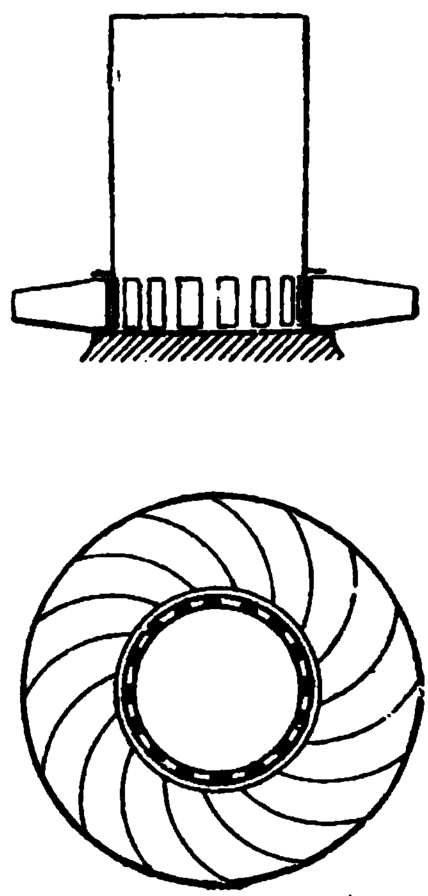


Fig. 571.

because there is pressure in the clearance space between. It will also be seen that the admission of water must take place over the whole circumference, and, further, that they must be always full of water. A typical case of such a turbine is shown in Fig. 576. These turbines may either discharge into the air above the tail-water, or the revolving wheel may discharge into a casing which is fitted with a long suction pipe, and a partial vacuum formed into which the water discharges.

In addition to the above distinctions, turbines are termed parallel flow, inward flow, outward flow, and mixed flow turbines, according as the water passes through the wheel parallel to the axis, from the circumference inwards towards the axis, from the axis outwards towards the circumference, or both parallel to the axis and either inwards or outwards.

We may tabulate the special features of the two forms of turbine thus :

Impulse.	Reaction.
All the energy of the water is converted into kinetic energy before being utilized.	Some of the energy of the water is converted into kinetic energy, and some into pressure energy.

IMPULSE.	REACTION.
The water impinges on curved wheel-vanes in free jets or streams, consequently the wheel passages must not be filled.	The water is under pressure in both the guide and wheel passages, also in the clearance space ; hence the wheel passages are always full.
The water is discharged freely into the atmosphere above the tail water ; hence the turbine must be at the foot of the fall.	As the wheel passages are always full, it will work equally well when discharging into the atmosphere or into water, *i.e.* above or below the tail water, or into suction pipes. The turbine may be placed 30 feet above the foot of the fall.
Water may be admitted on a portion or on the whole circumference of the wheel.	Water must be admitted on the whole circumference of the wheel.
Power easily regulated without much loss.	Power difficult to regulate without loss.

In any form of turbine, it is quite impossible to so arrange it that the water leaves with no velocity, otherwise the wheel would not clear itself. From 5 to 8 per cent. of the head is often required for this purpose, and is rejected in the tail-race.

Form of Blades for Impulse Turbine.—The form of blades required for the guide passages and wheel of a turbine are most easily arrived at by a graphical method. The main points to be borne in mind are—the water must enter the guide and wheel passages without shock. To avoid losses through sudden changes of direction, the vanes must be smooth easy curves, and the changes of section of the passages gradual (for reaction turbines specially). The absolute velocity of the water on leaving must be as small as is consistent with making the wheel to clear itself.

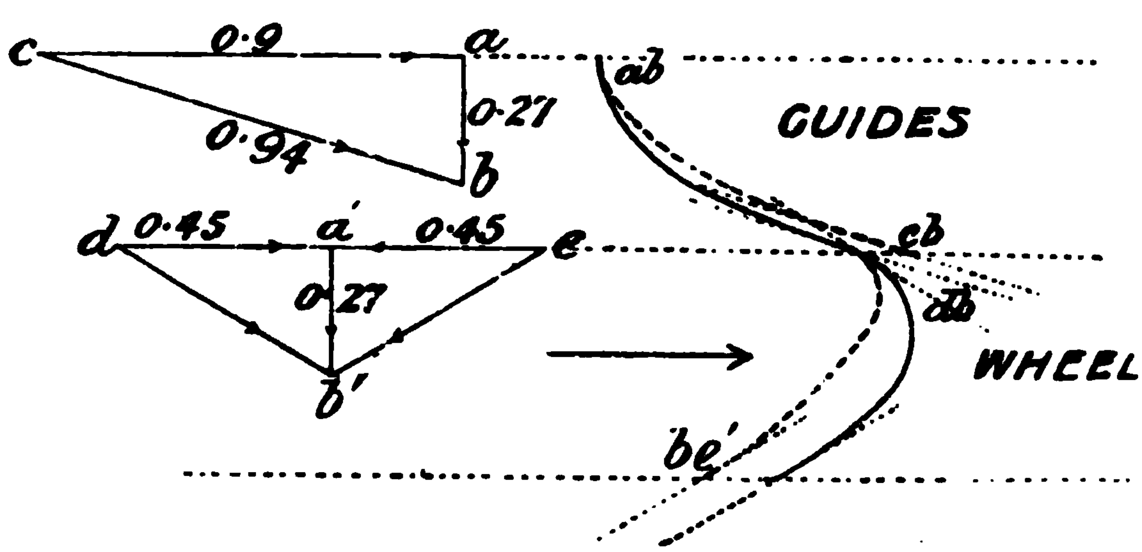


FIG. 572.

For simplicity we shall treat the wheel as being of infinite radius, and after designing the blades on that basis we shall, by a special construction, bend them round to the required

radius. We shall also treat the water as passing vertically down into the guide-blades. Let the velocity of the water be reduced 6 per cent. by friction in passing over the guide-blades, and let 7 per cent. of the head be rejected in the tail-race.

The water enters the guides vertically, hence the first tip of the guide-blade must be vertical as shown. In order to find the direction of the final tip, we proceed thus: We have decided that the water shall enter the wheel with a velocity due to 94 per cent. of the head, since 6 per cent. is lost in friction, whence—

$$V_0 = 0{\cdot}94\sqrt{2gH}$$

also the velocity of rejection—

$$Vr = \sqrt{\frac{2g \times 7H}{100}} = 0{\cdot}27\sqrt{2gH}$$

We now set down *ab* to represent the vertical velocity with which the water passes through the turbine wheel, and from *b* we set off *bc* to represent V_0; then *ac* gives us the horizontal component of the velocity of the water, and $= \sqrt{0{\cdot}94^2 - 0{\cdot}27^2} = 0{\cdot}9$[1] $= V_1$: *cb* gives us the direction in which the water leaves the guide-vanes; hence a tangent drawn to the last tip of the guide-vanes must be parallel to *cb*. We are now able to construct the guide-vanes, having given the first and last tangents by joining them up with a smooth curve as shown.

Let the velocity of the wheel be one-half the horizontal velocity of the entering stream, or $V_w = \frac{0{\cdot}9}{2} = 0{\cdot}45$; hence the horizontal velocity of the water relative to the wheel is also 0·45. Set off *a'b'* as before = 0·27, and *a'd* horizontal and = 0·45: we get *db'* representing the velocity of the water relative to the vane; hence, in order that there may be no shock, a tangent drawn to the first tip of the wheel-vane must be parallel to *db'*; but, as we want the water to leave the vanes with no absolute horizontal velocity, we must deflect it during its passage through the wheel, so that it has a backward velocity relative to the wheel of −0·45, and as it moves forward with it, the absolute velocity will be $-0{\cdot}45 + 0{\cdot}45 = 0$. To accomplish this, set off *a'e* = 0·45. Then *b'e* gives us the final velocity of the water relative to the wheel; hence the tangent to the last tip of the wheel-vane must be parallel to *b'e*. Then, joining up the two tangents with a smooth curve, we get the required form of vane.

[1] We omit $\sqrt{2gH}$ to save constant repetition.

It will be seen that an infinite number of guides and vanes could be put in to satisfy the conditions of the initial and final tangents, such as the dotted ones shown. The guides are, for frictional reasons, usually made as short as is consistent with a smooth easy-connecting curve, in order to reduce the surface to a minimum. The wheel-vanes should be so arranged that the absolute path of the water through the wheel is a smooth curve without a sharp bend, and the latter part must of course be vertical, thus—

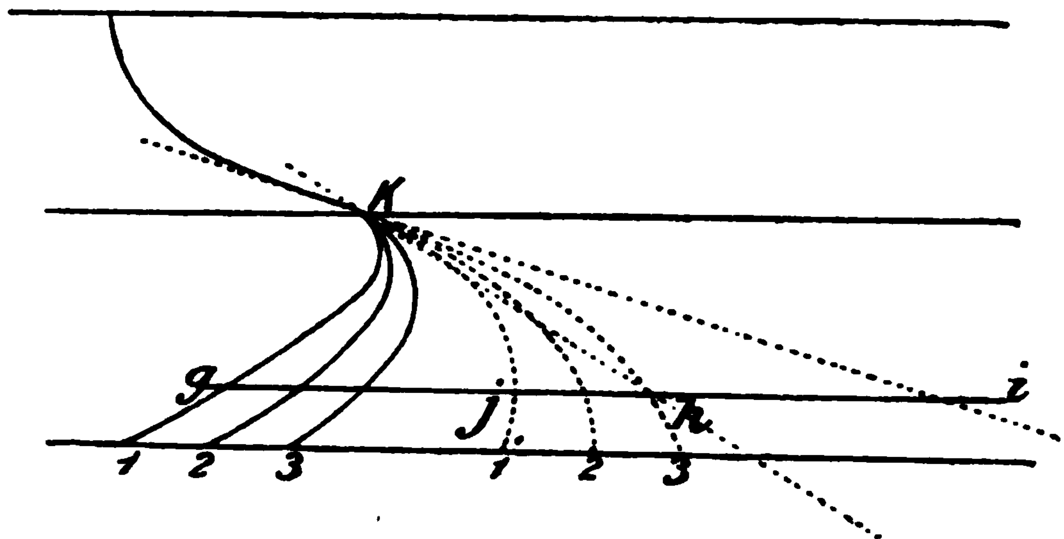


FIG. 573.

The water would move along the absolute path Ki and along the path Kh relative to the wheel if there were no vanes to deflect it, where hi is the distance moved by the vane while the water is travelling from k to i; but the wheel-vanes deflect it through a horizontal distance hg, hence the water at g would have been at j but for the vanes, where $gh = ij$. The three vanes 1, 2, 3 all fulfil the necessary tangential conditions, but it will be seen that 1 is curved too much back, and 3 not enough, whereas 2 is correct. In order to let the water get away very freely, and to prevent any possibility of them choking, the sides of the wheel-passages are usually provided with ventilation holes, and the wheel is flared out. The efficiency of the turbine is readily found thus:

The whole of the horizontal component of the velocity of the water has been imparted to the turbine wheel, hence—

the work done per pound of water $\left.\right\} = \dfrac{V_1^2}{2g} = \dfrac{(0·9V)^2}{2g}$

the energy per pound of the water on entering $\left.\right\} = H = \dfrac{V^2}{2g}$

$$\text{efficiency} = \frac{V_1^2}{2gH} = 0·9^2 = 81 \text{ per cent.}$$

This takes no account of the hydraulic resistances, such as eddying and friction in the wheel passages, shaft friction, etc., but an efficiency of 80 per cent. has been obtained from such a turbine when working under the most favourable conditions.

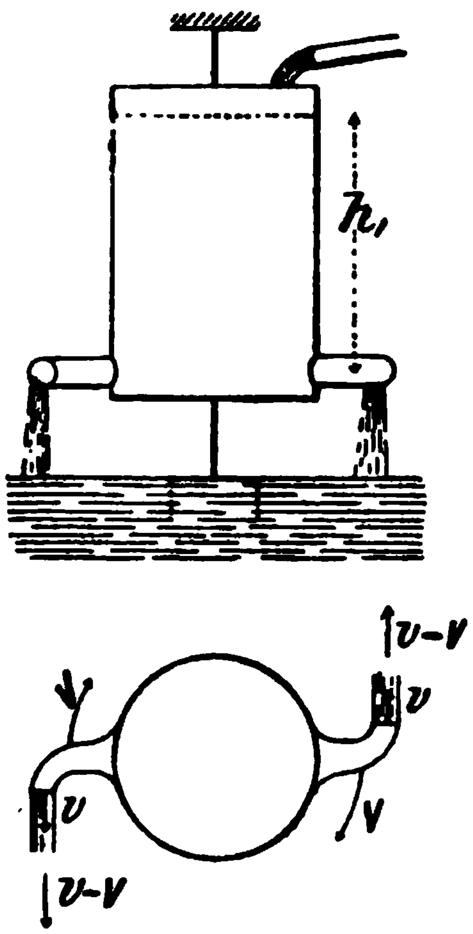


FIG. 574.

Pressure Turbine.—Before proceeding to consider the vanes for a pressure turbine, we will briefly look at its forerunner, the simple reaction wheel. Let the speed of the orifices be V; then, if the water were simply left behind as the wheel revolved, the velocity of the water relative to the orifices would be V, and the head required to produce this velocity—

$$h = \frac{V^2}{2g}$$

Let h_1 = the height of the surface of the water or the head above the orifices.

Then the total head producing flow, $H = h + h_1 = \dfrac{V^2}{2g} + h_1$

Let v = the relative velocity with which the water leaves the orifices.

$$\text{Then } v^2 = 2gH = V^2 + 2gh_1$$

The velocity of the water relative to the ground $= v - V$. If the jet impinged on a plate, the pressure would be $\dfrac{v - V}{g}$ per pound of water; but the reaction on the orifices is equal to this pressure, therefore the reaction—

$$R = \frac{v - V}{g}$$

and the work done per second by the jets in foot-lbs. $\Big\}\ RV = \dfrac{V(v - V)}{g}$ . . . . (i.)

energy wasted in discharge water per pound in foot-lbs. $\Big\} = \dfrac{(v - V)^2}{2g}$ . . . . (ii.)

total work done per pound of water $\Big\}$ i. + ii. $= \dfrac{v^2 - V^2}{2g}$

$$\text{efficiency } \frac{\text{i.}}{\text{i.} + \text{ii.}} = \frac{2V}{v + V}$$

When $v = V$, the efficiency is 100 per cent., but this entails a very excessive velocity of flow through the orifices, with a corresponding serious loss in friction. The hydraulic efficiency may reach 65 per cent., and the total 60 per cent. The loss is due to the water leaving with a velocity of whirl $v - V$. In order to reduce this loss, Fourneyron, by means of guide-blades, gave the water an initial whirl in the opposite direction before it entered the wheel, and thus caused the water to leave with little or no velocity of whirl, and a corresponding increase in efficiency.

The method of arranging such guides is shown in Fig. 575; they are simply placed in the central chamber of such a wheel

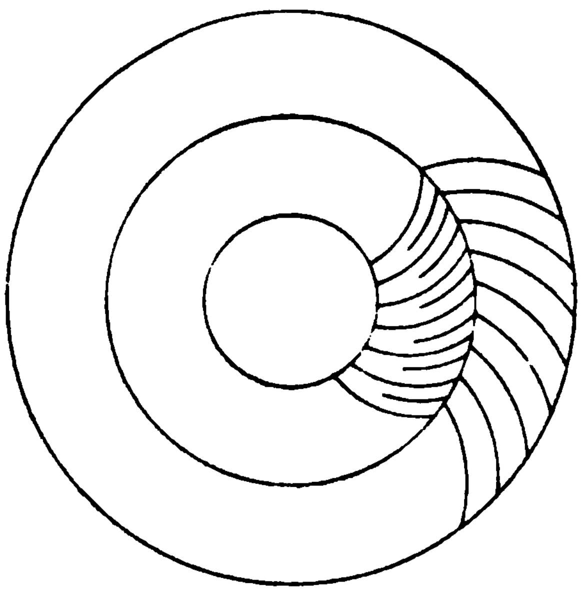


Fig. 575.

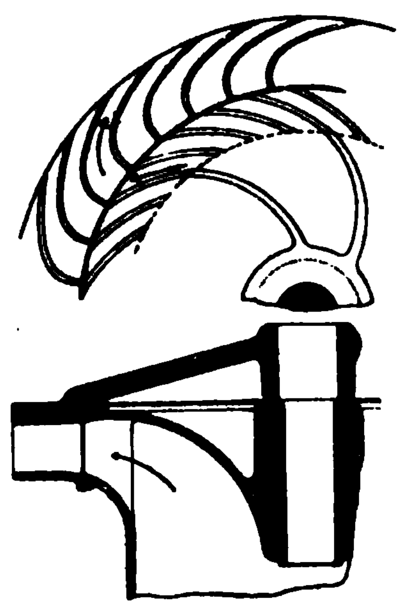


Fig. 576.

as that shown in Fig. 571. Sometimes, however, the guides are outside the wheel, and sometimes above, according to the type of turbine.

Form of Blades for Pressure Turbine.—As in the impulse turbine, let, say, 7 per cent. of the head be rejected in the tail-race, and say 13 per cent. is wasted in friction. Then we get 20 per cent. wasted, and 80 per cent. utilized.

Some of the head may be converted into pressure energy, and some into kinetic energy; the relation between them is optional. In this case say one-half is converted into pressure energy, and one-half into kinetic energy. If 80 per cent. of the *head* be utilized, the corresponding velocity will be—

$$V = \sqrt{2g\frac{H \times 80}{100}} = 0{\cdot}89\sqrt{2gH}$$

Thus 89 per cent. of the velocity will be utilized. To find the corresponding vertical or pressure component and the

horizontal or velocity component, we draw the triangle of velocities as shown (Fig. 577), and find that each is $0{\cdot}63\sqrt{2gH}$.

We will determine the velocity of the wheel by the principle of momentum. Water enters the wheel with a horizontal velocity $V_h = 0{\cdot}63$, and leaves with no horizontal momentum.

$$\text{Horizontal pressure per pound of water} = \frac{V_h}{g} \text{ lbs.}$$

$$\left.\begin{array}{l}\text{useful work done in foot-lbs. per second per}\\ \quad\text{pound of water}\end{array}\right\} = \frac{V_h V_w}{g} = 0{\cdot}8H$$

since we are going to utilize 80 per cent. of the head.

$$\text{hence } V_w = \frac{0{\cdot}8gH}{0{\cdot}63\sqrt{2gH}} = \frac{0{\cdot}8\sqrt{gH}}{0{\cdot}63\sqrt{2}}$$

$$V_w = 0{\cdot}64\sqrt{2gH}$$

We now have all the necessary data to enable us to determine the form of the vanes. Set down *ab* to represent

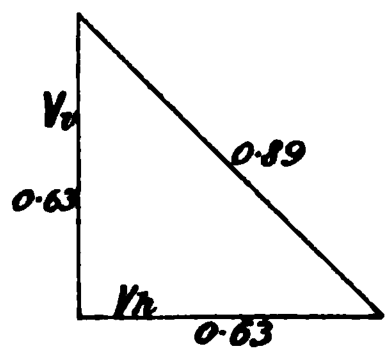


FIG. 577.

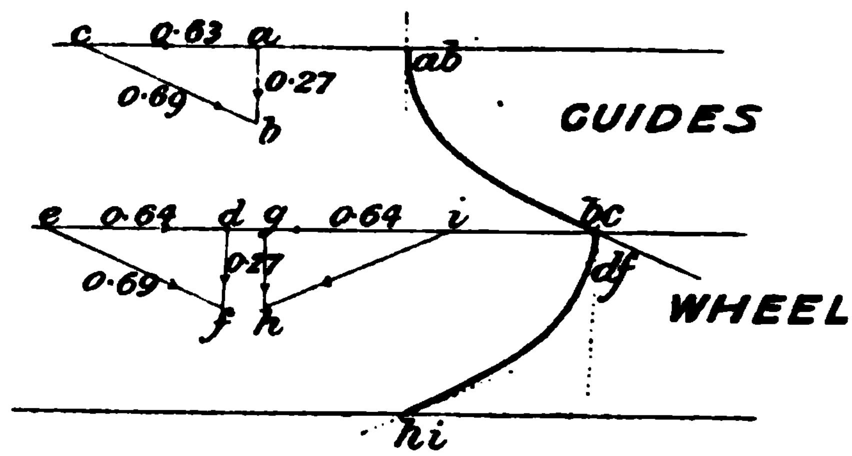


FIG. 578.

the vertical velocity of the water passing through the wheel, and *ac* the horizontal velocity. Completing the triangle, we find $cb = 0{\cdot}69$, which gives us the direction in which the water enters the wheel or leaves the guides. The guide-blade is then put in by the method explained for the impulse wheel.

To obtain the form of the wheel-vane. Set off *ed* to represent the horizontal velocity of the wheel, and *ef* the velocity of the water on leaving the guides; then *df* represents the velocity of the water relative to the wheel, which gives us the direction of the tangent to the first tip of the wheel-vanes. Then, in order that the water shall leave with no horizontal velocity, we must deflect it during its passage through the wheel so that it has a backward velocity relative to the wheel of $-0{\cdot}64$. Then, setting down $gh = 0{\cdot}27$, and $gi = -0{\cdot}64$, we

get *hi* as the final velocity of the water, which gives us the direction of the tangent to the last tip of the vane.

In Figs. 579, 580, 581, we show the form taken by the wheel-vanes for various proportions between the pressure and velocity energy.

$V = 0{\cdot}89$
$V_p = 0{\cdot}8$
$V_h = 0{\cdot}4$
$V_r = 0{\cdot}27$
$V_w = 1{\cdot}0$

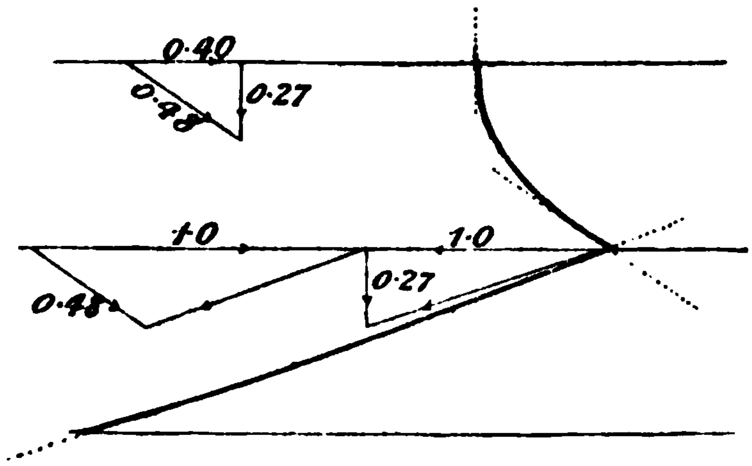


FIG. 579.

$V = 0{\cdot}89$
$V_p = 0{\cdot}4$
$V_h = 0{\cdot}8$
$V_r = 0{\cdot}27$
$V_w = 0{\cdot}5$

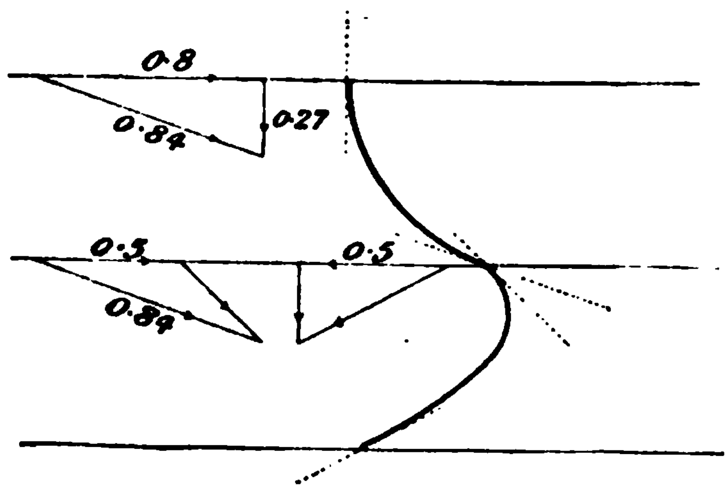


FIG. 580.

$V = 0{\cdot}89$
$V_p = 0$
$V_h = 0{\cdot}89$
$V_r = 0{\cdot}27$
$V_w = 0{\cdot}45$

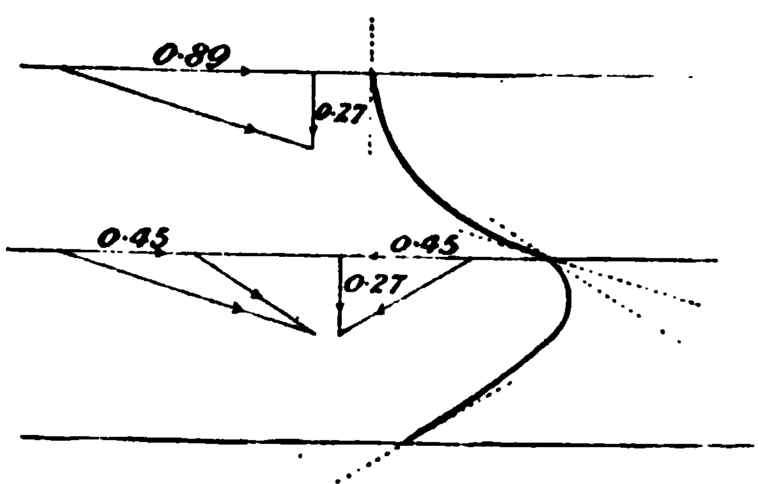


FIG. 581.

In this case, in which $V_p = 0$, the whole of the energy is converted into kinetic energy; then $V_h = 0{\cdot}89$, and—

$$V_w = \frac{0{\cdot}8gH}{0{\cdot}89\sqrt{2gH}}$$

$$V_w = 0{\cdot}45\sqrt{2gH}$$

Or the velocity of the wheel is one-half the horizontal velocity of the water, as in the impulse wheel. The form of blades in this case is precisely the same as in Fig. 572, but they are arrived at in a slightly different manner.

In one of the most perfect turbines in use in this country, viz. the Vortex, $V_h = 0.67$ and $V_v = 0.59$ approximately.

Projection of Turbine Blades.—In all the above cases we have constructed the vanes for a turbine of infinite radius, sometimes known as a "turbine rod." We shall now proceed to give a construction for bending the rod round to a turbine of small radius.

The blades for the straight turbine being given, draw a series of lines across as shown; in this case only one is shown for sake of clearness, viz. *ab*, which cuts the blade in the point *c*. Project this point on to the base-line, viz. *d*. From the centre *o* describe a circle *a'b'* touching the line *ab*. Join *od*, cutting the circle *a'b'* in the point *f*. This point *f* on the circular turbine blade corresponds to the point *c* on the straight blade. Other points are found in the same manner, and a smooth curve is drawn through them.

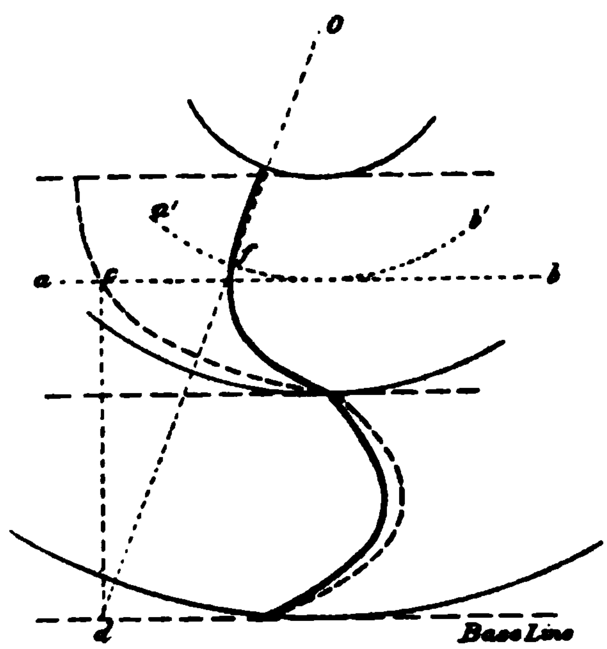


FIG. 582.

The blades for the straight turbine are shown dotted, and those for the circular turbine in full lines.

Efficiency of Turbines.—The following figures are taken from some curves given by Professor Unwin in a lecture delivered at the Institution of Civil Engineers in the Hydromechanics course in 1884–5 :—

Type of turbine.	Efficiency per cent. at various sluice-openings.						
	Full.	0·9.	0·8.	0·7.	0·6.	0·5.	0·4.
Impulse (Girard)	80	80	80	80	81	81	—
Pressure (Jonval) (throttle-valve) ...	71	59	46	35	25	16	—
Hercules	82	82	80	75	68	63	55

Losses in Turbines.—The various losses in turbines of course depend largely on the care with which they are designed and manufactured, but the following values taken from the source mentioned above will give a good idea of the magnitude of the losses.

Loss due to surface friction, eddying, etc., in the turbine	10 to 14 per cent.
Loss due to energy rejected in tail-race ...	3 to 7 ,,
,, shaft friction	2 to 3 ,,

CHAPTER XVIII.

PUMPS.

NEARLY all water-motors, when suitably arranged, can be made reversible—that is to say, that if sufficient power be supplied to drive a water-motor backwards, it will raise water from the tail-race and deliver it into the head-race, or, in other words, it will act as a pump.

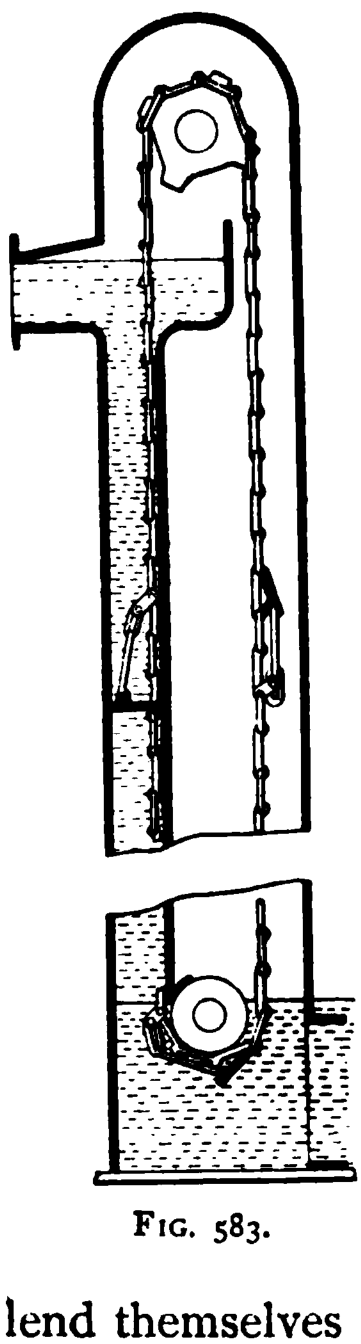


FIG. 583.

The only type of motor that cannot, for practical purposes, be reversed is an impulse motor, which derives its energy from a free jet or stream of water, such as a Pelton wheel.

We shall consider one or two typical cases of reversed motors.

Reversed Gravity Motors: Bucket Pumps, Chain Pump, Dredgers, Scoop Wheels, etc.—The two gravity motors shown in Figs. 549, 550, will act perfectly as pumps if reversed; an example of a chain pump is shown in Fig. 583. The floats are usually spaced about 10 feet apart, and the slip or the leakage past the floats is about 20 per cent. They are suitable for lifts up to 60 feet. The chain speed varies from 200 to 300 feet per minute, and the efficiency is about 63 per cent.

The ordinary dredger, is also another pump of the same type.

Reversed overshot water-wheels have been used as pumps, but they do not readily lend themselves to such work.

A pump very similar to the reversed undershot or breast wheel is largely used for low lifts, and gives remarkably good

results; such a pump is known as a "scoop wheel" (see Fig. 584).

The circumferential speed is from 6 to 10 feet per second. The slip varies from 5 per cent. in well-fitted wheels to 20 per cent. in badly fitted wheels.

The diameter varies from 20 to 50 feet, and the width from 1 to 5 feet; the paddles are pitched at about 18 inches. The total efficiency, including the engine, varies from 50 to 70 per cent.

Fig. 584.

Reversed Pressure Motors, or Reciprocating Pumps.—If a pressure motor be driven from some external source the feed pipe becomes a suction pipe, and the exhaust a delivery pipe; such a reversed motor is termed a plunger, bucket, or

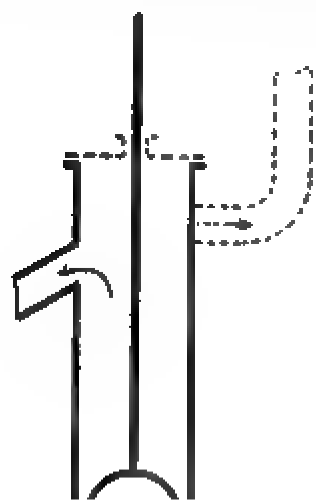


Fig. 585.

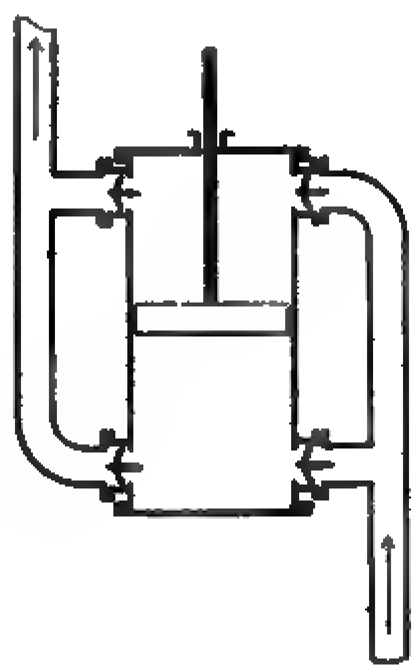


Fig. 586.

piston pump. They are termed single or double acting according as they deliver water at every or at alternate strokes of the piston or plunger. In Figs. 585, 586, 587, and 588 we show typical examples of various forms of reciprocating pumps.

Fig. 585 is a bucket pump, single acting, and gives an intermittent discharge. It is only suitable for low lifts. Some-

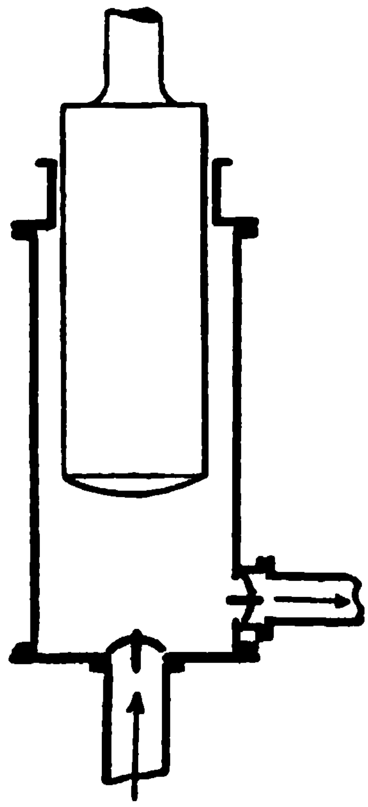


FIG. 587.

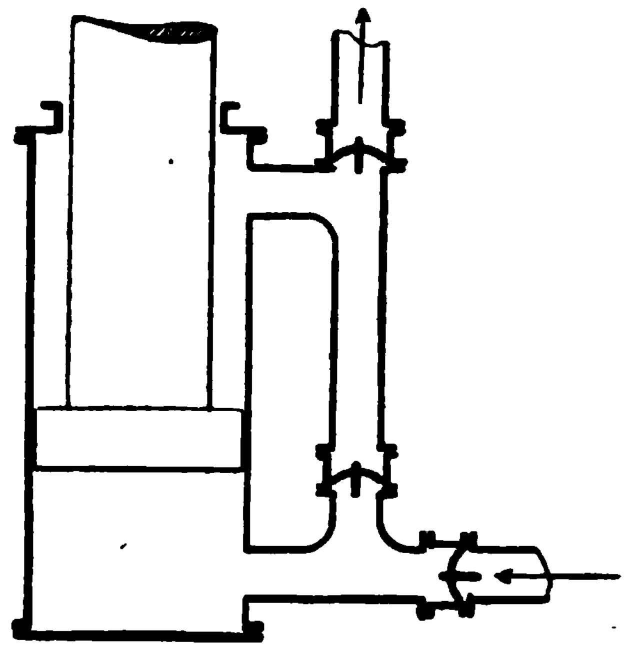


FIG. 588.

times this form of pump is modified as shown in dotted lines, when it is required to force water to a height.

Fig. 586 is a double-acting force or piston pump. When such pumps are made single acting, the upper set of valves are dispensed with. They can be used for high lifts. The manner in which the flow fluctuates will be dealt with in a future paragraph.

Fig. 587 is a plunger pump. It is single acting, and is the form usually adopted for very high pressures. The flow is intermittent.

Fig. 588 is a combined bucket-and-plunger pump. It is double acting, but has only one inlet and one delivery valve. The flow is similar to that of Fig. 586.

There is no need to enter into a detailed description of the manner in which these pumps work; it will be obvious from the diagrams. It may, however, be well to point out that if the velocity past the valves be excessive, the frictional resistance becomes very great, and the work done by the pump greatly exceeds the work done in simply lifting the water. Provided a pump is dealing with water only, and not air and water, the amount of clearance at each end of the stroke is a matter of no importance.

Fluctuation of Delivery.—In all forms of reciprocating

pumps there is more or less fluctuation in the delivery, both during the stroke and in single-acting pumps between the delivery and the suction strokes. The fluctuation during the stroke is entirely due to the variation in the speed of the piston, bucket, or plunger. As this fluctuation is in some instances a serious matter, *e.g.* on long lengths of mains, we shall carefully consider the matter.

When dealing with the steam-engine mechanism in Chapter VI. we gave a construction for finding the velocity of the piston at every part of the stroke. We repeat it in Fig. 589, showing the construction lines for only one or two points. The flow varies directly as the velocity, hence the velocity diagram is also a flow diagram.

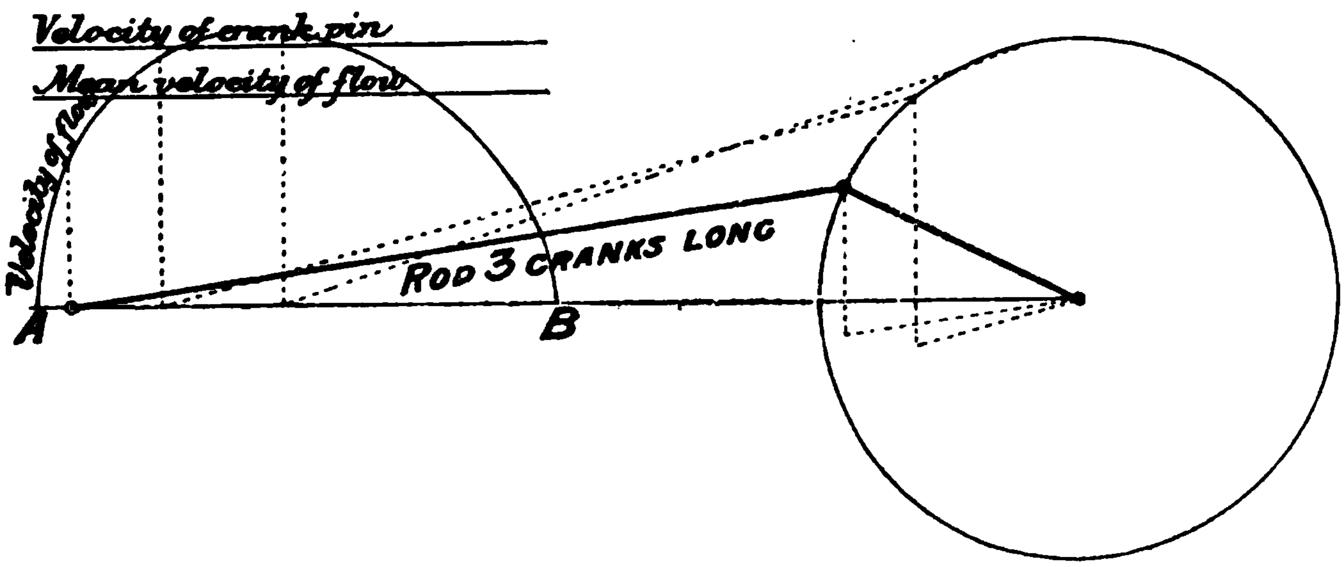


FIG. 589.

We give some typical flow diagrams below. The vertical height of the diagram represents the quantity of water being delivered at that particular part of the stroke. In all cases we have assumed that the crank revolves at a constant speed, and have adhered to the proportion of the connecting-rod = 3 cranks. The letters A and B refer to the particular end of the stroke, as in Fig. 589.

Single-acting Pump.—One barrel.

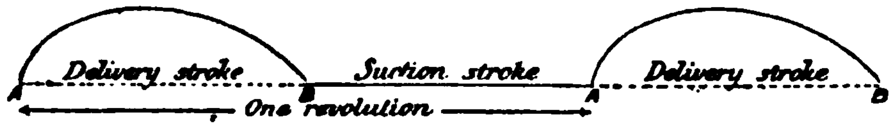


FIG. 590.

Single-acting Pump.—Two barrels, cranks Nos. 1 and 2 opposite one another or at 180°.

Each stroke is precisely the same as in the case above.

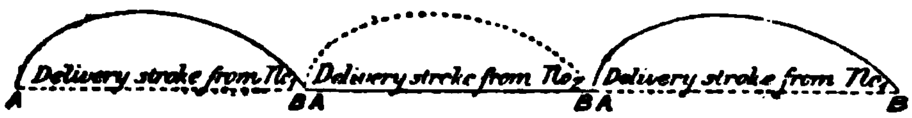


FIG. 591.

Single-acting Pump.—Two barrels, cranks Nos. 1 and 2 at 90°.

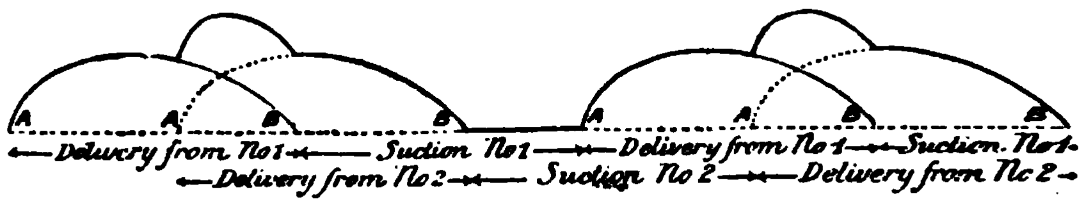


FIG. 592.

Where the two curves overlap, the ordinates are added. It will be observed that the fluctuation is very much greater than before. It should be noted that the second barrel begins to deliver just after the middle of the stroke.

Single-acting Pump.—Three barrels, Nos. 1, 2, 3 cranks at 120°.

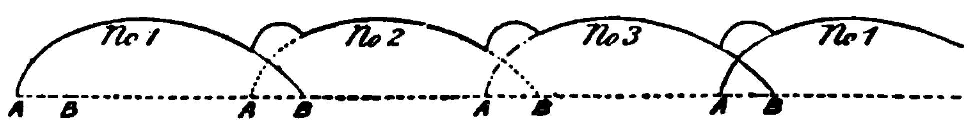


FIG. 593.

It will be observed that the flow is much more constant than in any of the previous cases.

Similar diagrams are easily constructed for double-acting pumps with one or more barrels; it should, however, be noticed that the diagram for the return stroke should be reversed end for end, thus—

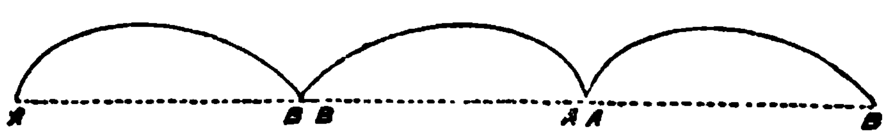


FIG. 594.

We shall shortly see the highly injurious effects that great changes of flow have on a long pipe. In order to partially compensate for it, air-vessels are usually placed on the delivery pipe close to the pump; the air is compressed and the water

stored during the period of maximum flow, to be sent on when the flow falls below the mean. These vessels vary greatly in size according to the requirements; their volume varies from 1 to 30 times the volume of the pump.

Speed of Pumps.—The term "speed of pump" is always used for the mean speed of the piston or plunger. The speed has to be kept down to moderate limits, or the resistance of the water in passing the valves becomes serious, and the shock due to the inertia of the water causes mischief by bursting the pipes or parts of the pump. The following are common maximum speeds for pumps:—

Large pumping engines and mining pumps	100 to 300 ft. min.
Exceptional cases with controlled valves ...	up to 600 ,,
Fire-engines	150 to 250 ,,
Slow-running pumps	30 to 50 ,,
Force pumps on locomotives	up to 900 ,,

The high speed mentioned above for the loco. force-pump would not be possible unless the pipes were very short and the valves very carefully designed; the valves in such cases are often 4-inch diameter with only $\frac{1}{8}$-inch lift. If a greater lift be given, the valves batter themselves to pieces in a very short time; but in spite of all precautions of this kind, the pressures due to the inertia of the water are very excessive. One or two instances will be given shortly.

Volume of Water delivered.—If there were no leakage past the valves and piston of a pump, and if the valves opened and closed instantly, the volume of water delivered would be the volume displaced by the piston. This, however, is never the case; generally speaking, the quantity delivered is less than that displaced by the piston, sometimes only 90 per cent. No figures that will apply to all cases can possibly be given, as it varies with every pump and its speed of working. As might be expected, the leakage is generally greater at very slow than at moderate speeds, and greater with high than with low pressures. This deficiency in the quantity delivered is termed the "slip" of the pump.

With a long suction pipe and a low delivery pressure, it is often found that both small and large pumps deliver more water than the displacement of the piston will account for; such an effect is due to the momentum of the water. During the suction stroke the delivery valves are of course closed, but just before the end of the stroke, when the piston is coming to rest, there may be a considerable pressure in the barrel due

to the inertia of the water (see p. 533), which, if sufficiently great, will force open the delivery valves and pass into the delivery pipes until the pressure in the barrel becomes equal to that in the uptake.

Suction.—We have shown above that the pressure of the atmosphere is equivalent to a head of water of about 34 feet. No pump will, however, lift water so great a height by suction, partly due to the leakage of air into the suction pipes, to the resistance of the suction valves, and to the fact that the water gives off vapour at very low pressures and destroys the vacuum that would otherwise be obtained. Under exceptionally favourable circumstances, a pump will lift by suction through a height of 30 feet, but it is rarely safe to reckon on more than 25 feet for pumps of average quality.

Inertia Effects in Pumps.—The hydraulic ramming action in reciprocating pumps due to the inertia of the water has to be treated in the same way as the inertia effects in water-pressure motors (see p. 499). In a pump the length of either the suction *or* the delivery pipe corresponds to the length of main, L.

As an illustration of the very serious effects of the inertia of the water in pumps, the following extreme case, which came under the author's notice, may be of interest. The force-pumps on the locomotives of one of the main English lines of railway were constantly giving trouble through bursting; an indicator was therefore attached in order to ascertain the pressures set up. After smashing more than one instrument through the extreme pressure, one was ultimately got to work successfully. The steam-pressure in the boiler was 140 lbs., but that in the pump sometimes amounted to 3500 lbs. per square inch; the velocity of the water was about 28 feet per second through the pipes, and still higher through the valves; the air-vessel was of the same capacity as the pump. After greatly increasing the areas through the valves, enlarging the pipes and air-vessels to five times the capacity of the pump, the pressure was reduced to 900 lbs. per sq. inch, but further enlargements failed to materially reduce it below this amount. The friction through the valve pipes and passages will account for about 80 lbs. per sq. inch, and the boiler pressure 140 or 220 lbs. per sq. inch due to both; the remaining 680 lbs. per sq. inch are due to inertia of the water in the pipes, etc.

The indicator diagrams given below were taken from an experimental reciprocating pump in the author's laboratory. The conditions when the diagrams were taken were—

Diameter of plunger	4 inches.
Stroke	6 ,,
Length of connecting-rod	12 ,,
Length of suction and delivery pipes	63 feet each.
Diameter of suction pipe	3 inches.

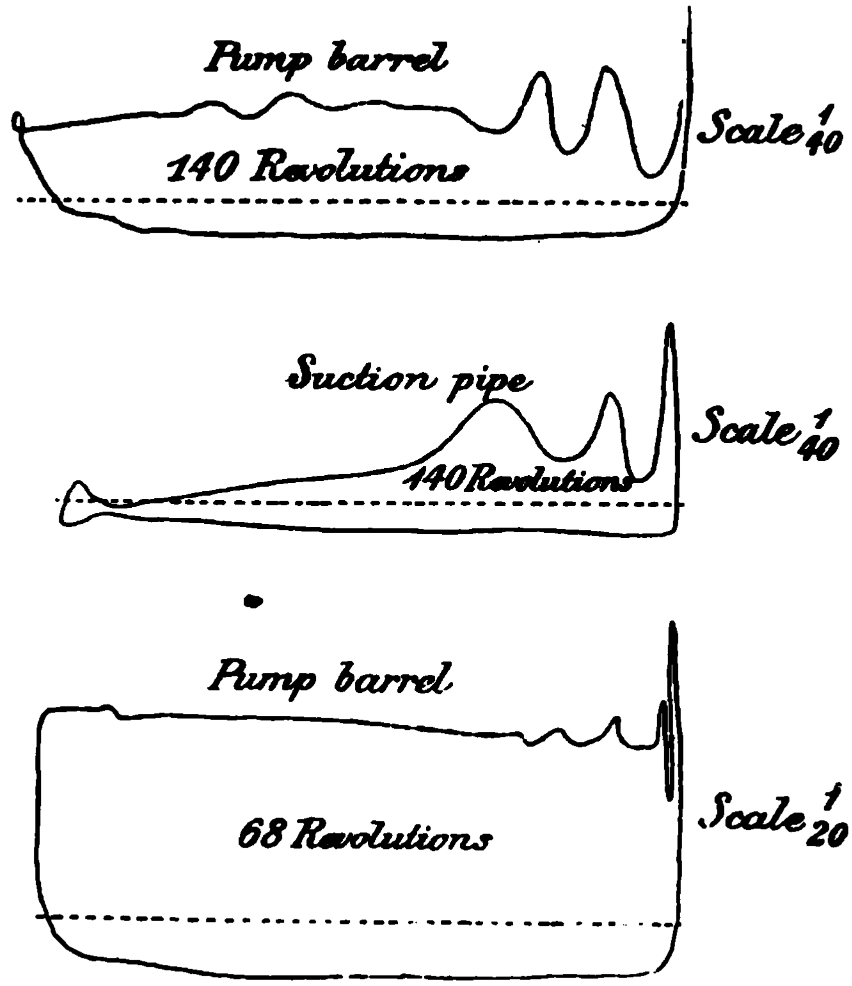


FIG. 595.[1]

Direct-acting Steam-pumps. — The term "direct-acting" is applied to those steam-pumps which have no crank or flywheel, and in which the water piston or plunger is on the same rod as the steam-piston; or, in other words, the steam and water ends are in one straight line. They are usually made double acting, with two steam and two water cylinders. The relative advantages and disadvantages are perhaps best shown thus:

ADVANTAGES.

Compact.
Small number of working parts.
No flywheel or crank-shaft.
Small fluctuation in the discharge.
Almost entire avoidance of shock in the pipes.

DISADVANTAGES.

Steam cannot be used expansively, except with special arrangements, hence—
Wasteful in steam.
Liable to run short strokes.

We have seen that, when a pump is driven by a uniformly revolving crank, the velocity of the piston, and consequently

[1] These diagrams have been reduced to one-half their original size.

the flow, varies between very wide limits, and, provided there is a heavy flywheel on the crank-shaft, the velocity of the piston (within fairly narrow limits) is not affected by the resistance it has to overcome; hence the serious ramming effects in the pipes and pump chambers. In a direct-acting pump, however, the pistons are free, hence their velocity depends entirely on the water-resistance to be overcome, provided the steam-pressure is constant throughout the stroke; therefore the water is very gradually put into motion, and kept flowing much more steadily than is possible with a piston which moves practically irrespectively of the resistance it has to overcome. A diagram of flow from a pump of this character is shown in Fig 596; it is

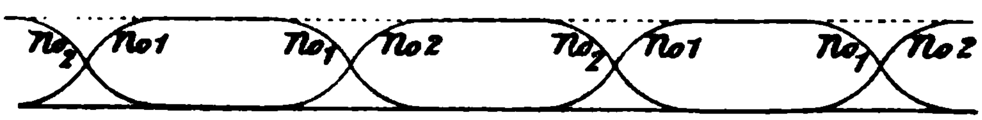


FIG. 596.

intended to show the regularity of flow from a Worthington pump, which owes much of its smoothness of running to the fact that the piston pauses at the end of each stroke.[1]

We will now look at some of the disadvantages of direct-acting pumps, and see how they can be avoided. The reason why steam cannot be used expansively in pumps of this type is because the water-pressure in the barrel is practically constant, hence the steam-pressure must at all parts of the stroke be sufficiently high to overcome the water-pressure; thus, if the steam were used expansively, it would be too high at the beginning of the stroke and too low at the end of the stroke. The economy, however, resulting from the expansive use of steam is so great, that this feature of a direct-acting pump is considered to be a very great drawback, especially for large sizes. Many ingenious devices have been tried with the object of overcoming this difficulty, and some with marked success; we shall consider one or two of them. Many of the devices consist of some arrangement for storing the excess energy during the first part of the stroke, and restoring it during the second part, when there is a deficiency of energy. It need hardly be pointed out that the work done in the steam-cylinder is equal to that done in the water-cylinder together with the friction work of the pump. It is easy enough to see how this is accomplished in the case of a pump fitted with a

[1] Taken from a paper on the Worthington Pump, *Proc. Inst. of Civil Engineers*, vol. lxxxvi. p. 293.

crank and heavy flywheel (see Chapter VI.) by accelerating the wheel during the first part of the stroke, and retarding

FIG. 597.—The line of real pressure has been put in as in Fig. 179.

during the latter part. Now, instead of using a rotating body such as a flywheel to store the energy, a reciprocating body such as a very heavy piston may be used; then the excess work during the early part of the stroke is absorbed in accelerating this heavy mass, which is given back during the latter part of the stroke, while the mass is being retarded, and a very nearly even driving pressure throughout the stroke can be obtained.

The heavy piston is, however, not a practical success for many reasons. A far better arrangement is D'Auria's pendulum pump, which is quite successful. It is in principle the same as

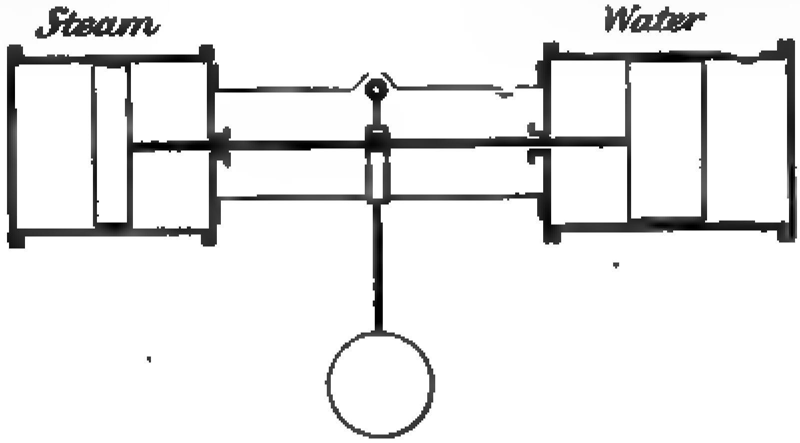


FIG. 598.

the heavy piston, only a much smaller reciprocating (or swinging) mass moving at a higher velocity is used. But far

the most elegant arrangement of this kind[1] is D'Auria's water-compensator, shown in Fig. 599. It consists of ordinary steam

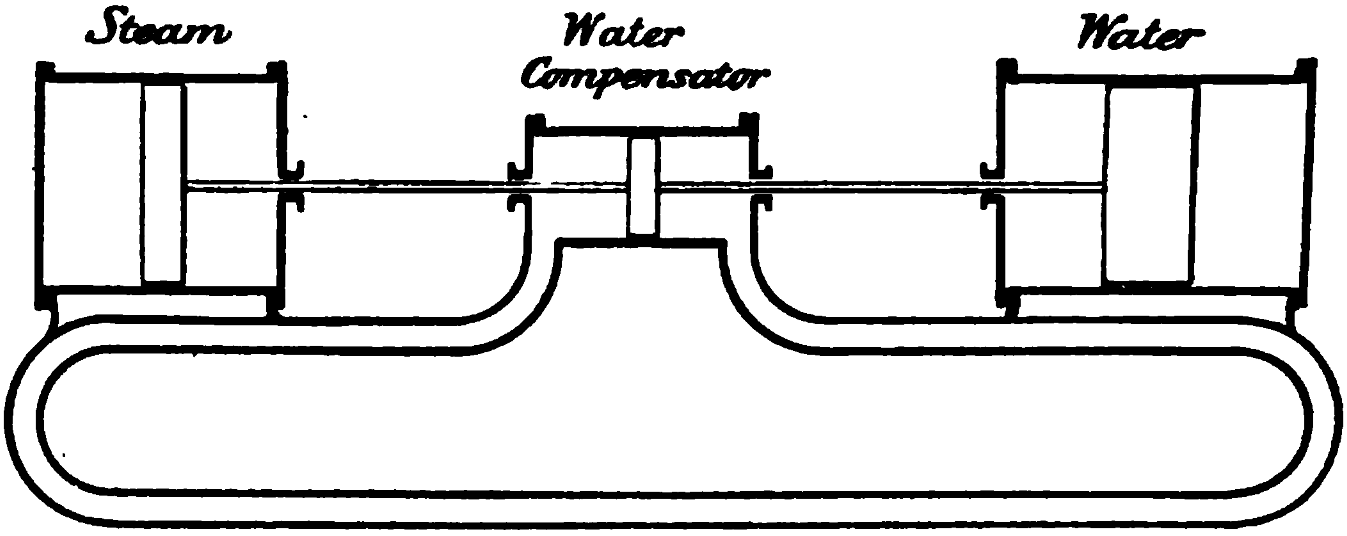


FIG. 599.

and water ends, with an intermediate water-compensator cylinder, which, together with the curved pipe below, is kept full of water. The piston in this cylinder simply causes the water to pass to and fro through the pipe, and thus transfers the water from one end of the cylinder to the other. Its action is precisely similar to that of a heavy piston, or rather the pendulum arrangement shown in Fig. 598, for the area of the pipe is smaller than that of the cylinder, hence the water moves with a higher velocity than the piston, and consequently a smaller quantity is required.

The indicator diagrams in Fig. 600 were taken from a pump of this type. The mean steam-pressure line has been added to represent the work lost in friction, and the velocity curve has been arrived at thus:

Let M = the moving mass of the pistons and water in the compensator, etc., per square inch of piston, each reduced to its equivalent velocity;

V = velocity of piston in feet per second.

Then the energy stored in the mass at any instant is $\frac{MV^2}{2}$; but this work is equal to that done by the steam over and above that required to overcome friction and to pump the water, hence the shaded area $px_1 = \frac{MV_1^2}{2}$, and likewise $px_2 = \frac{MV_2^2}{2}$, where p is the mean pressure acting during the interval x; but

[1] The author is indebted to Messrs. Thorp and Platt, of New York, for the particulars of this pump.

$\frac{M}{2}$ is a constant for any given case, hence $V = \sqrt{px} \times$ constant. When the steam-pressure falls below the mean steam-

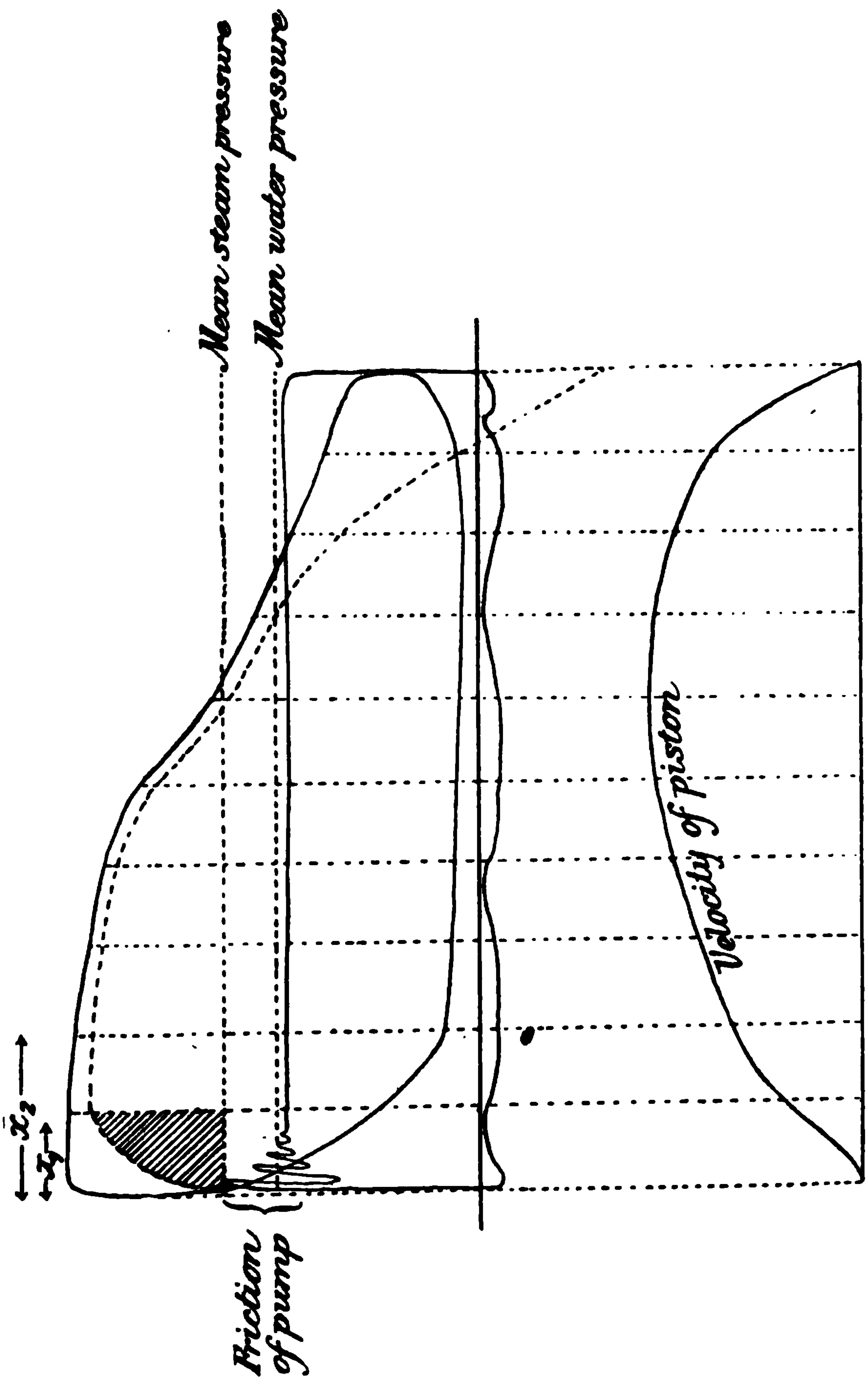


FIG. 600.

pressure line the areas are reckoned as negative. Sufficient data for the pump in question are not known, hence no scale can be assigned.

Another very ingenious device for the same purpose is the

compensator cylinders of the Worthington high duty pump (see Fig. 601). The high and low pressure steam-cylinders are shown to the left, and the water-plunger to the right. Midway between, the compensator cylinders A, A are shown; they are

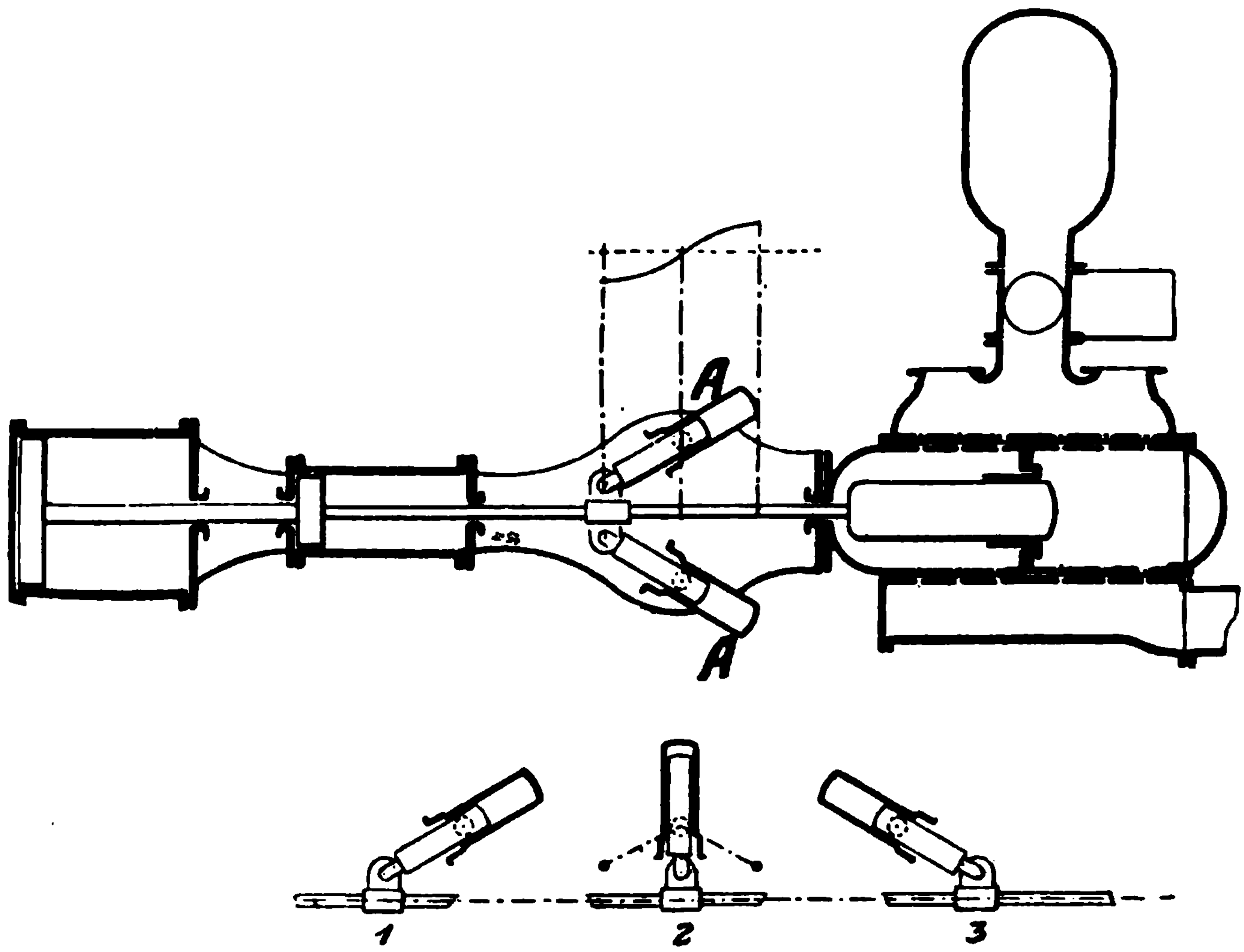


Fig. 601.

kept full of water, but they communicate with a small air-vessel not shown in the figure.

When the steam-pistons are at the beginning of the "out" stroke, and the steam-pressure is higher than that necessary to overcome the water-pressure, the compensator cylinder is in the position 1, and as the pistons move forward the water is forced from the compensator cylinder into the air-vessel, and thus by compressing the air stores up energy; at half-stroke the compensator is in position 2, but immediately the half-stroke is passed, the compensator plunger is forced out by the compressed air in the vessel, thus giving back the stored energy when the steam-pressure is lower than that necessary to overcome the water-pressure; at the end of the stroke the compensator is in position 3. A diagram of the effective compensator pressure on the piston-rods is shown to a small scale above the compensators, and complete indicator diagrams are shown in Fig. 602. No. 1 shows the two steam diagrams reduced to a common scale. No. 3 is the water diagram. In

No. 2 the dotted curved line shows the compensator pressures at all parts of the stroke; the light full line gives the combined diagram due to the high and low pressure cylinders, and the dark line the combined steam and compensator pressure diagram, from which it will be seen that the pressure due to the combination is practically constant and similar to the water-pressure diagram in spite of the variation of the steam-pressure.

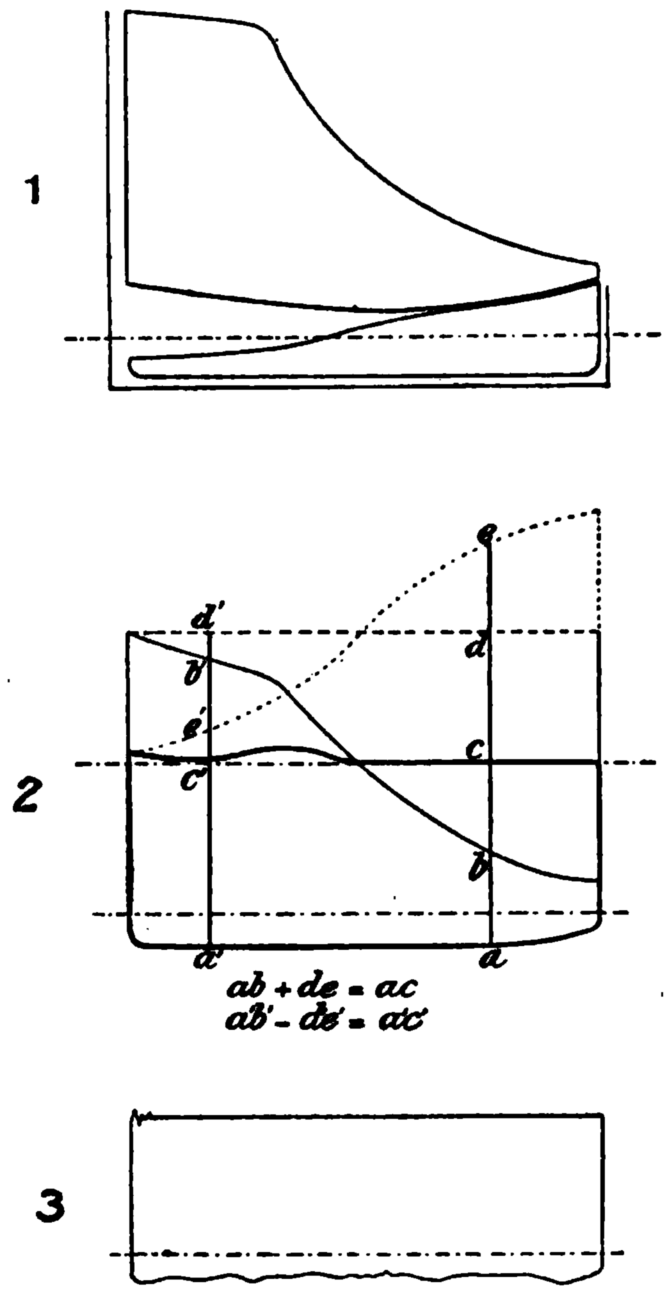


Fig. 602.

We must not leave this question without reference to another most ingenious arrangement for using steam expansively in a pump—the Davy differential pump. It is not, strictly speaking, a direct-acting pump, but on the other hand it is not a flywheel pump. We show the arrangement in Fig. 603.[1] The discs move to and fro through an angle rather less than a right angle.

In Fig. 604 we show a diagram which roughly indicates the manner in which the varying pressure in the steam-cylinder produces a tolerably uniform pressure in the water-cylinder. When the full pressure of steam is on the piston it moves slowly, and at the same time the water-piston moves rapidly; then when the steam expands, the steam piston moves rapidly and the water-piston slowly: thus in any given period of time the work done in each cylinder is approximately the same. We have moved the crank-pin through equal spaces, and shaded alternate strips of the water and steam diagrams. The corresponding positions of the steam-piston have been found by simple construction, and the corresponding strips of the steam diagram have also been shaded; they will be found to be equal to the corresponding water-diagram strips (neglecting

[1] Reproduced by the kind permission of the Editor of the *Engineer* and the makers, Messrs. Hathorn, Davy & Co., Leeds.

friction). It will be observed that by this very simple arrangement the variable steam-pressure on the piston is very nearly balanced at each portion of the stroke by the constant water-

Fig. 603.

pressure in the pump barrel. The pressure in the pump barrel is indicated by the horizontal width of the strips. We

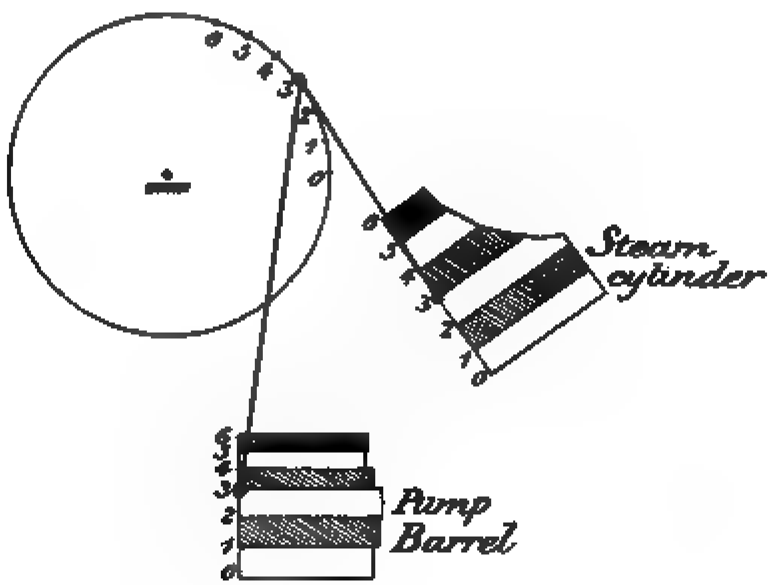


Fig. 604.

have neglected friction and the inertia of the moving masses, but our diagram will suffice to show the principle involved.

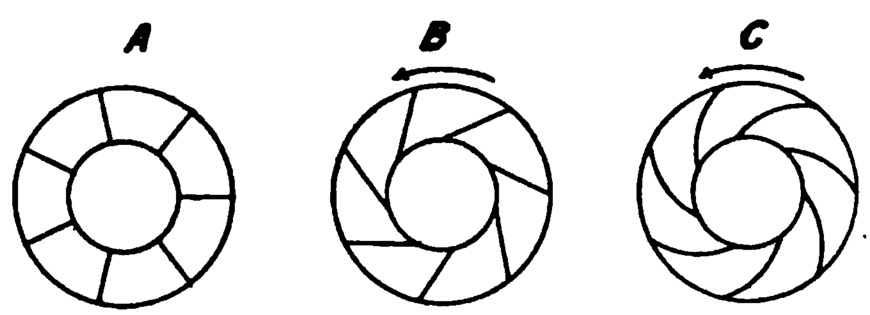
Reversed Reaction Wheel.—If a reaction wheel, such

as that shown in Fig. 570, be placed with the nozzles in water, and the wheel be revolved in the opposite direction from some external source of power, the water will enter the nozzles and be forced up the vertical pipe; the arrangement would, however, be very faulty, and a very poor efficiency obtained. A far better result would be, and indeed has been, obtained by turning it upside down with the nozzles revolving in the same sense as in a motor, and with the bottom of the central chamber dipping in water. The pipes have to be primed, *i.e.* filled with water before starting; then the water is delivered from the nozzles in the same manner as it would be from a motor. The arrangement is inconvenient in many respects. A far more convenient and equally efficient form of pump will be dealt with in the next paragraph.

Reversed Turbine or Centrifugal Pump.—An ordinary turbine simply reversed and turned by some external source of power would not make an efficient pump, but if the guide and wheel blades were suitably curved for a reversed axial-flow turbine, it would form a fairly efficient pump.

The Mather-Reynolds pump is designed on these lines, and gives much higher efficiencies than an ordinary centrifugal pump. For low lifts only one set of guides and wheel is used, for high lifts several low-lift sections are bolted together in series.

The form of centrifugal pump usually adopted is the reversed outward-flow turbine, in which the water enters the "eye" of the pump vanes and passes out more or less radially. In this type of pump there are usually no guide-blades either to direct the entering or leaving stream of water, although in large pumps they are occasionally used. Diagrammatic views of various forms of centrifugal pumps are shown below, with their approximate efficiency per cent.



	A.	B.	C.
With concentric casing ...	25	40	50
With spiral casing ...	30	50	65

Form of Vanes for Centrifugal Pump.—Let the figure represent a small portion of the disc and vanes of a

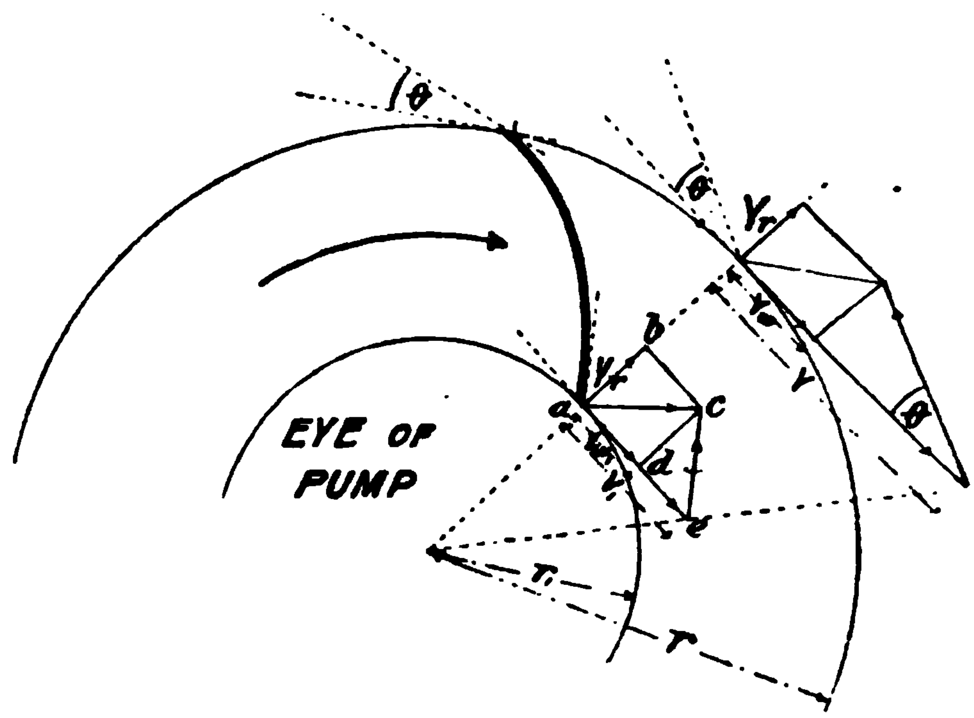


FIG. 605.

centrifugal pump. Symbols with the suffix 1 refer to the inner edge of the vanes.

Let the water have a radial velocity of flow V_r;

„ „ velocity of whirl V_w and V_{w1};

„ velocity of the edges of the vanes be V and V_1.

Then, setting off $ab = V_{r1}$ radially and $ad = V_{w1}$ tangentially, on completing the parallelogram we get *ac* representing the velocity of entry of the water; then, setting off $ae = V_1$ also tangentially, and completing the figure, we get *ec* representing the velocity of the water relative to the disc. Hence, in order that there shall be no shock on entry, the tangent to the first tip of the vane must be parallel to *ec*. In a similar manner, we find the direction of the tangent to the outer tip of the vanes; the angle that this makes to the tangent we term θ. Having found the first and last tangents, we connect them by a smooth curve. The radial velocity V_r is usually made from $\frac{1}{8}$ to $\frac{1}{4}$ $\sqrt{2gH}$, about the same as in a turbine (see p. 517).

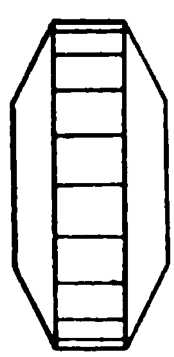


FIG. 606.

In order to keep this velocity constant as the water passes through the fan, the side discs are generally made hyperboloidal (Fig. 606).

The width at the outer radius being one-half the inner, and the radius of the "eye" being one-half the radius of the outer rim, thus the area of the waterway is constant. The area must be regulated to give the velocity of flow mentioned above.

If the pump do not run at the velocity for which it was designed, there will be a loss due to shock on entry; if the first tip of the vanes were made tangential to *fa*, and for the altered conditions of running it should have been tangential to *ga*, the loss of head due to shock will be $\frac{(fg)^2}{2g}$.

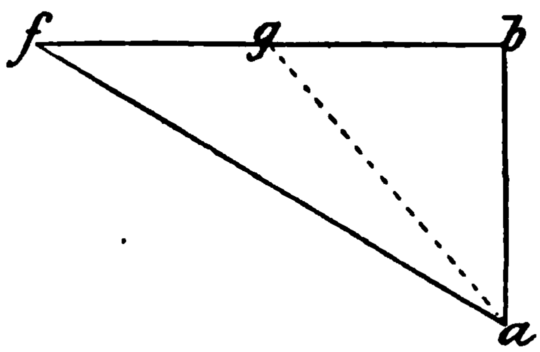


FIG. 607.

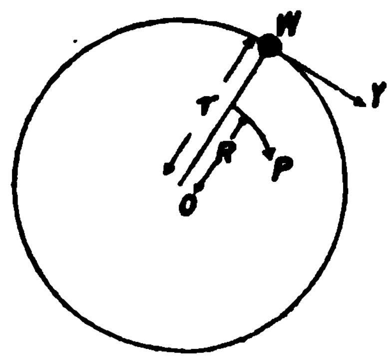


FIG. 608.

Efficiency of Centrifugal Pumps. *General Case.*—In all cases we shall neglect friction and shock losses.

Let a portion of water of weight W lbs. be revolving in a circular path with a linear velocity V_1 feet per second, and at a radius r_1 feet; also let the same portion, after an interval of time t, have a velocity V at a radius r.

$$\text{The momentum of W} = \frac{WV_1}{g} \text{ or } \frac{WV}{g}, \text{ as the case may be}$$

$$\left.\begin{array}{l}\text{Moment of momentum}\\ \text{of W about O}\end{array}\right\} = \frac{WV_1}{g}r_1 \text{ or } \frac{WV}{g}r$$

The water is being turned under the action of a force P acting for the time t.

$$\text{The impulse during the time } t = Pt$$
$$\text{Moment of the impulse about O} = PtR$$

But—

$$\left.\begin{array}{l}\text{Moment of impulse during}\\ \text{the time } t\end{array}\right\} = \text{change of moment of momentum in the time } t$$

$$\text{hence } PtR = \frac{WVr}{g} - \frac{WV_1r_1}{g}$$

Let W = the weight of water passing through the fan per second.

$$\text{Then } t = 1 \text{ second}$$
$$\text{and } \mathrm{PR} = \frac{\mathrm{W}}{g}(\mathrm{V}r - \mathrm{V}_1 r_1)$$

If the point of application of the force P move with a velocity V_0, then the work done per second in foot-lbs. is—

$$\mathrm{PV}_0 = \mathrm{PR}\omega$$

where ω is the angular velocity of the *water*;

$$\text{then } \mathrm{PR}\omega = \frac{\mathrm{W}\omega}{g}(\mathrm{V}r - \mathrm{V}_1 r_1)$$

But ωr is the velocity of whirl V_w of the water at the radius r; likewise with ωr_1—

$$\mathrm{PR}\omega = \frac{\mathrm{W}}{g}(\mathrm{VV}_w - \mathrm{V}_1\mathrm{V}_{w1})$$

The velocity of whirl V_{w1}, on entering the vanes proper, has been imparted to the water by the revolving arms of the fan, hence the energy expended in imparting this velocity has been supplied from the same source as that from which the fan is driven, and therefore, as far as the efficiency is concerned, V_{w1} is zero, and the work done by the disc per pound of water is therefore—

$$\frac{\mathrm{VV}_w}{g} \text{ foot-lbs.}$$

The useful work done per pound of water = H

$$\text{efficiency} = \frac{g\mathrm{H}}{\mathrm{VV}_w}$$

Investigation of the Efficiency of Various Types of Centrifugal Pumps. CASE I. *Pump with no Volute* (Fig. 609).—In this case the water is discharged from the fan at a high velocity into a chamber in which there is no provision made for utilizing its energy of motion, consequently nearly the whole of it is dissipated in eddies before it reaches the discharge pipe. A small portion of the velocity of whirl V_w may be utilized, but it is so small that we shall neglect it.

The whole of the energy due to the radial component of the velocity V_r is necessarily wasted, because it impinges

normally against the casing (see Chap. XVI. Assuming the whole of the energy of motion on leaving the fan to be wasted,

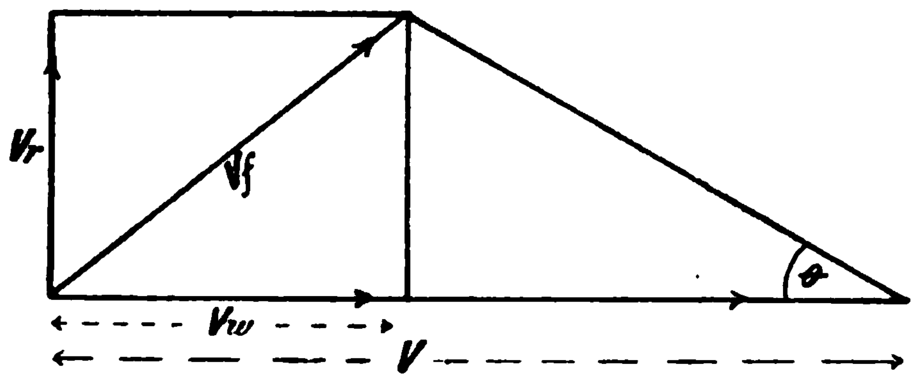


FIG. 609.

we can readily arrive at the efficiency of the pump. We shall neglect friction and minor losses due to shock. In Fig. 609 we have reproduced a small portion of Fig. 605.

The work wasted on leaving the fan per pound of water $= \dfrac{V_r^2 + V_w^2}{2g} = \dfrac{V_f^2}{2g}$

The work done by the disc per pound of water $= \dfrac{VV_w}{g}$

The useful work done in raising the water per pound $= H$

Then we have—

Total work done by the disc = useful work + wasted work

$$\text{or } \frac{VV_w}{g} = H + \frac{V_f^2}{2g}$$

$$\text{whence } V = \frac{2gH + V_f^2}{2V_w}$$

V is the velocity of the tip of the disc required to raise the water a height H.

$$\text{Useful work } H = \frac{2VV_w - V_f^2}{2g}$$

$$\text{efficiency} = \frac{H}{\dfrac{VV_w}{g}} = \frac{2VV_w - V_f^2}{2VV_w} = 1 - \frac{V_f^2}{2VV_w}$$

Neglecting, as we have done, the friction on the disc, the efficiency will be found to increase as the angle θ is decreased (see a paper by Professor Unwin, *I.C.E. Proc.*, vol. liii.). But when friction is taken into account, experiments show

that the best results are obtained when θ lies between $30°$ and $40°$, or when $V = \text{about } 1{\cdot}1\sqrt{2gH}$.

CASE II. *Pump with a Volute* (Fig. 610).—The velocity of the water in the volute is constant, hence the area of the volute at any section must be proportional to the quantity of water passing that section in any given time. To secure the uniform velocity of flow, the form of the volute is arranged thus:

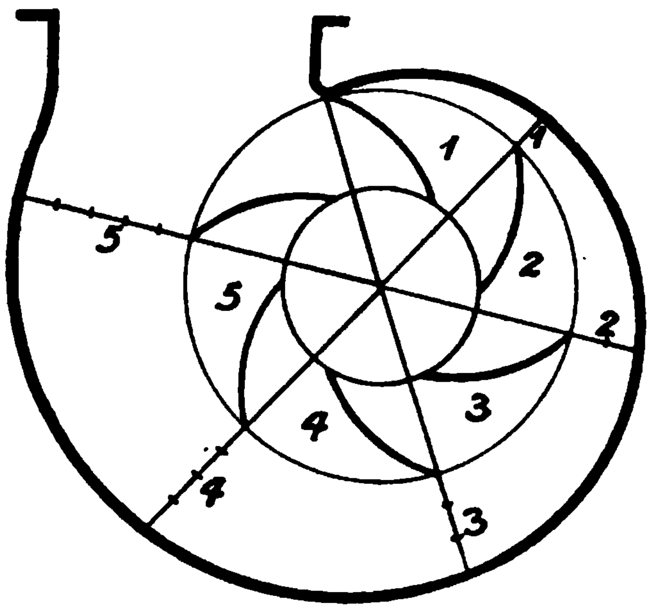


FIG. 610.

The volute is shown in radial sections. The first section has to carry off the water from section 1 of the fan, the second section from sections 1 and 2 of the fan; therefore at the radius 2 it is made twice as wide as at 1; likewise at 3 it is three times as wide as at 1, and so on. As these radii enclose equal angles, it will be seen that the form of the casing is an Archimedian spiral if the width be made constant.

Let V_v be the velocity of whirl in the volute, which is

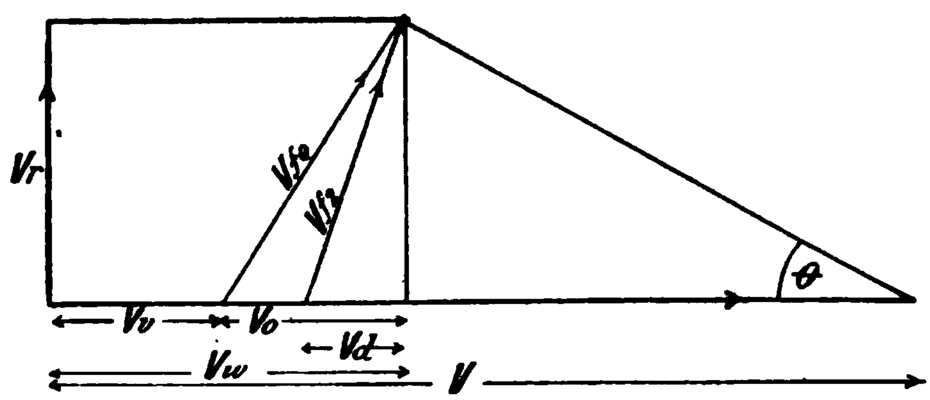


FIG. 611.

generally less than V_w. We shall now deal with the two sources of loss separately, the loss on entry into the volute, and the loss at discharge from the discharge pipe.

The loss on entry into the volute $= \dfrac{V_r^2 + V_0^2}{2g} = \dfrac{V_{f0}^2}{2g}$, which is least when $V_{f0} = V_r$; but then $V_v = V_w$, and the loss of head at discharge is then $\dfrac{V_w^2}{2g}$.

$$\left.\begin{array}{l}\text{Total loss of head at entry in volute and}\\ \quad\text{at discharge}\end{array}\right\} = \frac{V_r^2 + V_w^2}{2g}$$

which is the same as in the case of the pump with no volute; but if the velocity of discharge V_d be made less than V_w by means of a bell mouth on the discharge pipe, the loss at discharge may be very materially reduced, but not entirely overcome (for the limit, see p. 469); we then have—

$$\left.\begin{array}{l}\text{The total loss of head due}\\ \quad\text{to both causes}\end{array}\right\} = \frac{V_r^2 + V_d^2}{2g} = \frac{V_{f2}^2}{2g}$$

$$\text{then } \frac{VV_w}{g} = H + \frac{V_{f2}^2}{2g}$$

$$\text{whence } V = \frac{2gH + V_{f2}^2}{2V_w}$$

$$\text{useful work} = H = \frac{2VV_w - V_{f2}^2}{2g}$$

$$\text{efficiency} = \frac{H}{\frac{VV_w}{g}} = \frac{2VV_w - V_{f2}^2}{2VV_w} = 1 - \frac{V_{f2}^2}{2VV_w}$$

If no bell mouth be used—

$$\left.\begin{array}{l}\text{Loss of head at entry into volute}\\ \quad\text{and at discharge}\end{array}\right\} = \frac{V_{f0}^2 + V_v^2}{2g}$$

$$\text{or } \frac{V_r^2}{2g} + \frac{V_0^2 + V_v^2}{2g}$$

The latter term has its least value when $V_0 = V_v = \frac{V_w}{2}$. The losses then amount to—

$$\frac{V_r^2}{2g} + \frac{V_w^2}{4g} = \frac{V_{f1}^2}{2g}$$

where V_{f1} is the hypotenuse of the triangle in Fig. 611 when $V_1 = \frac{V_w}{\sqrt{2}}$ (omitted in the figure for sake of clearness).

$$\text{Then } \frac{VV_w}{g} = H + \frac{V_{f1}^2}{2g}$$

$$\text{whence } V = \frac{2gH + V_{f1}}{2V_w}$$

$$\text{useful work} = H = \frac{2VV_w - V_{f1}}{2g}$$

$$\text{efficiency} = \frac{H}{\frac{VV_w}{g}} = \frac{2VV_w - V_{f1}^2}{2VV_w} = 1 - \frac{V_{f1}}{2VV_w}$$

When the full advantage is taken of the bell mouth, the efficiency, especially with low lifts, is greater than that possible with the pump arranged as in the last instance.

Case III. *Pump with Whirlpool Chamber.*—We have just seen how the loss of energy due to the velocity of whirl on entry into the volute may be partially avoided; we shall now see how the loss of energy due to shock may be minimized by the use of a whirlpool chamber. The employment of such a chamber for this purpose is due to the late Professor James Thompson.

On p. 479 we showed that the velocity at any ring in a free vortex varied inversely as the radii.

Let r_e = external radius of the whirlpool chamber;

r_i = internal radius of the whirlpool chamber, *i.e* the radius of the fan.

Fig. 612.

Then—

$$\frac{V_e}{V_i} = \frac{r_i}{r_e}$$

$$\text{hence } V_e^2 = \frac{V_i^2 r_i^2}{r_e^2} = n^2 V_i^2$$

where $n = \frac{r_i}{r_e}$.

Take the case given above, in which there is no bell mouth on the discharge pipe. The total loss was then $\frac{V_{f1}^2}{2g}$; but with the whirlpool chamber the velocity V_{f1} is reduced to nV_{f1}, and the total loss is proportionately reduced—

$$\text{Total loss at entry into volute and at discharge} = \frac{n^2 V_{f1}^2}{2g}$$

and by similar reasoning to that given above, we have—

$$V = \frac{2gH + n^2V_{f1}^2}{2V_w}$$

$$H = \frac{2VV_w - n^2V_{f1}^2}{2g}$$

$$\text{efficiency} = 1 - \frac{n^2V_{f1}^2}{2VV_w}$$

In order to show at a glance the relative magnitude of the losses in the various types of pump, we give the following diagram :—

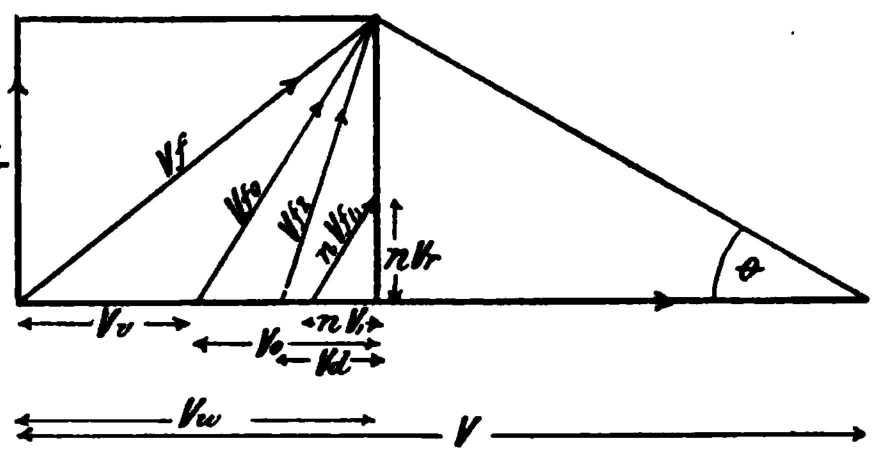


FIG. 613.

V_f is the loss in the case in which there is no volute.

V_{f0} is the loss in the case in which there is a volute, but no bell mouth.

V_{f2} is the loss in the case in which there is a volute with a bell mouth.

nV_{f1} is the loss in the case in which there is a volute with a whirlpool chamber and no bell mouth.

nV_{f1} is drawn parallel to V_{f0}, the sides of the triangle being n times as great; n is usually about 0·7 to 0·8. Smaller values give slightly better results, but the weight and size of the pump have to be increased. When a whirlpool chamber is used, there is no need to have a bell mouth on the discharge pipe, because by means of it the velocity can be reduced as low as we please.

APPENDIX

THE reader should get into the habit of checking the units in any expression he may arrive at, for the sake of preventing errors and for getting a better idea of the quantity he is dealing with.

A mere number or a constant may be struck out of an expression at once, as it in no way affects the units: thus, the length of the circumference of a circle is $2\pi r$ (p. 22); or—

$$\begin{aligned}\text{The length} &= 2\pi\ (\textit{a constant}) \times r\ (\textit{expressed in length units})\\ &= \text{a certain number of length units}\end{aligned}$$

Similarly, the area of a circle $= \pi r^2$ (p. 28); or—

$$\begin{aligned}\text{The area} &= \pi\ (\textit{a constant}) \times r\ (\textit{in length units}) \times r\ (\textit{in length units})\\ &= \text{a certain number of (length units)}^2\end{aligned}$$

Similarly, the volume of a sphere $= \frac{4}{3}\pi r^3$; or—

$$\begin{aligned}\text{The volume} &= \frac{4\pi}{3}\ (\textit{a constant}) \times r\ (\textit{in length units}) \times r\ (\textit{in length units}) \times r\ (\textit{in length units})\\ &= \text{a certain number of (length units)}^3\end{aligned}$$

The same relation holds in a more complex case, *e.g.* the slice of a sphere (p. 44).

$$\begin{aligned}\text{The volume} &= \frac{\pi}{3}[3R(Y_2^2 - Y_1^2) - Y_2^3 + Y_1^3]\\ &= \frac{\pi}{3}(\textit{a constant})\ \{3\ (\textit{a constant}) \times R\ (\textit{in length units})\\ &\qquad [Y_2^2 \text{ (in length units)}^2 - Y_1^2\ (\textit{in length units})^2] - Y_2^3\\ &\qquad (\textit{length units})^3 + Y_1^3\ (\textit{in length units})^3\}\\ &= \textit{a constant}\ \{\textit{a constant}\ [(\textit{length units})^3 - (\textit{length units})^3 - (\textit{length units})^3 + (\textit{length units})^3]\\ &= \textit{a constant} \times (\textit{length units})^3\\ &= \text{a certain number of (length units)}^3\}\end{aligned}$$

One more example may serve to make this question of units quite clear.

The weight of a flywheel rim is given by the following expression (p. 155):—

$$\text{Weight of rim} = \frac{g \,.\, E_n \,.\, R^2}{KY^2R_w^2}$$

$$\text{The weight} = \frac{g\,(\textit{an acceleration}) \times E_n\,(\textit{force} \times \textit{space}) \times R^2\,(\textit{in length units})^2}{K\,(\textit{a constant}) \times V^2\,(\textit{velocity units})^2 \times R_w^2\,(\textit{in length units})^2}$$

$$= \frac{g\,(\textit{acceleration}) \times E_n\left(\textit{mass} \times \dfrac{\textit{space}}{\textit{time}^2} \times \textit{space}\right)}{V^2\left(\dfrac{\textit{space}}{\textit{time}^2}\right)^2}$$

[N.B.—The (length units)^2 cancel out, and the constant is omitted, as it does not affect the units.]

$$\text{The weight} = \frac{\text{acceleration} \times \text{mass} \times \text{space}^2 \times \text{time}^2}{\text{time}^2 \times \text{space}^2}$$

$$= \text{mass} \times \text{acceleration of gravity (see p. 9)}$$

Thus showing that the *form* or the *dimensions* of our flywheel equation is correct.

Operations of Differentiating and Integrating.—Not one engineer in a thousand ever requires to make use of any mathematics beyond an elementary knowledge of the calculus, but any one who wishes to get beyond the Kindergarten stage of the principles that underlie engineering work must have such an elementary knowledge. The labour spent in acquiring sufficient knowledge of the calculus to enable him to solve practically all the problems he is likely to come across, can be acquired in a fraction of the time that he would have to spend in dodging it by some roundabout process. There is no excuse for ignorance of the subject now that we have such excellent and simple books on the calculus for engineers as Perry's, Barker's, Smith's, Miller's, and others. For the benefit of those who know nothing whatever of the subject, the following treatment may be of some assistance; but it must be distinctly understood that it is only given here for the sake of helping beginners to get some idea of what such processes as differentiation and integration mean, not that they may stop when they can mechanically perform them, but rather to encourage them to read up the subject.

Suppose we have a square the length of whose sides is x units, then the area of the square is $a = x^2$. Now let the side be increased by a small amount Δx, and in consequence let the area be increased by a corresponding amount Δa; then we have—

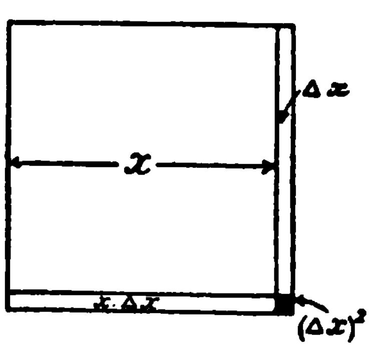


FIG. 614.

$$a + \Delta a = (x + \Delta x)^2 = x^2 + 2x\Delta x + (\Delta x)^2$$

Subtracting our original value of a, we have—

$$\Delta a = 2x\Delta x + (\Delta x)^2$$

Thus by increasing the side of the square by an amount Δx, we

have increased the area by the two narrow strips of length x and of width Δx, and by the small black square at the corner $(\Delta x)^2$.

For many problems we want to know how the area $\left\{\begin{matrix}\text{varies, } \textit{i.e.} \\ \text{increases or} \\ \text{decreases}\end{matrix}\right\}$ with any given $\left\{\begin{matrix}\text{variation, } \textit{i.e.} \\ \text{increase or} \\ \text{decrease}\end{matrix}\right\}$ in the length of the side; or, expressed in symbols, we want to know the value of $\frac{\Delta a}{\Delta x}$, which we can obtain from the expression above thus—

$$\frac{\Delta a}{\Delta x} = 2x + \Delta x.$$

Now, the fraction $\frac{\Delta a}{\Delta x}$ will always have the same value, whether Δx be taken large or small, because Δa varies at the same time as Δx; hence, if we make Δx smaller and smaller, the fraction $\frac{\Delta a}{\Delta x}$ will retain its same value, but the quantity $2x + \Delta x$ will get nearer and nearer to $2x$, and ultimately (which we shall term the limit) it will be $2x$ exactly. We then substitute da and dx for Δa and Δx, and write it thus—

$$\frac{da}{dx} = 2x$$

or, expressed in words, the area of the square varies $2x$ times as fast as the length of the side. This is actually and absolutely true, not a mere approximation, as we shall show later on by some examples.

The fraction $\frac{da}{dx}$ is termed the *differential coefficient* of a with regard to x. Then, having given $a = x^2$, we are said to differentiate a with regard to x when we write $\frac{da}{ax} = 2x$.

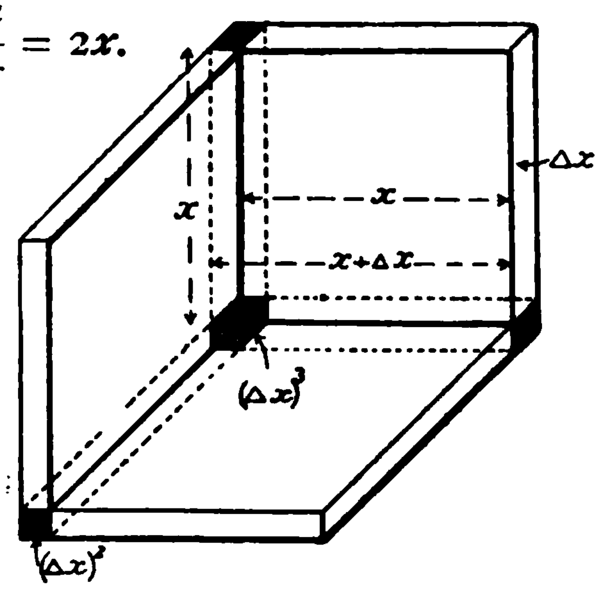


FIG. 615.

We will now go one step further, and consider a cube of side x; its volume $V = x^3$.

As before, let its side be increased by a very small amount Δx, and in consequence let the volume be increased by a corresponding amount ΔV. Then we get—

$$V + \Delta V = (x + \Delta x)^3 = x^3 + 3x^2\Delta x + 3x(\Delta x)^2 + (\Delta x)^3$$

Subtracting our original value of V, we have—

$$\Delta V = 3x^2\Delta x + 3x(\Delta x)^2 + (\Delta x)^3$$

The increase is shown in the figure, viz. three flat portions of area x^2 and thickness Δx, three long prisms of length x and section $(\Delta x)^2$, and one small cube of volume $(\Delta x)^3$. From the above, we have—

$$\frac{\Delta V}{\Delta x} = 3x^2 + 3x\Delta x + \Delta x^2$$

which in the limit (*i.e.* when Δx becomes infinitely small) becomes—

$$\frac{dV}{dx} = 3x^2$$

By a similar process, we can show that if $s = t^4$—

$$\frac{ds}{dt} = 4t^3$$

Likewise, if we expand by the binomial theorem, or if we work out a lot of results and tabulate them, we shall find that the following relation holds :—

$$\text{when } x = y^n$$
$$\frac{dx}{dy} = ny^{n-1}$$

Suppose we had such a relation as—

$$x = Ay^2 + C$$

where A and C are constant quantities, and which of course do not vary ; then, if x increases by an amount Δx, and in consequence y increases by a corresponding amount Δy, we have—

$$\begin{aligned} x + \Delta x &= A(y + \Delta y)^2 + C \\ &= A[y^2 + 2y\Delta y + (\Delta y)^2] + C \end{aligned}$$

Subtracting the original value of x, we have—

$$\Delta x = 2Ay\Delta y + A(\Delta y)^2$$
$$\text{and } \frac{\Delta x}{\Delta y} = 2Ay + A\Delta y$$

which becomes in the limit—

$$\frac{dx}{dy} = 2Ay$$

It should be noticed that the constant C has disappeared, while the constant A is multiplied by the differential of y^2. Similarly for the constants in the following case.

Let $x = Ay^m \pm By^n \pm Cy \pm D$, where the capital letters are

constants. Then by a similar process to the one just given, we get—

$$\frac{dx}{dy} = m\text{A}y^{m-1} \pm n\text{B}y^{n-1} \pm \text{C}$$

It sometimes happens that x increases to a certain value, its maximum, and then decreases to a certain value, its minimum, and possibly increases again, and so on. If at any one instant it is found to be increasing, and the next to be decreasing, it is certain that there must be a point between the two when it neither increases nor decreases; this may occur at either the maximum or the minimum. At that instant we know that $\frac{dx}{dy} = 0$. Then, in order to find when a given quantity has a maximum or a minimum value, we have to find the value of $\frac{dx}{dy}$ and equate it to zero.

Thus, suppose we want to divide a given number N into two parts such that the product is a maximum. Let x be one part, and $\text{N} - x$ the other, and y be the product. Then—

$$y = (\text{N} - x)x = \text{N}x - x^2$$

$$\text{and } \frac{dy}{dx} = \text{N} - 2x = 0 \text{ when } y \text{ is a maximum}$$

$$\text{or } 2x = \text{N}$$

$$\text{and } x = \frac{\text{N}}{2}$$

As an example, let N = 10:

$\text{N} - x$	x	y
9	1	9
8	2	16
7	3	21
6	4	24
5	5	25
4	7	24

and so on.

Thus we see that y has its maximum value when $x = \frac{\text{N}}{2} = 5$.

In some cases we may be in doubt as to whether the value we arrive at is a maximum or a minimum; in such cases the beginner had better assume one or two numerical values near the maximum or minimum, and see whether they increase or decrease, or what is very often a convenient method—to plot a diagram. This question is very clearly treated in either of the books mentioned above.

The process of integration follows quite readily from that of differentiation.

Let the line *ae* be formed by placing end to end a number of

short lines *ab* all of the same length. When the line gets to *c*, let its length be termed L_c, and when it gets to *e*, L_e. The length *ce* we have already said is equal to *ab*. Now, what is *ce*? It is simply the difference between the length of the line *ae* and the line *ac*, or $L_e - L_c$; then, instead of writing in full that *ce* is the difference in length of the two lines, we will write it Δl, *i.e.* the difference in the length after adding or subtracting one of the short lengths. Now, however long or however short the line may be, the difference in the length after adding or subtracting one of the lengths will be Δl. The whole length of the line L_e is the sum of all the short lengths of which it is composed; this we usually write briefly thus:

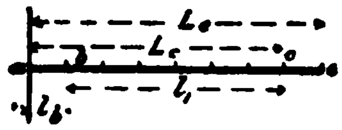


FIG. 616.

$$\Sigma_0^{L_e} \Delta l = \Big[l \Big]_0^{L_e} = L_e$$

or, the sum of (Σ) all the short lengths Δl between the limits when the line is of length L_e and of zero length is equal to L_e. Sometimes the sign of summation is written—

$$\sum_{l=0}^{l=L_e} \Delta l = L_e$$

Similarly, the length L_1 is the sum of all the short lengths between *b* and *c*, and may be written—

$$\sum_{L_b}^{L_c} \Delta l = \Big[l \Big]_{L_b}^{L_c} = L_c - L_b = L_1$$

the upper limit L_c being termed the superior, and the lower limit L_b the inferior limit: the superior limit is always the larger quantity. The lower limit of *l* is subtracted from the upper limit.

In the case of a line, the above statements are perfectly true however large or however small the short lengths Δl are taken, but in some cases which we shall shortly consider it will be seen that if Δl be taken large, an error will be introduced, and that the error becomes smaller as Δl becomes smaller, and it disappears when Δl becomes infinitely small; then we substitute *dl* for Δl, but it still means the difference in length between the two lines L_e and L_c, although *ce* has become infinitely small. We shall now use a slightly different sign of summation, or, as we shall term it, integration. For the Greek letter Σ (sigma) we shall use the old English *s*, viz. $\int$, and—

$$\Sigma_0^{L_e} \Delta l \text{ becomes } \int_0^{L_e} dl = L_e$$

$$\text{and } \sum_{L_b}^{L_c} \Delta l \text{ becomes } \int_{L_b}^{L_c} dl = L_c - L_b = L_1$$

The expression still means that the sum of all the infinitely short lengths dl between the limits of length L_o and L_1 is L_1, which is, of course, perfectly evident from the diagram.

Now that we have explained the meaning of the symbols, we will show a general connection between the processes of differentiation and integration. We have shown that when—

$$x = y^n \quad \ldots \ldots \ldots \quad \text{(i.)}$$

$$\frac{dx}{dy} = ny^{n-1}$$

$$\text{and } dx = ny^{n-1}dy \quad \ldots \ldots \ldots \quad \text{(ii.)}$$

But the sum of all such quantities dx, viz. $\int dx, = x = y^n$ (iii.): hence, substituting the value of dx from (ii.), we have—

$$\int dx = \int ny^{n-1}dy$$

But from (iii.) we have—

$$\int dx = y^n$$

$$\text{hence } \int ny^{n-1}dy = y^n$$

This operation of integrating a function of a variable y may be expressed in words thus :

Add 1 *to the index of the power of y, and divide by the index so increased, and by the differential of the variable.*

Thus—

$$\int ny^{n-1}dy = \frac{ny^{n-1+1}dy}{(n-1+1)dy} = y^n$$

Similarly—

$$\int my^n dy = \frac{my^{n+1}}{n+1}$$

Let $m = 4$, $n = 2$. Then—

$$\int 4y^2dy = \frac{4y^3}{3}$$

or when $M = 3{\cdot}2$, $n = \frac{1}{2}$—

$$\int 3{\cdot}2y^{\frac{1}{2}}dy = \frac{3{\cdot}2y^{1+\frac{1}{2}}}{1+\frac{1}{2}} = \frac{3{\cdot}2y^{1{\cdot}5}}{1{\cdot}5}$$

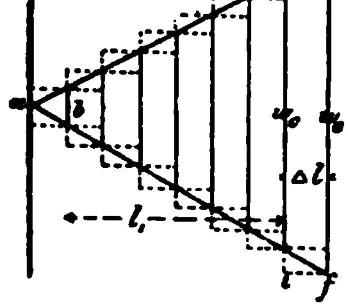


FIG. 617.

In order to illustrate this method of finding the sum of a large number of small quantities we will consider one or two simple cases.

Let the triangle be formed of a number of strips all of equal length, ab or Δl. The width w of each strip varies ; not only is each one of different width from the next, but its own width is not

constant. If we take the greater width of, say, the strip *cefi*, viz. w_e, the area $w_e \Delta l$ is too great, and if we take the smaller width w_c, the area $w_c \Delta l$ will be too small. The narrower we take the strips the nearer will $w_e = w_c$, and when the strips are infinitely narrow, $w_e = w_c = w_0$, and the area will be $w_0 dl$ exactly, and the steps in the stepped figure gradually merge into a straight line, and the figure becomes a triangle.

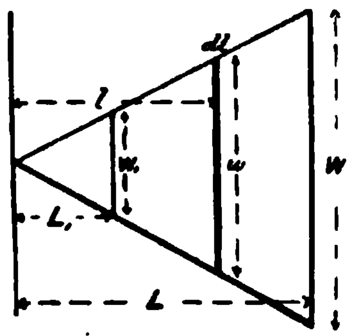


FIG. 618.

The area of the triangle is the sum of the areas of all the small strips $w \,.\, dl$ (Fig. 618).

But $\frac{w}{W} = \frac{l}{L}$, or $w = \frac{Wl}{L}$. Hence the area of the triangle is the sum of all the small strips of area $\frac{Wl}{L}dl$, which we write—

$$\int_{l=0}^{l=L} \frac{Wl}{L}dl, \text{ or } \frac{W}{L}\int_0^L l\,.\,dl = \frac{W}{L} \times \frac{L^2}{2}$$

$$= \frac{WL}{2}$$

The $\frac{W}{L}$ is placed outside the sign of integration, because neither W nor L varies. Now, *from other sources* we know that the area of the triangle is $\frac{WL}{2}$. Thus we have an independent proof of the accuracy of our reasoning.

Suppose we wanted the area of the trapezium bounded by W and W_1, distant L and L_1 from the apex. We have—

$$\frac{W}{L}\int_{L_1}^{L} l\,.\,dl = \frac{W}{L}\left(\frac{L^2 - L_1^2}{2}\right)$$

But here again we know, *from other sources*, that the area of the trapezium is—

$$\frac{WL}{2} - \frac{W_1L_1}{2} = \frac{W}{L}\left(\frac{L^2 - L_1^2}{2}\right)$$

$$\text{since } W_1 = \frac{WL_1}{L}$$

which again corroborates our rule for integration.

By way of further illustrating the method, we will find the volume of a triangular plate of uniform thickness *t*. The volume of the plate is the sum of the volumes of the thin slices of thickness *dl*.

Volume of a thin slice (termed an elementary slice) $\left.\right\} = w \,.\, t \,.\, dl$

$$= \frac{Wl}{L} t \,.\, dl$$

Vol. of whole plate $\dfrac{Wt}{L}\displaystyle\int l \,.\, dl = \frac{Wt}{L}\left(\frac{L^2}{2}\right) = \frac{WtL}{2}$

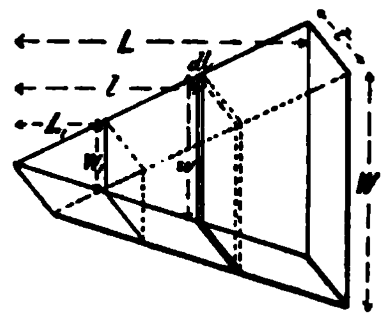


FIG. 619.

a result which we could have obtained by taking the product of the area $\dfrac{WL}{2}$ and the thickness t.

Similarly, the volume of the trapezoidal plate is—

$$\frac{Wt}{L}\int_{L_1}^{L} l \,.\, dl = \frac{Wt}{L}\left(\frac{L^2 - L_1^2}{2}\right)$$

One more case yet remains—that in which t varies, as in a pyramid.

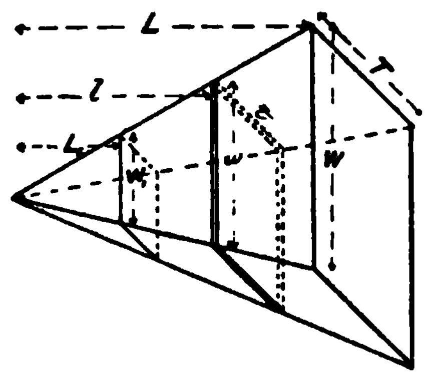


FIG. 620.

Volume of an elementary slice $= w \,.\, t \,.\, dl$

$$w = \frac{Wl}{L} \qquad t = \frac{Tl}{L}$$

hence the volume of slice $= \dfrac{WT}{L^2} l^2 \,.\, dl$

$$\text{Volume of pyramid} = \frac{WT}{L^2}\int l^2 \,.\, dl = \frac{WT}{L^2} \times \frac{L^3}{3}$$
$$= \frac{WTL}{3}$$

We know, however, from other sources, that the volume of a pyramid is one-third the volume of the circumscribing solid. This is easily shown by making a solid and the corresponding pyramid of the same material, and comparing the weights or the volumes of the two; or a cube can be so cut as to form three similar pyramids. In this case the volume of the circumscribing solid is WTL, and the volume of the pyramid $\dfrac{WTL}{3}$. Thus we have another proof of the accuracy of our method of integration. Similarly with the volume of the frustum of a pyramid.

$$\frac{WT}{L^2}\int_{L_1}^{L} l^2 \,.\, dl = \frac{WT}{L^2}\left(\frac{L^3 - L_1^3}{3}\right)$$

Until the reader has had an opportunity of following up the subject a little further, we must ask him to accept on faith the following results :—

$$\int \frac{1}{x} dx = \log_e x$$

$$\int_{x}^{x_1} \frac{1}{x} dx = \log_e x_1 - \log_e x_2 = \log_e \frac{x_1}{x_2}$$

and if $y = \log_e x$

$$\frac{dy}{dx} = \frac{1}{x}$$

Where $\log_e = 2{\cdot}3 \times$ the ordinary log. of the number, $\log_e$ is known as the hyperbolic logarithm. The reader should also read up the elements of conic sections, especially the parabola and hyperbola.

Checking Results.—The more experience one gets the more one realizes how very liable one is to make slips in calculations, and how essential it is to check results. "Cultivate the habit of checking every calculation," is, perhaps, the best advice one can give a young engineer.

First and foremost in importance is the habit of mentally reasoning out roughly the sort of result one would expect to get; this will prevent huge errors such as this. In an examination paper a question was asked as to the deflection of a beam of 10 feet span under a certain load. One student gave it as 34·2 inches, whereas it should have been 0·342 inches—an easy slip to make when working with a slide rule; but if he had thought for one moment, he would have seen that it is impossible to get nearly 3 feet elastic deflection on a 10-feet beam.

If an approximate method of arriving at a result is known, it should be used as a check on the more accurate method; or if there are two different ways of arriving at a result, it should be worked out by both to see if they agree. A graphical method is often an excellent check on an analytical method, or *vice versâ*. For example, suppose the deflection of an irregularly loaded beam has been arrived at by a graphical process; it can be roughly checked by calculating the deflection on the assumption that the load is evenly distributed. If the load be mostly placed in the middle, the graphical process should show a rather greater deflection; but if most of the load be near the two abutments, the graphical process should show a rather smaller deflection. In this way a very good check may be obtained. A little ingenuity will suggest ways of checking every result one obtains.

In solving an equation or in simplifying a vulgar fraction with several terms, rough cancelling can often be done which will enable one by inspection to see the sort of result that should be obtained, and calculations made on the slide rule can be checked by taking the terms in a different order.

Then, lastly, all very important work should be independently checked by another worker, or in some cases by two others.

Real and Imaginary Accuracy.—Some vainly imagine that the more significant figures they use in expressing the numerical

value of a quantity, the greater is the accuracy of their work. This is an exceedingly foolish procedure, for if the data upon which the calculations are based are not known to within 5 per cent., then all figures professing to give results nearer than 5 per cent. are false and misleading. For example, very few steam-engine indicators, with their attendant reducing gears, etc., are accurate to within 3 per cent., and yet one often sees the I.H.P. of an engine given as, say, 2345·67; the possible error here is $2345·67 \times \frac{3}{100} = 70·37$ I.H.P. Thus we are not even certain of the 45 in the 2345; hence the figures that follow are not only meaningless, but liable to mislead by causing others to think that we can measure the I.H.P. of an engine much more accurately than we really can. In the above case, it is only justifiable to state the I.H.P. as 2340, or rather nearer 2350; that is, to give one significant figure more than we are really certain of, and adding o's after the uncertain significant figure. In some cases a large number of significant figures is justifiable; for example, the number of revolutions made by an engine in a given time, say a day. The counter is, generally speaking, absolutely reliable, and therefore the digits are as certain as the millions. Generally speaking, engineering calculations are not reliable to anything nearer than one per cent., and not unfrequently to within 10 or even 20 per cent. The number of significant figures used should therefore vary with the probable accuracy of the available data.

Note on Beams.—Since the printing of Chapter X., a friend has pointed out that a reader may not see the connection between the T lever on p. 290 and a beam. The model in Fig. 336 is really the same as an I-section beam, Fig. 349, in which the strength of the web is neglected.

In the vector polygon for finding the position of the resultant, Fig. 407, the l_1 should be l_2, and l_2 should be l_1.

Experimental Proof of the Accuracy of the Beam Theory.—Each year the students in the author's laboratory make a tolerably complete series of experiments on the strength and elasticity of beams, in order to show the discrepancies (if any) between theory and experiment. The following results are those made during the past session, and are not selected for any special reason, but they serve well to show that there is no material error involved in the usual assumptions made in the beam theory. The deflection due to the shear has been neglected. The gear for measuring the deflection was attached at the neutral plane of the beam just over the end supports, in order to prevent any error due to external causes. If there had been any material error in the theory, the value of Young's Modulus E, found by bending, would not have agreed so closely with the values found by the tension and compression experiments. The material was mild steel; all the specimens were cut from one bar, and all annealed together. The section was approximately 2 inches square, but the exact dimensions are given in each case.

Bending.				Tension.		Compression.	
Central load. $E_b = \frac{WL^3}{48\delta I}$		Two loads dividing span into three equal parts. $E_b = \frac{23WL^3}{648\delta I}$		$E_t = \frac{WL}{Ax}$		$E_c = \frac{WL}{Ax}$	
Depth of sect. (h) 2·026″ Breadth ,, (b) 2·035″ I . . . . . . 1·41 Span (L) . . 36·25″		Depth of sect. (h) 2·020″ Breadth ,, (b) 2·026″ I . . . . . . 1·40 Span (L) . . 36·25″		Sect. 1·066″ × 2·038″ Area (A) 2·172 sq. ins. Length between datum points (L) 10″		Sect. 2·041″ × 2·037″ Area (A) 4·157 sq. ins. Length between datum points (L) 10″	
Load in tons (W).	Deflection in inches (δ).	Load in tons (W).	Deflection in inches (δ).	Load in tons (W).	Strain in inches (x).	Load in tons (W).	Strain in inches (x).
0·1	0·007	0·1	0·004	1	0·0003	2	0·0003
0·2	0·012	0·2	0·009	2	0·0006	4	0·0006
0·3	0·017	0·3	0·014	3	0·0010	6	0·0010
0·4	0·023	0·4	0·019	4	0·0013	8	0·0013
0·5	0·028	0·5	0·023	5	0·0017	10	0·0017
0·6	0·033	0·6	0·028	6	0·0020	12	0·0021
0·7	0·037	0·7	0·032	7	0·0023	14	0·0024
0·8	0·043	0·8	0·037	8	0·0026	16	0·0028
0·9	0·047	0·9	0·042	9	0·0029	18	0·0031
1·0	0·053	1·0	0·047	10	0·0033	20	0·0035
1·1	0·058	1·1	0·051	11	0·0037	22	0·0038
1·2	0·064	1·2	0·056	12	0·0040	24	0·0042
1·3	0·069	1·3	0·060	13	0·0044	26	0·0046
1·4	0·075	1·4	0·065	14	0·0048	28	0·0050
1·5	0·080	1·5	0·070	15	0·0051	30	0·0053
1·6	0·086	1·6	0·075	16	0·0054	32	0·0057
1·7	0·091	1·7	0·079	17	0·0057	34	0·0061
1·8	0·097	1·8	0·083	18	0·0060	36	0·0065
1·9	0·103	1·9	0·088	19	0·0063	38	0·0069
2·0	0·109	2·0	0·093	20	0·0068	40	0·0072
2·1	0·114	2·1	0·098	21	0·0072	42	0·0075
2·2	0·120	2·2	0·103	22	0·0075	44	0·0080
2·3	0·126	2·3	0·108	23	0·0079	46	0·0084
2·4	0·131	2·4	0·112	24	0·0083	48	0·0088
2·5	0·136	2·5	0·117	25	0·0087	50	0·0093
2·6	0·142	2·6	0·122	26	0·0098	52	0·0109
2·7	0·148	2·7	0·126	27	0·0130	53	0·0175
2·8	0·154	2·8	0·133	28	0·0200	54	0·02
2·9	0·160	2·9	0·138	29	0·0300	56	0·10
3·0	0·168	3·0	0·142	30	0·0400	58	0·11
3·1	0·176	3·1	0·147	31	0·14	60	0·13
3·2	0·183	3·2	0·153	32	0·16	62	0·15
3·3	0·194	3·3	0·157	33	0·18	64	0·17
3·4	0·208	3·4	0·162	34	0·20	68	0·21
3·5	0·236	3·5	0·168	36	0·22	72	0·25
3·6	0·57	3·6	0·173	38	0·30	76	0·30

Load in tons (W).	Deflection in inches (δ).	Load in tons (W).	Deflection in inches (δ).	Load in tons (W).	Strain in inches (x).	Load in tons (W).	Strain in inches (x).
3·7	0·82	3·7	0·177	40	0·35	80	0·35
3·8	1·05	3·8	0·180	42	0·42	84	0·40
3·9	1·18	3·9	0·184	44	0·52		
4·0	1·59	4·0	0·189	46	0·62		
4·1	1·91	4·1	0·196	48	0·75		
4·2	2·19	4·2	0·203	50	0·95		
4·3	2·60	4·3	0·210	52	1·30		
4·4	2·77	4·4	0·218	53	1·74		
4·5	3·01	4·5	0·228	53·8	3·01		
4·6	3·40	4·6	0·250	48·0	Broke		
4·7	4·12	4·7	0·300				
4·8	4·97	4·8	0·35				
4·9	6·41	4·9	0·45				
5·0	8·88	5·0	0·50				
		5·1	0·60				
		5·2	0·80				
		5·3	1·10				
		5·4	1·45				
		5·5	2·50				
		6·0	3·65				
		6·5	5·25				
E_b = 13,000 tons sq. inch.		E_b = 12,940 tons sq. inch.		E_t = 13,420 tons sq. inch.		E_c = 13,180 tons sq. inch.	

It will be seen that the E obtained by the bending experiments agrees closely with that found by tension and compression; the mean of the former is 2·5 per cent. lower than the latter. If we had taken into account the deflection due to the shear, the discrepancy would have been even smaller. Further, the E_c in this case is 1·8 per cent. less than the E_t, hence we should expect the E_b to be 0·9 per cent. less than the E_t (see p. 323).

The elastic limit of the tension bar occurs at about 27 tons, corresponding to a tensile stress of 12·43 tons per square inch; in compression it occurs at about 53 tons, corresponding to a compressive stress of 12·75 tons per square inch. For reasons stated on p. 324, we cannot arrive at the true elastic limit in bending.

Tension Tests of Cast Iron.—The following results of elastic tensile tests of cast iron by the author will serve to illustrate the point mentioned on p. 323 (Fig. 380). The reader should plot the results on a sheet of scale paper, and note that the strain does not vary directly as the stress for high stresses; but for such low stresses as are permissible in structures, the proportionality between the stress and the strain holds very closely.

Loads in tons.	Extension in inches.		Loads in tons.	Extension in inches.	
Total.	Old cold blast iron.	Soft cast iron.	Total.	Old cold blast iron.	Soft cast iron.
0·25	0·0005	—	4·25	0·0084	
0·5	0·0010	0·008	4·5	0·0091	0·0103
0·75	0·0015	—	4·75	0·0098	0·0112
1·00	0·0020	0·0018	5·0	0·0105	0·0122
1·25	0·0025	—	5·25	0·0112	0·0133
1·50	0·0029	0·0029	5·5	0·0121	0·0147
1·75	0·0034	—	5·75	0·0130	0·0160
2·0	0·0038	0·0040	6·0	0·0141	0·0176
2·25	0·0043	—	6·25	0·0152	0·0193
2·5	0·0048	0·0049	6·5	0·0169	0·0212
2·75	0·0053	—	6·75	0·0182	0·0235
3·0	0·0058	0·0060	7·0	0·0196	0·0260
3·25	0·0063	—	7·25	0·0218	Broke
3·50	0·0068	0·0072	7·5	0·0276	
3·75	0·0073	—	7·77	Broke	
4·0	0·0078	0·0086			

Note on Struts.—Long cast-iron struts usually fail by tension on the convex side when bent. The formula given on p. 395 may be modified for such failure by tension ; it then becomes—

$$P = \frac{T}{ar^2 - 1}$$

where T = the tensile strength of the material, or rather the *nominal* tensile strength as found by a bending experiment. The values of f and T then vary from 30,000 to 45,000 lbs. per square inch, and E (at the breaking-point) is somewhat lower than in compression; it varies from 11,000,000 to 16,000,000 lbs. per square inch. If a new value of a be obtained with these modified constants, it will be found that the value of P for struts of over 40 diams. will agree fairly well with the value given by the original formula, hence there is no need to complicate matters by introducing this modification.

Note on Water Pressure.—On p. 502, we speak of the pressure set up in a pipe by the gradual uniform closing of a valve. The velocity of the water at each instant can be found thus:

Let h = head of water above the valve in feet ;
L = length of main in feet ;
V = velocity of flow in the main in feet per second ;
K = a constant depending on the roughness of the pipe (see p. 487) ;
D = diameter of the pipe in feet ;

V_1 = velocity of flow through the valve opening ;
n = the ratio of the valve opening to the area of the pipe,
or $n = \dfrac{V}{V_1}$.

The total energy per pound of water is h foot-lbs. This is expended (i.) in overcoming friction, (ii.) in imparting kinetic energy to the water issuing from the valve ; or—

$$h = \frac{LV^2}{KD} + \frac{V_1^2}{2g} \quad \text{(see p. 487)}$$

Substituting the value of $V_1 = \dfrac{V_1}{n}$, we have—

$$V = \sqrt{\frac{h}{\dfrac{L}{KD} + \dfrac{1}{2gn^2}}}$$

The change of velocity at any instant $= \dfrac{dV}{dt}$.

If various values of V be calculated and plotted on a time-velocity curve, it will be seen that the velocity decreases very slowly at first, but at the last it is very rapid. By graphically differentiating this curve, a curve showing the variation of pressure at each instant is obtained. The reader who is interested in this question is recommended to plot such curves.

EXAMPLES

Chapter I.

1. Express 25 tons per square inch in kilos. per square millimetre. *Ans.* 39·4.

2. A train is running at 30 miles per hour. What is its speed in feet per second? *Ans.* 44.

3. What is the angular speed of the wheels in the last question? Diameter = 6 feet. *Ans.* $14\frac{2}{3}$ radians per second.

4. A train running at 30 miles per hour is pulled up gradually and evenly in 100 yards by brakes. What is the negative acceleration? *Ans.* 3·23 feet per second per second.

5. What is the negative acceleration, if the train in Question 4 be pulled up from 40 miles to 5 miles per hour in 80 yards? *Ans.* 7·06 feet per second per second.

6. If the train in Question 5 weighed 300 tons, what was the total retarding force on the brakes? *Ans.* 65·8 tons.

(Check your result by seeing if the work done in pulling up the train is equal to the change of kinetic energy in the train.)

7. Water at 60° Fahr. falls over a cliff 1000 feet high. What would be the temperature in the stream below if there were no disturbing causes? *Ans.* 61·29° Fahr.

8. A body weighing 10 lbs. is attached to the rim of a rotating pulley of 8 feet diameter. If a force of 50 lbs. were required to detach the body, calculate the speed at which the wheel must rotate in order to make it fly off. *Ans.* 61 revolutions per minute.

9. (I.C.E., October, 1897.) Taking the diameter of the earth as 8000 miles, calculate the work in foot-pounds required to remove to an infinite distance from the earth's surface a stone weighing 1 lb. *Ans.* 21,120,000 foot-lbs.

10. (I.C.E., October, 1897). If a man coasting on a bicycle down a uniform slope of 1 in 50 attains a limiting speed of 8 miles per hour, what horse-power must he exert to drive his machine up the hill at the same speed, there being no wind in either case? The weight of man and bicycle together is 200 lbs. *Ans.* 0·17.

11. Find the maximum and minimum speeds with which the body represented in Fig. 2 is moving. Express the result in feet per second and metres per minute.

Ans. Maximum, 13 feet per second, or 238 metres per minute at about the fifth second; minimum, 0·5 foot per second, or 9·15 metres per minute at about the third second.

CHAPTER II.

1. A locomotive wheel 8 feet 6 inches diameter slips 17 revolutions per mile. How many revolutions does it actually make per mile?

Ans. 214·7.

2 In Fig. 13, $C = 200$ feet, $C_1 = 120$ feet. Find the length of the arc by both methods. *Ans.* 256 feet, 253·3 feet.

3. Find the length of a semicircular arc of 2 inches radius by means of the method shown in Fig. 15, and see whether it agrees with the actual length, viz. 6·28 inches.

4. Find the weight of a piece of $\frac{3}{8}$-inch boiler plate $12' \, 3'' \times 4' \, 6''$, having an elliptical manhole $15'' \times 10''$ in it. (One-inch plate weighs 40 lbs. per square foot.) *Ans.* 1629 lbs., or 0·73 ton.

5. Find the area of a regular hexagon inscribed in a circle of 50 feet radius. *Ans.* 6495 square feet.

6. Find the area in square yards of a quadrilateral ABCD. Length of AB = 200 yards, BC = 650 yards, CD = 905 yards, DA = 570 yards, AC = 800 yards.

(N.B.—The length BD can be calculated by working backwards when the area is known, but the work is very long. The tie-lines AC and DB in a survey are often checked by this method.)

Ans. 272,560 square yards.

7. A piece of sheet iron 5 feet wide is about to be corrugated. Assuming the corrugations to be semicircular of 3-inch pitch, what will be the width when corrugated? *Ans.* 3·18 feet.

8. Find the error per cent. involved in using the first formula of Fig. 25, when $H = \frac{r}{2}$ for a circle 5 feet diameter, assuming the second formula to be exact. *Ans.* 5·8.

9. Find the area of the shaded portion of Fig. 351. S = 4 inches; the outlines are parabolic. *Ans.* 2·67 square inches.

10. Cut an irregular-shaped figure out of a piece of thin cardboard or metal with smooth edges. Find its area by—(1) the method shown in Fig. 32; (2) the mean ordinate method; (3) Simpson's method; (4) the planimeter; (5) by weighing.

11. Find the area of Figs. 29. 31, 32, 33, 34 expressed in square inches. *Ans.* (29) 0·29, (31) 0·16, (32) 1·14, (33) 0·83, (34) 0·86.

12. Find the sectional area of a hollow circular column, 12 inches diameter outside, 9 inches diameter inside. *Ans.* 49·48 square inches.

13. Bend a piece of thin wire to form the quarter of an arc of a circle, balance it to find the c. of g., and then find the surface of a hemisphere by the method shown in Fig. 35. Check the result by calculation by the method given in Fig. 36.

14. Find the heating surface of a taper boiler-tube—length, 3 feet; diameter at one end, 5 inches; at the other, 8 inches.

Ans. 5·1 square feet.

15. Find the mean height of an indicator diagram in which the initial height Y is 2 inches, and cut-off occurs at $\frac{1}{4}$ stroke, log 4 = 0·602—(i.) with hyperbolic expansion; (ii.) with adiabatic expansion. ($n = 1·4$.)

Ans. (i.) 1·19 inch; (ii.) 1·02 inch.

16. Find the weight of a cylindrical tank 4 feet diameter inside, and 7 feet deep, when full of water. Thickness of sides $\frac{3}{8}$ inch; the bottom, which is $\frac{1}{2}$ inch thick, is attached by an internal angle $3'' \times 3'' \times \frac{1}{2}''$. There are two vertical seams in the sides, having an overlap of 3 inches; pitch of all rivets, $2\frac{1}{4}$ inches; diameter, $\frac{3}{4}$ inch. Water weighs 62·5 lbs. per cubic foot, and the metal weighs 480 lbs. per cubic foot.
Ans. 7270 lbs.

17. (Victoria, 1893.) A series of contours of a reservoir bed have the following areas:—

At water-level	... ...	4,800,000 sq. yards.
2 feet down	... ...	3,700,000 ,,
4 ,,	... ...	2,000,000 ,,
6 ,,	... ...	700,000 ,,
8·4 ,,	... ...	Zero

Find the volume of the reservoir in cubic yards. *Ans.* 24,300,000.

18. The area of a half-section of a concrete building consisting of cylindrical walls and a dome is 12,000 square feet. The c. of g. of the section is situated 70 feet from the axis of the building. Find the number of cubic yards of concrete in the structure. *Ans.* 203,500.

19. Find the weight of water in a spherical vessel 6 feet diameter, the depth of water being 5 feet. *Ans.* 6545 lbs.

20. In the last question, how much water must be taken out in order to lower the level by 6 inches. *Ans.* 581 lbs.

(Roughly check the result by making a drawing of the slice, measure the mean diameter of it, then calculate the volume by multiplying the mean area by the thickness of the slice.)

21. Calculate the number of cubic feet of water in an egg-ended boiler, diameter, 6 feet; total length, 30 feet; depth of water, 5·3 feet.
Ans. 763.

22. Find the number of cubic feet of solid stone in a heap having a rectangular base $60' \times 18'$, standing on level ground; slope of sides, $1\frac{1}{2}$ to 1 foot; flat-topped height, 4 feet; the voids being 35 per cent.
Ans. 1881.

23. Find the weight of a steel projectile with solid body and tapered point, assumed to be a paraboloid. Material weighs 0·28 lb. per cubic inch; diameter, 6 inches; length over all, 24 inches; length of tapered part, 8 inches. *Ans.* 158 lbs.

24. Find the weight of a wrought-iron anchor ring, 6 inches internal, and 10 inches external diameter, section circular. *Ans.* 22 lbs.

Chapter III.

1. A uniform bar of iron, $\frac{3}{4}$ inch square section and 6 feet long, rests on a support 18 inches from one end. Find the weight required on the short arm at a distance of 15 inches from the support in order to balance the long arm. *Ans.* 13·5 lbs.

2. In the case of a lever such as that shown in Fig. 65, $W_1 = 1\frac{1}{4}$ ton, $l_1 = 28$ inches, $l = 42·5$ inches, $l_2 = 50$ inches, $l_3 = 4$ inches, $w_2 = 1$ ton.

Find W when $w_3 = 0$, and find w_3 when $w_2 l_2$ is clockwise and $l_2 = 12{\cdot}5$ feet. *Ans.* W = 2 tons; w_3 = 50 tons.

(N.B.—In the Buckton testing-machine, w_3 is the load on the bar under test.)

3. A lever safety-valve is required to blow off at 70 lbs. per square inch. Diameter of valve, 3 inches; weight of valve, 3 lbs.; short arm of lever, $2\frac{1}{2}$ inches; weight of lever, 11 lbs.; distance of c. of g. of lever from fulcrum, 15 inches. Find the distance at which a cast-iron ball 6 inches diameter must be placed from the fulcrum. The weight of the ball-hook = 0·6 lb. *Ans.* 35·5 inches.

4. Find the value of w_1 in Fig. 71. W = 30 lbs., w_2 = 4 lbs. *Ans.* 10·2 lbs.

5. Find the position of the c. of g. of a beam section such as that shown in Fig. 355.

Top flange	3 inches wide, $1\frac{1}{2}$ inch thick.
Bottom flange	15 „ wide, $1\frac{3}{4}$ „ thick.
Web	$1\frac{1}{2}$ „ thick.
Total height	18 „

Ans. 5·72 inches from the bottom edge.

6. Find the height of the c. of g. of a T-shaped section from the foot, the top cross-piece being 12 inches wide, 4 inches deep, the stem 3 feet deep, 3 inches wide. *Ans.* 24·16 inches.

(Check by seeing if the moments are equal about a line passing through the section at the height found.)

7. Cut out of a piece of thin card or metal such a figure as is shown in Figs. 78 and 87. Find the c. of g. by calculation or by the graphical process, and check the result by balancing as shown on p. 75.

8. In such a figure as 77, the width of the base is 4 inches, and at the base of the top triangle 1·5 inch. The height of the trapezium = 2 inches, and the total height = 5 inches. Find the height of the c. of g. from the apex. *Ans.* 3·54 inches.

9. A trapezoidal wall has a vertical back and a sloping front face: width of base, 10 feet; width of top, 7 feet; height, 30 feet. What horizontal force must be applied at a point 20 feet from the top in order to overturn it, *i.e.* to make it pivot about the toe? Width of wall, 1 foot; weight of masonry in wall, 130 lbs. per cubic foot. *Ans.* 18,900 lbs.

10. Find the height of the c. of g. of a column 4 feet square and 40 feet high, resting on a tapered base forming a frustum of a square-based pyramid 10 feet high and 8 feet square at the base.

Ans. 20·4 feet from base.

11. Find the position of the c. of g. of a piece of wire bent to form three-fourths of an arc of a circle of radius R.

Ans. On a line drawn from the centre of the circle to a point bisecting the arc, and at a distance 0·3R from the centre.

12. Find the position of the c. of g. of a balance weight having the form of a circular sector of radius R, subtending an angle of 90°.

Ans. 0·6R from centre of circle.

13. Find the second moment (I_0) of a thin door about its hinges: height, 7 feet; width, 4 feet. *Ans.* 457 in feet^4 units.

14. Find the second moment (I) of a rectangular section 9 inches

deep, 3 inches wide, about an axis passing through the c. of g., and parallel to the short side. *Ans.* 183 inch^4 units.

15. Find the second moment (I_0) of a square section of 4-inch side about an axis parallel to one side and 5 inches from the nearest edge. *Ans.* 805·3 inch^4 units.

16. Find the second moment (I_0) of a triangle 0·9 inch high, base 0·6 inch wide as in Fig. 100. *Ans.* 27 inch^4 units.

17. Find the second moment (I) of a triangle 4 inches high and 3 inches base, about an axis passing through the c. of g. of the section and parallel with the base. *Ans.* 3 inch^4 units.

Ditto (I_0) about the base of the triangle. *Ans.* 9 inch^4 units.

Ditto (I_0) of a trapezium (Fig. 103); $B = 3$ inches, $B_1 = 2$ inches, $h = 2$ inches. *Ans.* 7·3 inch^4 units.

Ditto (I_0) ditto, as shown in Fig. 104. *Ans.* 6 inch^4 units.

Ditto (I) ditto, as shown in Fig. 105. *Ans.* 1·64 inch^4 units.

Ditto ditto, by the approximate method in Fig. 106. *Ans.* 1·67 inch^4 units.

18. Find the second moment (I) of a square of 6 inches edge about its diagonal. *Ans.* 108 inch^4 units.

19. Find the second moment (I) of a circle 6 inches diameter about a diameter. *Ans.* 63·6 inch^4 units.

20. Find the second moment (I) of a hollow circle 6 inches external and 4 inches internal diameter about a diameter. *Ans.* 51 inch^4 units.

21. Find the second moment (I) of a hollow eccentric circle, as shown in Fig. 110. $D_e = 6$ inches, $D_i = 4$ inches. The metal is $\frac{1}{2}$ inch thick on the one side, and $1\frac{1}{2}$ inch on the other side. *Ans.* 45·4 inch^4 units.

22. Find the second moment (I) of an ellipse about its minor axis. $D_2 = 6$ inches, $D_1 = 4$ inches. *Ans.* 42·4 inch^4 units.

Ditto, ditto about the major axis. *Ans.* 18·84 inch^4 units.

23. Find the second moment (I) of a parabola about its axis. $H = 6$ inches, $B = 4$ inches. *Ans.* 102·4 inch^4 units.

Ditto, ditto (I_0) about its base, as in Fig. 114. *Ans.* 263·3 inch^4 units.

24. Take an irregular figure, such as that shown in Fig. 115, and find its second moment (I_0) by calculation. Check the result by the graphical method given on p. 96.

25. Find the second polar moment (I_p) of a rectangular surface as shown in Fig. 117. $B = 4$ inches, $H = 6$ inches. *Ans.* 104 inch^4 units.

Ditto, ditto, ditto for Fig. 118. $D = 6$ inches. *Ans.* 127·2 inch^4 units.

Ditto, ditto, ditto for Fig. 119. $D_e = 6$ inches, $D_i = 4$ inches. *Ans.* 102 inch^4 units.

26. Find the second polar moment (I_p) of such a bar as that shown in Fig. 120. $B = 3$ inches, $H = 2$ inches, $L = 16$ inches. *Ans.* 2120 inch^5 units.

27. Find the second polar moment (I_p) for a cylinder as shown in Fig. 121. $D = 16$ inches, $H = 2$ inches. *Ans.* 12,870 inch^5 units.

28. Find the second polar moment (I_p) for a hollow cylinder as shown in Fig. 122. $R_e = 8$ inches, $R_i = 5$ inches, $H = 2$ inches. *Ans.* 10,903 inch^5 units.

29. Find the second polar moment (I_p) of a disc flywheel about its axis.

External radius of rim	= 8 inches
Internal radius of rim	= 6 ,,
Internal radius of web	= 1·5 ,,
Thickness of web	= 0·7 ,,
Internal radius of boss	= 0·75 ,,
Thickness of boss	= 3 ,,
Width of rim	= 3 ,,

Ans. 14,640 inch^5 units.

30. Find the second polar moment (I_p) of a sphere 6 inches diameter about its diameter. *Ans.* 407 inch^5 units.

Find the second polar moment (I_{op}) about a line situated 12 inches from the centre of the sphere. *Ans.* 16,690 inch^5 units.

Find the second polar moment (I_p) of a cone about its axis. H = 12 inches, R = 2 inches. *Ans.* 60·3 inch^5 units.

Chapter IV.

1. Set off on a piece of drawing-paper six lines representing forces acting on a point in various directions. Find the resultant in direction and magnitude by the two methods shown in Fig. 128.

2. Set off on a piece of drawing-paper six lines as shown in Fig. 130. Find the magnitude of the forces required to keep it in equilibrium in the position shown.

3. In the case of a suspension bridge loaded with eight equal loads of 1000 lbs. each placed at equal distances apart. Find the forces acting on each link by means of the method shown in Fig. 131.

4. (I.C.E., October, 1897.) A pair of shear legs make an angle of 20° with one another, and their plane is inclined at 60° to the horizontal. The back stay is inclined at 30° to the plane of the legs. Find the force on each leg, and on the stay per ton of load carried.

Ans. Each leg, 0·88; stay, 1·0 tons.

5. A telegraph wire $\frac{1}{10}$ inch diameter is supported on poles 170 feet apart and dips 2 feet in the middle. Find the pull on the wire.

Ans. 47·4 lbs.

6. A crane has a vertical post DC as in Fig. 135. 8 feet high. The tie AC is 10 feet long, and the jib AB 14 feet long. Find the forces acting along the jib and tie when loaded with 5 tons simply suspended from the extremity of the jib. *Ans.* Tie, 6·3 tons; jib, 8·8 tons.

7. In the case of the crane in Question 6, find the radius, viz. l, by means of a diagram. If the distance x be 4 feet, find the pressures p_1 and p_2. *Ans.* $p_1 = p_2 = 12·2$ tons.

8. Again referring to Question 6. Taking the weight of the tie as 50 lbs., and that of the jib as 350 lbs., and the pulley, etc., at the end of the jib as 50 lbs., find the forces acting on the jib and tie.

Ans. Jib, 8·99; tie, 6·39 tons.

9. In the case of the crane in Question 6, find the forces when the crane chain passes down to a barrel as shown in Fig 137. Let the sloping chain bisect the angle between the tie and the jib. Find W_1 if $l_1 = 5$ feet; also find the force acting along the back stay.

Ans. Jib, 11·4; tie = 3·6; W_1 = 4·9; back stay, 5·8 tons.

10. Each leg of a pair of shear legs is 50 feet long. They are spread out 20 feet at the foot. The back stay is 75 feet long. Find the forces acting on each member when lifting a load of 20 tons at a distance of 20 feet from the foot of the shear legs, neglecting the weight of the structure. *Ans.* Legs, 17·2; back stay, 18·8 tons.

11. In the last question, find the horizontal pull on the screw and the total upward pull on the guides, also the force tending to thrust the feet of the shear legs apart.

Ans. Horizontal pull, 16·7; upward pull, 8·75; thrust, 4·1 tons.

12. A simple triangular truss of 30 feet span and 5 feet deep supports a load of 4 tons at the apex. Find the force acting on each member.

Ans. 6 tons on the tie; 6·32 tons on the rafters.

13. (I.C.E., October, 1897.) Give a reciprocal diagram of the stress in the bars of such a roof as that shown in Fig. 139, loaded with 2 tons at each joint of the rafters; span, 40 feet; total height, 10 feet; depth of truss in the middle, 8 feet $7\frac{1}{2}$ inches.

14. The platform of a suspension foot bridge 100 feet span is 10 feet wide, and supports a load of 150 lbs. per square foot, including its own weight. The two suspension chains have a dip of 20 feet. Find the force acting on each chain close to the tower and in the middle, assuming the chain to hang in a parabolic curve.

Ans. 60,000 lbs. close to tower; 47,000 lbs. in middle.

CHAPTER V.

1. Construct the centrodes of $O_{b.d}$ and $O_{a.c}$ for the mechanism shown in Fig. 148, when the link *d* is fixed.

2. In Fig. 149, if the link *b* be fixed, the mechanism is that of an oscillating cylinder engine, the link *c* representing the cylinder, *a* the crank, and *d* the piston-rod. Construct a diagram to show the relative angular velocity of the piston and the rod for one stroke when the crank rotates uniformly.

3. Construct a curve to show the velocity of the cross-head at all parts of the stroke for a uniformly revolving crank of 1 foot radius—length of connecting rod = 3 feet—and from this curve construct an acceleration curve, scale 2 inches = 1 foot. State the scale of the acceleration curve when the crank makes 120 revolutions per minute.

4. The bars in a four-bar mechanism are of the following lengths: *a* 1·2, *b* 2, *c* 1·9, *d* 1·4. Find the angular velocity of *c* when normal to *d*, having given the angular velocity of *a* as 2·3 radians per second.

5. In Fig. 153, find the weight that must be suspended from the point 6 in order to keep the mechanism in equilibrium, when a weight of 30 lbs. is suspended from the point 5, neglecting the friction and the weight of the mechanism itself.

6. Construct velocity and acceleration curves, such as those shown in Fig. 163, for the mechanism given in Question 4.

7. Construct (i.) an epicycloidal, (ii.) a hypocycloidal, tooth for a spur wheel; width of tooth on pitch line, 1 inch; depth below pitch line, $\frac{3}{4}$ inch; do. above, $\frac{5}{7}$ inch; diameter of pitch circle, 18 inches; do. of rolling circle, 6 inches. Also construct an involute and a cycloidal tooth for a straight rack.

CHAPTER VI.

1. Find the acceleration pressure at each end of the stroke of a vertical inverted high-speed steam-engine when running at 500 revolutions per minute; stroke, 9 inches; weight of reciprocating parts, 110 lbs.; diameter of cylinder, 8 inches; length of connecting-rod, 1·5 feet.

Ans. 54·8 lbs. sq. inch at top; 85·5 lbs. sq. inch at bottom.

2. Find the acceleration pressure at the end of the stroke of a pump having a slotted cross-head as shown in Fig. 156; speed, 100 revolutions per minute; stroke, 8 inches; diameter of cylinder, 5 inches; weight of reciprocating parts, 95 lbs. *Ans.* 5·49 lbs. sq. inch.

3. To what pressure should compression be carried in the case of a horizontal engine running at 60 revolutions per minute; stroke, 4 feet; weight of reciprocating parts per square inch of piston, 3·2 lbs.; length of connecting-rod, 9 feet.

Ans. 9·57 lbs. sq. inch at "in" end; 6·09 lbs. sq. inch at "out" end.

4. In a horizontal engine, such as that shown in Fig. 177, the weights of the parts were as follows:—

Piston	54 lbs.
Piston and tail rods	40 ,,
Both cross-heads	100 ,,
Small end of connecting-rod	18 ,,
Plain part of ,, ...	24 ,,
Air-pump plunger	18 ,,
l_1	3 feet
l_2	1 foot
w_1	20 lbs.
w_2	7 ,,
Diameter of cylinder	8 inches
Revolutions per minute	140
Length of connecting-rod	40 inches
Length of stroke	18 ,,

Find the acceleration pressure at each end of the stroke.

Ans. 27·8, 17·6 lbs. sq. inch.

5. Take the indicator diagrams from a compound steam-engine (if the reader does not happen to possess any, he can frequently find some in the Engineering papers), and construct diagrams similar to those shown in Figs. 179, 181, 182*a*, and 183. Also find the value of m, p. 156, and the weight of flywheel required, taking the diameter of the wheel as five times the stroke.

6. Taking the average piston speed of an engine as 400 feet per minute, and the value of m as $\frac{1}{3}$; the fluctuation of velocity as 1 % on either side of the mean, namely, $K = 0·02$; the radius of the flywheel as twice the stroke; show that the weight of the rim may be taken as 50 lbs. per I.H.P. for a single-cylinder engine, running at 100 revolutions per minute.

7. A two-cylinder engine with cranks at right angles indicates 120 H.P. at 40 revolutions per minute. The fluctuation of energy per stroke is 12 % (*i.e.* $m = 0·12$); the percentage fluctuation of velocity is 2 % on either side of the mean (*i.e.* $K = 0·04$); the diameter of the flywheel is 10 feet. Find the weight of the rim. *Ans.* 4·87 tons.

8. Taking the average value of the piston speed of a gas-engine as 600 feet per minute, and the value of m as $8·5$, the fluctuation of velocity as 2 % (*i.e.* K = 0·04) ; on either side of the mean, the radius of the flywheel as twice the stroke, show that the weight of the rim may be taken at about 300 lbs. per I.H.P. for a single cylinder engine running at 200 revolutions per minute.

9. Find the weight of rim required for the flywheel of a punching machine, intended to punch holes $1\frac{1}{8}$ inch diameter through $1\frac{1}{4}$-inch plate, speed of rim 30 feet per second. *Ans.* 1180 lbs.

10. (I.C.E., October, 1897.) A flywheel supported on a horizontal axle 2 inches in diameter is pulled round by a cord wound round the axle carrying a weight. It is found that a weight of 4 lbs. is just sufficient to overcome the friction. A further weight of 16 lbs., making 20 in all, is applied, and after two seconds starting from rest it is found that the weight has gone down 12 feet. Find the moment of inertia of the wheel.

Ans. 0·014 mass feet^2 units.

(N.B.—For the purposes of this question, you may assume that the speed of the wheel when the weight is released is twice the mean.)

11. (I.C.E., October, 1897.) In a gas-engine, using the Otto cycle, the I.H.P. is 8, and the speed is 264 revolutions per minute. Treating each fourth single stroke as effective and the resistance as uniform, find how many foot.-lbs. of energy must be stored in the flywheel in order that the speed shall not vary by more than one-fortieth of its mean value.

Ans. 30,000 foot-lbs.

12. Find the stress due to centrifugal force in the rim of a cast-iron wheel 8 feet diameter, running at 160 revolutions per minute.

Ans. 431 lbs. sq. inch.

13. (S. & A., 1896.) A flywheel weighing 5 tons has a mean radius of gyration of 10 feet. The wheel is carried on a shaft of 12 inches diameter, and is running at 65 revolutions per minute. How many revolutions will the wheel make before stopping, if the coefficient of friction of the shaft in its bearing is 0·065? (Other resistances may be neglected.)

Ans. 352.

14. Find the bending stress in the middle section of a coupling-rod of rectangular section when running thus: Radius of coupling-crank, 11 inches; length of coupling-rod, 8 feet; depth of coupling-rod, 4·5 inches; width, 2 inches; revolutions per minute, 200. *Ans.* 2·4 tons per sq. inch.

15. Find the bending stress in the rod given in the last question if the sides were fluted to make an I section; the depth of fluting, 3 inches; thickness of web, 1 inch. *Ans.* 1·87 ton sq. inch.

16. Assuming that one-half of the force exerted by the steam on one piston of a locomotive is transmitted through a coupling-rod, find the maximum stress in the rod mentioned in the two foregoing questions. Diameter of cylinder, 16 inches; steam-pressure, 140 lbs. per sq. inch.

Ans. 3·09 tons sq. inch for rectangular rod; 2·92 tons sq. inch for fluted rod.

17. If the rod in Question 14 had been tapered off to each end instead of being parallel in side elevation, find the bending stress at the middle section of the rod. The rod has a straight taper of $\frac{1}{8}$ inch per foot.

Ans. 2·22 tons sq. inch.

18. Find the skin stress due to bending in a connecting-rod when

running thus: Radius of crank, 10 inches; length of rod, 4 feet; diameter of rod, 3 inches; number of revolutions per minute, 120.

Ans. 505 lbs. sq. inch.

19. A railway carriage wheel is found to be out of balance to the extent of 3 lbs. at a radius of 18 inches. What will be the amount of "hammer blow" on the rails when running at 60 miles per hour? Diameter of wheels, 3 feet 6 inches. *Ans.* 354 lbs.

20. In Mr. Hill's paper (I.C.E., vol. civ. p. 265) on locomotive balancing, the following problem is set:—

The radius of the crank = 12 inches; radius of balance-weight, 33 inches; weight of the reciprocating parts, 500 lbs.; weight of the rotating parts, 680 lbs.; distance from centre to centre of cylinders = 24 inches; ditto of balance-weights, 60 inches. Find the weight of the balance-weight if both the reciprocating and rotating weights are fully balanced. *Ans.* 325 lbs.

21. Find the weight and position of the balance-weights of an inside cylinder locomotive working under the following conditions:—

Weight of connecting rod, big end	150 lbs.
,, ,, ,, small end ...	70 ,,
,, ,, ,, plain part ...	180 ,,
,, cross-head and slide blocks ...	170 ,,
,, piston and rod	200 ,,
,, crank-web and pin	330 ,,
Radius of c. of g. of crank-web and pin ...	10 inches.
,, ,, crank (R)	12 ,,
,, ,, balance-weight (R_b) ...	30 ,,
Cylinder centres (C)	24 ,,
Wheel centres (y)	72 ,,

Find the requisite balance-weight and its position for balancing the rotating and two-thirds of the reciprocating parts.

Ans. 259 lbs.; $\theta = 26\frac{1}{2}°$.

22. An English railway company balances for the whole of the reciprocating parts of a locomotive only. Find the amount of each balance-weight for such a condition, taking the values given in Question 20.

Ans. 138 lbs.

23. Find the amount of balance-weight required for the conditions given in Question 21 for an uncoupled outside-cylinder engine. Weight of coupling-crank and pin, 80 lbs.; radius of c. of g. of ditto, 11 inches; R_c = 12 inches. *Ans.* 266 lbs.

24. Find the amount and position of the balance-weight required for a six-wheel coupled inside-cylinder engine, where w from a to b and c to d = 140 lbs., and w from b to c = 300 lbs.; weight of coupling-crank and pin, 80 lbs.; radius of c. of g. of ditto, 11 inches; R_c = 12 inches. The other weights may be taken from Question 21.

Ans. 85 lbs. on trailing and leading wheels; 140 lbs. on driving wheel; $\theta = 55°$.

25. Find the balance-weights for such an engine as that shown in Fig. 196. The weight of the coupling-rod = 250 lbs. For other details see Questions 21 and 24.

Ans. 79 lbs. on trailing wheel; 270 lbs. on driving wheel.

26. Find the speed at which a simple Watt governor runs when the arm

makes an angle of 30° with the vertical. Length of arm from centre of pin to centre of ball = 18 inches. *Ans.* 47·5 revs. per minute.

27. (Victoria, 1896.) A loaded Porter's governor geared to an engine with a velocity ratio 2 has rods and links 1 foot long, balls weighing 2 lbs. each, and a load of 100 lbs.; the valve is full open when the arms are at 30° to the vertical, and shut when at 45°. Find the extreme working speeds of the engine. *Ans.* 92 and 83·2 revs. per minute.

28. (Victoria, 1898.) The balls of a Porter governor weigh 4 lbs. each, and the load on it is 44 lbs. The arms of the links are so arranged that the load is raised twice the distance that the balls rise for any given increase of speed. Calculate the height of this governor for a speed of 180 revolutions per minute. Calculate also the force required to hold the sleeve for an increase of speed of 3 %. *Ans.* Height, 13·05 inches; force, 2·92 lbs.

29. Find the amount the sleeve rises in the case of a simple Watt governor when the speed is increased from 40 to 41 revolutions per minute. The sleeve rises twice as fast as the balls. Find the weight of each ball required to overcome a resistance on the sleeve of 1 lb. so that the increase in speed shall not exceed the above-mentioned amount.
Ans. 2·14 inches; 20 lbs.

30. Find the speed at which a crossed-arm governor runs when the arms make an angle of 30° with the vertical. The length of the arms from centre of pin to centre of ball = 29 inches; the points of suspension are 7 inches apart. *Ans.* 43 revolutions per minute.

31. In a Wilson-Hartnell governor (Fig. 209) r_0 and $r = 6$ inches when the lower arm is horizontal. $K = 12{,}000$, $W = 2$ lbs., $n = 1$. Find the speed at which the governor will float.
Ans. 133 revolutions per minute.

32. In a McLaren governor (Fig. 210), the weight W weighs 60 lbs., the radius of W about the centre of the shaft is 8 inches, the load on the spring $s = 1000$ lbs., the radius of the c. of g. of W about the point J is 9 inches, and the radius at which the spring is attached is 8 inches. Find the speed at which the governor will begin to act.
Ans. 256 revolutions per minute.

33. A simple Watt governor 11 inches high lags behind to the extent of 10 % of its speed when just about to lift. The weight of each ball is 15 lbs. The sleeve moves up twice as fast as the balls. Calculate the equivalent frictional resistance on the sleeve.
Ans. 3·2 lbs.

34. In the governor mentioned in Question 27, find the total amount of energy stored. For what size of engine would such a governor be suitable if controlled by a throttle valve?
Ans. 32·44 foot-lbs. About 35 I.H.P.

35. Find the energy stored in the McLaren governor mentioned in Question 32. There are two sets of weights and springs; the weight moves out 3 inches, and the final tension on the spring is 1380 lbs.
Ans. 529 foot-lbs.

Chapter VII.

1. (S. and A., 1896.) A weight of 5 cwt., resting on a horizontal plane, requires a horizontal force of 100 lbs. to move it against friction. What is the coefficient of friction? *Ans.* 0·179.

2. (S. and A., 1891.) The saddle of a lathe weighs 5 cwt.; it is moved along the bed of the lathe by a rack-and-pinion arrangement. What force, applied at the end of a handle 10 inches in length, will be just capable of moving the saddle, supposing the pinion to have twelve teeth of $1\frac{1}{4}$ inch pitch, and the coefficient of friction between the saddle and lathe bed to be 0·1, other friction being neglected?

Ans. 13·37 lbs.

3. A block rests on a plane which is tilted till the block commences to slide. The inclination is found to be 8·4 inches at starting, and afterwards 6·3 inches on a horizontal length of 2 feet. Find the coefficient of friction when the block starts to slide, and after it has started.

Ans. 0·35; 0·26.

4. If the block in the last question weighed 80 lbs., what would be the acceleration if the plane were kept in the first-mentioned position?

Ans. 2·9 feet per second per second.

5. A block of wood weighing 300 lbs. is dragged over a horizontal metal plate. The frictional resistance is 126 lbs. What would be the probable frictional resistance if it be dragged along when a weight of 600 lbs. is placed on the wooden block, (i.) on the assumption that the value of μ remains constant; (ii.) that it decreases with the intensity of pressure as given on p. 193. *Ans.* (i.) 378 lbs.; (ii.) 308 lbs.

6. In an experiment, the coefficient of friction of metal on metal was found to be 0·2 at 3 feet per second. What will it probably be at 10 feet per second? Find the value of K as given on p. 194.

Ans. 0·11; K = 0·83.

7. (S. and A., 1897.) A locomotive with three pairs of wheels coupled weighs 35 tons; the coefficient of friction between wheels and rails is 0·18. Find the greatest pull which the engine can exert in pulling itself and a train. What is the total weight of itself and train which it can draw up an incline of 1 in 100, if the resistance to motion is 12 lbs. per ton on the level? *Ans.* 6·3 tons; 410 tons.

8. (Victoria, 1896.) Taking the coefficient of friction to be μ, find the angle θ at which the normal to a plane must be inclined to the vertical so that the work done by a horizontal force in sliding a weight w up the plane to a height h may be $2wh$. *Ans.* 19° 25′.

9. A body weighing 1000 lbs. is pulled along a horizontal plane, the coefficient of friction being 0·3; the line of action being (i.) horizontal; (ii.) at 45°; (iii.) such as to give the least pull. Find the magnitude of the pull, and the normal pressure between the surfaces.

Ans. Pull, (i.) 300 lbs., (ii.) 598 lbs., (iii.) 287 lbs. Normal pressure, (i.) 1000 lbs., (ii.) 576 lbs., (iii.) 917 lbs.

10. A horse drags a load weighing 35 cwt. up an incline of 1 in 20. The resistance on the level is 100 lbs. per ton. Find the pull on the traces when they are (i.) horizontal, (ii.) parallel with the incline, (iii.) in the position of least pull.

Ans. (i.) 371·7 lbs., (ii.) 370·67 lbs., (iii.) 370·04.

11. A cotter, or wedge, having a taper of 1 in 8, is driven into a cottered joint with an estimated pressure of 600 lbs. Taking the coefficient of friction between the surfaces as 0·2, find the force with which the two parts of the joint are drawn together, and the force required to withdraw the wedge. *Ans.* 1128 lbs.; 307 lbs.

12. Find the mechanical efficiency of a screw-jack in which the load rotates with the head of the jack in order to eliminate collar friction. Threads per inch, 3; mean diameter of threads, $1\frac{3}{4}$ inch; coefficient of friction, 0·14. Also find the efficiency when the load does not rotate.

Ans. 29·7 per cent.; 17·1 per cent.

13. (Victoria, 1897.) Find the turning moment necessary to raise a weight of W lbs. by a vertical square-threaded screw having a pitch of 6 inches, the mean diameter of the thread being 4 inches, and the coefficient of friction $\frac{1}{6}$. *Ans.* 1·93 W lbs.-inches.

14. (Victoria, 1898.) Find the tension in the shank of a bolt in terms of the twisting moment T on the shank when screwing up, (i.) when the thread is square, (ii.) when the angle of the thread is 55°; neglecting the friction between the head of the bolt and the washer; taking $r =$ the outside radius of the thread, $a =$ the depth of thread, $n =$ the number of threads to an inch, $f =$ the coefficient of friction.

Ans. (i.) $\dfrac{2\pi T n}{1 + 2\pi n f\left(r - \dfrac{a}{2}\right)}$ (ii.) $\dfrac{2\pi T n}{1 + 2{\cdot}26\pi n f\left(r - \dfrac{a}{2}\right)}$

15. The mean diameter of the threads of a $\frac{1}{2}$-inch bolt is 0·45 inch, the slope of the thread 0·07, and the coefficient of friction 0·16. Find the tension on the bolt when pulled up by a force of 20 lbs. on the end of a spanner 12 inches long. *Ans.* 1920 lbs.

16. The rolling resistance of a waggon is found to be 73 lbs. per ton on the level; the wheels are 4 feet 6 inches diameter. Find the value of K.

Ans. 1·76.

17. A 4-inch axle makes 400 revolutions per minute on anti-friction wheels 30 inches diameter, which are mounted on 3-inch axles. The load on the axle is 5 tons; $\mu = 0{\cdot}1$; $\theta = 30°$. Find the horse-power absorbed. *Ans.* 0·16.

18. A horizontal axle 10 inches diameter has a vertical load upon it of 20 tons, and a horizontal pull of 4 tons. The coefficient of friction is 0·02. Find the heat generated per minute, and the horse-power wasted in friction, when making 50 revolutions per minute.

Ans. 155 thermal units; 3·63 H.P.

19. Calculate the length required for the two necks in the case of the axle given in the last question, if placed in a tolerably cool place ($t =$ 0·3). *Ans.* 26 inches.

20. Calculate the horse-power of the bearing mentioned in Question 18 by the rough method given on p. 221, taking the resistance as 2 lbs. per square inch. *Ans.* 4·13 H.P.

21. Calculate the horse-power absorbed by a footstep bearing 8 inches diameter when supporting a load of 4000 lbs., and making 100 revolutions per minute. $\mu = 0{\cdot}03$, with (i.) a flat end; (ii.) a conical pivot; $\alpha = 30°$; (iii.) a Schiele pivot. $t = \dfrac{R_1}{2}$, $R_2 = \dfrac{R_1}{4}$.

Ans. (i.) 0·51; (i.) 1·02; (iii.) 0·72.

22. The efficiency of a single-rope pulley is found to be 94 %. Over how many of such pulleys must the rope pass in order to make it self-sustaining, *i.e.* to have an efficiency of under 50 %. *Ans.* 12 pulleys.

23. In a three-sheaved pulley-block, the pull W on the rope was 110 lbs., and the weight lifted, $W_{\prime\prime}$, was 369 lbs. What was the

mechanical efficiency? and if the friction were 80 °/₀ of its former value when reversed, what would be the reversed efficiency, and what resistance would have to be applied to the rope in order to allow the weight to gently lower? *Ans.* 55·9 per cent.; 36·9 per cent., 22·7 lbs.

(N.B.—The probable friction when reversed may be roughly arrived at thus: The total load on the blocks was $369 + 110 = 479$ lbs. when raising the load. Then, calculating the resistance when lowering, assuming the friction to be the same, we get $369 + 22·7 = 391·7$ lbs. Assuming the friction to vary as the load, we get the friction when lowering to be $\frac{392}{479} = 0·82$. This is only a rough approximation, but experiments on pulleys show that it holds fairly well. In the question above, the experiment gave 23·4 lbs. resistance against 22·7 lbs., and the efficiency as 38 °/₀ against 36·9 °/₀. Many other experiments agree equally well.)

24. (S. and A., 1896.) A lifting tackle is formed of two blocks, each weighing 15 lbs.; the lower block is a single movable pulley, and the upper or fixed block has two sheaves. The cords are vertical, and the fast end is attached to the movable block. Sketch the arrangement, and determine what pull on the cord will support 200 lbs. hung from the movable block, and also what will then be the pressure on the point of support of the upper block. *Ans.* $71\frac{2}{3}$ lbs. pull; $301\frac{2}{3}$ lbs. on support.

25. (S. and A., 1896.) If in a Weston pulley block only 40 per cent. of the energy expended is utilized in lifting the load, what would require to be the diameter of the smaller part of the compound pulley when the largest diameter is 8 inches in order that a pull of 50 lbs. on the chain may raise a load of 550 lbs.? *Ans.* 7·42 inches.

26. Find the horse-power that may be transmitted through a conical friction clutch, the slope of the cone being 1 in 6, and the mean diameter of the bearing surfaces 18 inches. The two portions are pressed together with a force of 170 lbs. The coefficient of friction between the surfaces is 0·15. Revolutions per minute, 200. *Ans.* 4·4.

27. An engine is required to drive an overhead travelling crane for lifting a load of 30 tons at 4 feet per minute. The power is transmitted by means of $2\frac{1}{2}$-inch shafting, making 160 revolutions per minute. The length of the shafting is 250 feet; the power is transmitted from the shaft through two pairs of bevel wheels (efficiency 90 °/₀ each including bearings) and one worm and wheel having an efficiency of 85 °/₀ including its bearings. Taking the mechanical efficiency of the steam-engine at 80 °/₀, calculate the required I.H.P. of the engine. *Ans.* 5·69.

28. (I.C.E., October, 1897.) When a band is slipping over a pulley, show how the ratio of the tensions on the tight and slack sides depends on the friction and on the angle in contact. Apply your result to explain why a rope exerting a great pull may be readily held by giving it two or three turns round a post.

29. (S. and A., 1896.) What is the greatest load that can be supported by a rope which passes round a drum 12 inches in diameter of a crab or winch which is fitted with a strap friction brake worked by a lever, to the long arm of which a pressure of 60 lbs. is applied? The diameter of the brake pulley is 30 inches, and the brake-handle is 3 feet in length from its fulcrum; one end of the brake strap is immovable, being attached to the pin forming the fulcrum of the brake-handle, while the other end of the strap is attached to the shorter arm, 3 inches in length, of the brake-lever. The angle $\alpha = \frac{3\pi}{2}$. The gearing of the crab is as follows: On the shaft

which carries the brake-wheel is a pinion of 15 teeth, and this gears into a wheel of 50 teeth on the second shaft; a pinion of 20 teeth on this latter shaft gears into a wheel with 60 teeth carried upon the drum or barrel shaft. $\mu = 0{\cdot}1$. *Ans.* 4·83 tons.

30. (S. and A., 1896.) The table of a small planing-machine, which weighs 1 cwt., makes six double strokes of $4\frac{1}{2}$ feet each per minute. The coefficient of friction between the sliding surfaces is 0·07. What is the work performed in foot-pounds per minute in moving the table? *Ans.* 423·3.

31. (S. and A., 1897.) A belt laps 150° round a pulley of 3 feet diameter, making 130 revolutions per minute; the coefficient of friction is 0·35. What is the maximum pull on the belt when 20 H.P. is being transmitted and the belt is just on the point of slipping? *Ans.* 898 lbs.

32. (Vict., 1897.) Find the width of belt necessary to transmit 10 H.P. to a pulley 12 inches in diameter, so that the greatest tension may not exceed 40 lbs. per inch of the width when the pulley makes 1500 revolutions per minute, the weight of the belt per square foot being 1·5 lbs., taking the coefficient of friction as 0·25. *Ans.* 8 inches.

33. A strap is hung over a fixed pulley, and is in contact over an arc of length equal to two-thirds of the total circumference. Under these circumstances a pull of 475 lbs. is found to be necessary in order to raise a load of 150 lbs. Determine the coefficient of friction between the strap and the pulley-rim. *Ans.* 0·275.

34. Power is transmitted from a pulley 5 feet in diameter, running at 110 revolutions per minute, to a pulley 8 inches in diameter. Thickness of belt = 0·24 inch; modulus of elasticity of belt, 9000 lbs. per square inch; tension on tight side per inch of width = 60 lbs.; ratio of tensions, 2·3 to 1. Find the revolutions per minute of the small pulley. *Ans.* 792.

35. How many ropes, 4 inches in circumference, are required to transmit 200 H.P. from a pulley 16 feet in diameter making 90 revolutions per minute? *Ans.* 10.

Chapter VIII.

1. Plot a stress-strain and a real stress diagram for the following test: Scales—elastic strains, 2000 times full size; permanent strains, twice full size; loads, 10,000 lbs. to an inch.

Calculate the stress at the elastic limit, the maximum stress; the percentage of extension on 10 inches; the reduction in area; the work done in fracturing the bar. Compare the calculated work with that obtained from the diagram; the modulus of elasticity (mean up to 28,000 lbs.). Original dimensions: Length, 10 inches; width, 1·753 inch; thickness, 0·611 inch. Final dimensions: Length, 12·9 inches; width, 1·472 inch; thickness, 0·482 inch.

Loads in pounds ...		4000	8000	12,000	16,000	20,000
Extensions in inches		0·0009	0·0020	0·0033	0·0044	0·0056
	24,000	28,000	32,000	34,000	36,000	40,000
	0·0070	0·0082	0·0103	0·016	0·07	0·19
	44,000	48,000	52,000	56,000	59,780	54,900
	0·30	0·47	0·75	1·36	2·5	2·9

Ans. Elastic limit, 15 tons square inch. Maximum stress, 24·92. Extension, 29 per cent. on 10 inches. Reduction, 34 per cent. Work (by calculation), 6·27 inch-tons per cubic inch. Modulus of elasticity, 31,000,000 lbs. per square inch; 13,840 tons per square inch.

2. (I.C.E., October, 1897.) Distinguish between stress and strain. An iron bar 20 feet long and 2 inches in diameter is stretched $\frac{1}{20}$ of an inch by a load of 7 tons applied along the axis. Find the intensity of stress on a cross-section, and the coefficient of elasticity of the material (E).

Ans. Stress, 2·23 tons per sq. inch; E = 10,700 tons per sq. inch.

3. (I.C.E., October, 1897.) A bar 4" × 2" in cross-section is subjected to a longitudinal tension of 40 tons. Find the normal and shearing stresses on a section inclined at 30° to the axis of the bar.

Ans. Normal, 3·75 tons per square inch; tangential, 2·16 tons per square inch.

4. A bar of steel 4" × 1" is rigidly attached at each end to a bar of brass 4" × $\frac{3}{8}$"; the combined bar is then subjected to a load of 23 tons. Find the load taken by each bar. E for steel = 13,000 tons per square inch; brass, 4000 tons per square inch.

Ans. Load on steel bar, 17·93 tons; load on brass bar, 2·07 tons.

5. The nominal tensile stress (reckoned on the original area) of a bar of steel was 32·4 tons per square inch, the reduction in area at the point of fracture was 54 %. What would be the approximate tensile strength of hard drawn wire made from such steel? *Ans.* 70 tons per square inch.

6. Plot a stress-strain and a real stress diagram for the following compression test of a specimen of copper. Scale of loads, 5 tons per inch; strain twice full size.

Loads in tons	13	14	15	16	18	20	22	24	26	28	30	32	34	36	38	40	42	44	46	48	50
Length of specimen in inches	2·50	2·47	2·29	2·19	2·02	2·86	1·73	1·61	1·52	1·43	1·35	1·30	1·23	1·17	1·11	1·08	1·02	0·99	0·96	0·92	0·90

Original length, 2·42 inches; diameter, 2·968 inches.

7. (S. and A., 1896.) A bar of iron is at the same time under a direct tensile stress of 5000 lbs. per square inch, and to a shearing stress of 3500 lbs. per square inch. What would be the resultant equivalent tensile stress in the material? *Ans.* 6800 lbs. per square inch.

8. (I.C.E., October, 1897.) Explain what is meant by Poisson's Ratio. A cube of unit length of side has two simple normal stresses p_1 and p_2 on pairs of opposite faces. Find the length of the sides of the cube when deformed by the stresses (tensile).

Ans. $1 + \frac{1}{E}\left(p_1 - \frac{p_2}{n}\right)$; $1 + \frac{1}{E}\left(p_2 - \frac{p_1}{n}\right)$; $1 - \frac{1}{En}\left(p_1 + p_2\right)$

9. Find the pitch of the rivets for a double row lap joint. Plates $\frac{1}{2}$ inch thick; rivets, 1 inch diameter; clearance, $\frac{1}{16}$ inch; thickness of ring damaged by punching, $\frac{1}{16}$ inch; $f_t = 23$ tons per square inch; $f_r = 25$ tons per square inch. *Ans.* 4·27 inches.

10. Calculate the bearing pressure on the rivets when the above-mentioned joint is just about to fracture. *Ans* 33·3 tons per square inch.

11. Calculate the pitch of rivets for a double cover plate riveted joint with diamond riveting as in Fig. 321, the one cover plate being wide enough to take three rows of rivets, but the other only two rows; thickness of plates, $\frac{7}{8}$ inch; drilled holes; material steel; the pitch of the outer row of the narrow cover plate must not exceed six times the diameter of the rivet. What is the efficiency of the joint and the bearing pressure?

Ans. 10·84 inches outer row, 5·42 inches inner row; 88·7 per cent., 44·2 tons per square inch.

12. Two lengths of a flat tie-bar are connected by a lap riveted joint. The load to be transmitted is 50 tons. Taking the tensile stress in the plates at 5 tons per square inch, the shear stress in the rivets at 4 tons per square inch, and the thickness of the plates as $\frac{3}{4}$ inch; find the diameter and the number of rivets required, also the necessary width of bar for both the types of joint as shown in Figs. 329 and 330. What is the efficiency of each, and the working bearing pressure?

Ans. 0·94 inch. 18 are sufficient, but 20 must be used for convenience; 17 inches and $14\frac{1}{4}$ inches; 78 and 93·4 per cent.; 3·6 tons per square inch.

13. Find the thickness of plates required for a boiler shell to work at a pressure of 160 lbs. per square inch: diameter of shell, 8 feet; efficiency of riveted joint, 89 %; stress in plates, 5 tons per square inch.

Ans. 0·77 inch, or say $\frac{25}{32}$ inch.

14. Find the maximum and minimum stress in the walls of a thick cylinder; internal diameter, 8 inches; external diameter, 14 inches; internal fluid pressure, 2000 lbs. per square inch.

Ans. 4670 lbs. per square inch; 1520 lbs. per square inch.

15. A thick cylinder is built up in such a manner that the initial tensile stress on the outer skin and the compressive stress on the inner skin are both 3 tons per square inch. Calculate the resultant stress on both the outer and the inner skin when under pressure. Internal fluid pressure, 4·5 tons per square inch. *Ans.* 4·2 and 4·5 tons per square inch.

Chapter IX.

1. (Victoria, 1897.) Find the greatest stress which occurs in the section of a beam resting on two supports, the beam being of rectangular section, 12 inches deep, 6 inches wide, carrying a uniform load of 5 cwt. per foot run. *Ans.* 3·12 tons square inch.

2. Rolled joists are used to support a floor which is loaded with 150 lbs. per square foot including its own weight. The pitch of the joists is 3 feet; span, 20 feet; skin stress, 5 tons per square inch. Find the required Z and a suitable section for the joists taking the depth at not less than $\frac{1}{25}$ of the span. *Ans.* $Z = 24{\cdot}1$; say $10'' \times 5'' \times \frac{1}{2}''$.

3. A rectangular beam, 9 inches deep, 3 inches wide, supports a load of $\frac{1}{2}$ ton, concentrated at the middle of an 8-foot span. Find the maximum skin stress. *Ans.* 0·3 ton square inch.

4. A beam of circular section is loaded with an evenly-distributed load of 200 lbs. per foot run; span, 14 feet; skin stress, 5 tons per square inch. Find the diameter. *Ans.* 3·77 inches.

5. Calculate approximately the safe central load for a simple web riveted girder, 6 feet deep; flanges, 18 inches wide, $2\frac{1}{8}$ inches thick.

The flange is attached to the web by two $4'' \times \frac{1}{2}''$ angles. Neglecting the strength of the web, and assuming that the section of each flange is reduced by two rivet-holes $\frac{7}{8}$ inch diameter passing through the flange and angles. Span of girder, 50 feet; stress in flanges, 5 tons per square inch.

Ans. 110 tons.

6. Find the relative weights of beams of equal strength having the following sections: Rectangular, $h = 3b$; Square; circular; rolled joist, $h_1 = 2b_1 = 12t$.

Ans. Rectangular, 1·00; square, 1·44; circular, 1·61; joist, 0·54.

7. Find the safe distributed load for a cast-iron beam of the following dimensions: Top flange, $3'' \times 1''$; bottom flange, $8'' \times 1{\cdot}5''$; web, 1·25 inch thick; total depth, 10 inches; with (i.) the bottom flange in tension, (ii.) when inverted; span, 12 feet; skin stress, 3000 lbs. per square inch.

Ans. (i.) 5·7; (ii.) 3·2 tons.

8. In the last question, find the safe central load for a stress of 3000 lbs. per square inch, including the stress due to the weight of the beam. The beam is of constant cross-section. *Ans.* (i.) 2·65; (ii.) 1·4 ton.

9. (S. and A., 1896.) If a bar of cast iron, 1 inch square and 1 inch long, when secured at one end, breaks transversely with a load of 6000 lbs. suspended at the free end, what would be the safe working pressure, employing a factor of safety of 10, between the two teeth which are in contact in a pair of spur-wheels whose width of tooth is 6 inches, the depth of the tooth, measured from the point to the root, being 2 inches, and the thickness at the root of the tooth $1\frac{1}{2}$ inch? (Assume that one tooth takes the whole load.) *Ans.* 4050 lbs.

10. (S. and A., 1897.) Compare the resistance to bending of a wrought-iron I section when the beam is placed like this, I, and like this, ⊢⊣. The flanges of the beam are each 6 inches wide and 1 inch thick, and the web is $\frac{3}{4}$ inch thick and measures 8 inches between the flanges.

Ans. 4·56 to 1.

11. A trough section, such as that shown in Fig. 350, is used for the flooring of a bridge; each section has to support a uniformly distributed load of 150 lbs. per square foot, and a concentrated central load of 4 tons. Find the span for which such a section may be safely used. Skin stress = 5 tons per square inch; pitch of corrugation, 2 feet; depth, 1 foot; width of flange (B, Fig. 350) = 8 inches; thickness = $\frac{3}{8}$ inch.

Ans. 19 feet.

12. A $4'' \times 4'' \times \frac{1}{2}''$ ⊥ section is used for a roof purlin, the load being applied on the flanges; the span is 12 feet; the evenly distributed load is 100 lbs. per foot run. Find the skin stress at the top and the bottom of the section.

Ans. Top, 4·9 tons square inch compression; bottom, 2·1 tons square inch tension.

13. A triangular knife-edge of a weighing-machine overhangs $1\frac{1}{2}$ inch, and supports a load of 2 tons (assume evenly distributed). Taking the triangle to be equilateral, find the requisite size for a tensile stress at the apex of 10 tons per square inch. *Ans.* S = 1·53 inch.

14. A cast-iron water main, 30 inches inside diameter and 32 inches outside, is unsupported for a length of 12 feet. Find the stress in the metal due to bending. *Ans.* 180 lbs. square inch.

15. In the case of a tram-rail, the area A of one part of the modulus figure is 4·12 square inches, and the distance D between the two centres

of gravity is 5·55 inches; the neutral axis is situated at a distance of 3·1 inches from the skin of the bottom flange. Find the I and Z.

Ans. 70·9; 22·86.

16. Find the Z for the sections given on pp. 314, 315, and 316, which are drawn to the following scales: Figs. 363, 364, and 365, 4 inches = 1 foot; Fig. 366, 3 inches = 1 foot; Fig. 367, 1 inch = 1 foot; Fig. 368, $\frac{1}{4}$ full size; Fig. 369, $\frac{1}{3\cdot5}$ full size; Fig. 370, $\frac{1}{2}$ inch = 1 foot. (You may assume that the crosses correctly indicate the c. of g. of each figure.) The areas must be measured by a planimeter or by one of the methods given in Chapter II.

Ans. 363, 4·86; 364, 6·03; 365, 17·3; 366, 8·78; 367, 36·3; 368, 15·5; 369, 11·4; 370, 460.

17. When testing a 9" × 9" timber beam, the beam split along the grain by shearing along the neutral axis under a central load of 15 tons. Calculate the shear stress. *Ans.* 310 lbs. square inch.

18. Calculate the ratio of the maximum shear stress to the mean in the case of a square beam loaded with one diagonal vertical.

Ans. 1; *i.e.* they are equal.

19. Calculate the central deflection of a tram rail due to (i.) bending, (ii.) shear, when centrally loaded on a span of 3 feet 6 inches with a load of 10 tons. E = 12,300 tons square inch. I = 80·5 inch units; A = 10·5 square inches. G = 4900 tons square inch; K = 4·03.

Ans. (i.) 0·016 inch; (ii.) 0·008 inch.

Chapter X.

1. (Victoria, 1897.) The total load on the axle of a truck is 6 tons. The wheels are 6 feet apart, and the two axle-boxes 5 feet apart. Draw the curve of bending moment on the axle, and state what it is in the centre. *Ans.* 18 tons-inches.

2. (Victoria, 1896.) A beam 20 feet long is loaded at four points equidistant from each other, and the ends with equal weights of 3 tons. Find the bending moment at each of these points, and draw the curve of shearing force. *Ans.* 24, 36, 36, 24 tons-feet.

3. (I.C.E., October, 1898.) In a beam ABCDE, the length (AE) of 24 feet is divided into four equal panels of 6 feet each by the points B, C, D. Draw the diagram of moments for the following conditions of loading, writing their values at each panel-point: (i.) Beam supported at A and E, loaded at D with a weight of 10 tons; (ii.) beam supported at B and D, loaded with 10 tons at C, and with a weight of 2 tons at each end A and E; (iii.) beam *encastré* from A to B, loaded with a weight of 2 tons at each of the points C, D, and E.

Ans. (i.) $M_B = 15$, $M_C = 30$, $M_D = 45$ tons-feet.
(ii.) M_B and $M_D = 12$, $M_C = 18$ tons-feet.
(iii.) $M_B = 72$, $M_C = 36$, $M_D = 12$ tons-feet.

4. Find the bending moment and shear at the abutment, also at a section situated 4 feet from the free end in the case of a cantilever loaded thus: length, 12 feet; loads, 3 tons at extreme end, $1\frac{1}{2}$ ton 2 feet from end, 4 tons 3 feet 6 inches from end, 8 tons 7 feet from end.

Ans. M at abutment, 125 tons-feet; ditto 4 feet from end, 17 tons-feet.
Shear ,, 16·5 tons ,, ,, 8·5 ,,

5. Construct bending moment and shear diagrams for a beam 20 feet long resting on supports 12 feet apa't. The left-hand support is 3 feet from the end. The beam is loaded thus: 2 tons at the extreme left-hand end, 1 ton 2·5 feet from it, 3 tons 5 feet from it, 8 tons 12 feet from it, and 6 tons on the extreme right-hand end. Write the values of the bending moment under each load.

Ans. 30 tons-feet at the right abutment; 4·62 tons-feet under the 8-ton load; 1·42 under the 3-ton load; 6·5 over the left abutment.

6. A beam arranged with symmetrical overhanging ends, as in Fig. 404, is loaded with three equal loads—one at each end and one in the middle. What is the distance apart of the supports, in terms of the total length l, when the bending moment is equal over the supports and in the middle, and at what sections is the bending moment zero?

Ans. $\frac{4}{5}l$. At sections situated between the supports distant $\frac{l}{5}$ from them.

7. A square timber beam of 12 inches side and 20 feet long supports a load of 2 tons at the middle of its span. Calculate the skin stress at the middle section, allowing for its own weight. The timber weighs 50 lbs. per cubic foot. *Ans.* 537 lbs. per square inch.

8. The two halves of a flanged coupling on a line of shafting are accidentally separated so that there is a space of 2 inches between the faces. Assuming the bolts to be a driving fit in each flange, calculate the bending stress in the bolts when transmitting a twisting moment of 10,000 lbs.-inches. Diameter of bolt circle, 6 inches; diameter of bolts, $\frac{3}{4}$ inch; number of bolts, four. *Ans.* 9 tons square inch.

Chapter XI.

1. (Victoria, 1897.) A round steel rod $\frac{1}{2}$ inch in diameter, resting upon supports A and B, 4 feet apart, projects 1 foot beyond A, and 9 inches beyond B. The extremity beyond A is loaded with a weight of 12 lbs., and that beyond B with a weight of 16 lbs. Neglecting the weight of the rod, investigate the curvature of the rod between the supports, and calculate the greatest deflection between A and B. Find also the greatest intensity of stress in the rod due to the two applied forces. ($E = 30{,}000{,}000$.)

Ans. The rod bends to the arc of a circle between A and B. $\delta = 0·45$ inch; $f = 11{,}750$ lbs. square inch.

2. A cantilever of length l is built into a wall and loaded evenly. Find the position at which it must be propped in order that the bending moment may be the least possible, and find the position of the virtual joints or points of inflection.

(N.B.—The solution of this is very long.)

Ans. Prop must be placed $0·265l$ from free end. Virtual joints $0·55l$, $0·58l$ from wall.

3. Find an expression for the maximum deflection of a beam supported at the ends, and loaded in such a manner that the bending-moment diagram is (i.) a semicircle, the diameter coinciding with the beam; (ii.) a rectangle; height = $\frac{1}{2}$ the span (L).

Ans. (i.) $\frac{L^3}{17·7EI}$; (ii.) $\frac{L^3}{16EI}$.

4. What is the height of the prop relative to the supports in a centrally

propped beam with an evenly distributed load, when the load on the prop is equal to that on the supports?

Ans. $\frac{7}{1152} \frac{WL^3}{EI}$ below the end supports.

5. (S. and A., 1897.) A uniform beam is fixed at its ends, which are 20 feet apart. A load of 5 tons in the middle; loads of 2 tons each at 5 feet from the ends. Construct the diagram of bending moment. State what the maximum bending moment is, and where are the points of inflection.

Ans. $M_{max} = 20$ tons-feet close to built-in ends. Points of inflection 4·4 feet from ends.

6. Find an expression in the usual terms for the end deflection of a cantilever of uniform depth, but of variable width, the plan of the beam being a triangle with the apex loaded and the base built in.

Ans. $\delta = \frac{WL^3}{2EI}$.

(NOTE.—In this case and in that given in Question 12, the I varies directly as the bending moment, hence the curvature is constant along the whole length, and the beam bends to the arc of a circle.)

7. Find an expression for the deflection at the free end of a cantilever of length L under a uniformly distributed load W and an *upward* force W_0 acting at the free end. What proportion must W_0 bear to the whole of the evenly distributed load in order that the end deflection may be zero? What is the bending moment close to the built-in end?

Ans. $\frac{L^3}{EI}\left(\frac{W}{8} - \frac{W_0}{3}\right)$. W_0 must be $\frac{3}{8}$W; M at wall, $\frac{WL}{8}$.

8. Find graphically the maximum deflection of a beam loaded thus: Span, 15 feet; load of 2 tons, 2 feet from the left-hand end, 4 tons 3 feet from the latter, 1 ton 2 feet ditto, 3 tons 4 feet ditto, *i.e.* at 4 feet from the right-hand end. Take I = 242; E = 12,000 tons square inch.

Ans. 0·32 inch.

9. (I.C.E., October, 1898.) In a rolled steel beam (symmetrical about the neutral axis), the moment of inertia of the section is 72 inch units. The beam is 8 inches deep, and is laid across an opening of 10 feet, and carries a distributed load of 9 tons. Find the maximum fibre stress, also the central deflection, taking E at 13,000 tons square inch.

Ans. 7·5 tons square inch; 0·215 inch.

10. (I.C.E., February, 1898.) A rolled steel joist, 40 feet in length, depth 10 inches, breadth 5 inches, thickness throughout ½ inch, is continuous over three supports, forming two spans of 20 feet each. What uniformly distributed load would produce a maximum stress of 5½ tons per square inch? Sketch the diagrams of bending moments and shear force.

Ans. 0·31 ton foot run.

11. (I.C.E., October, 1898.) A horizontal beam of uniform section, whose moment of inertia is I, and whose total length is 2 L, is supported at the centre, one end being anchored down to a fixed abutment. Neglecting the weight of the beam, suppose it to be loaded at the other end with a single weight W. Find an expression for the vertical deflection at that end below its unstrained position.

Ans. $\delta = \frac{2WL^3}{3EI}$

12. If the plates of a laminated spring, such as is used for railway

rolling stock, be laid out flat side by side, they will form an approximately triangular surface. Such a spring may be regarded as a triangular beam whose depth = the thickness of one leaf, t, and whose breadth at the base = nb, where b = the breadth of one leaf, and n = the number of leaves, L = the length of the spring, all in inches. Find an expression for the deflection in terms of the weight W supported by the spring, the modulus of elasticity (E), and the above-mentioned quantities.

(N.B.—Experiments on the deflection of such springs show that the above formula is fairly correct when they are thoroughly well greased; but when the leaves are rough and dirty they usually lag behind, both when loading and when unloading, to almost 25 per cent. of the deflection.)

Ans. $\dfrac{ML^2}{8EI}$, or $\dfrac{3WL^3}{8Enbt^3}$.

Chapter XII.

1. A brickwork pier, 18 inches square, supports a load of 4 tons; the resultant pressure acts at a distance of 4 inches from the centre of the pier. Calculate the maximum and minimum stresses in the brickwork.

Ans. 4·15 tons square foot compression.
0·59 ,, ,, tension.

2. (I.C.E., 1897.) The total vertical pressure on a horizontal section of a wall of masonry is 100 tons per foot length of wall. The thickness of the wall is 4 feet, and the centre of stress is 6 inches from the centre of thickness of the wall. Determine the intensity of stress at the opposite edges of the horizontal joint. *Ans.* Outer edge, 43·75 tons square foot.
Inner ,, 6·25 ,, ,,

3. (I.C.E., October, 1898.) A hollow cylindrical tower of steel plate, having an external diameter of 3 feet, a thickness of $\frac{1}{2}$ inch, and a height of 60 feet, carries a central load of 50 tons, and is subjected to a horizontal wind-pressure of 56 lbs. per foot of its height. Calculate the vertical stresses at the fixed base of the tower on the windward and on the leeward side. (Allow for the weight of the tower itself.)

Ans. Windward side, 0·11 tons square inch.
Leeward ,, 2·09 ,, ,,

4. (Victoria, 1897.) A tension bar, 8 inches wide, $1\frac{1}{4}$ inch thick, is slightly curved in the plane of its width, so that the mean line of the stress passes 2 inches from the axis at the middle of the bar. Calculate the maximum and minimum stress in the material. Total load on bar, 25 tons.

Ans. Maximum, 6·25 tons square inch tension.
Minimum, 1·25 ,, ,, compression.

5. (Victoria, 1896.) If the pin-holes for a bridge eye-bar were drilled out of truth sideways, and the main body of the bar were 5 inches wide and 2 inches thick, what proportion would the maximum stress bear to the mean over any cross-section of the bar at which the mean line of force was $\frac{1}{8}$ inch from the middle of the section. *Ans.* 1·15.

6. The cast-iron column of a 100-ton testing-machine has a sectional area of 133 square inches. $Z_t = 712$; $Z_c = 750$. The distance from the line of loading to the c. of g. of the section is 17·5 inches. Find the maximum tensile and compressive skin stresses.

Ans. 3·08 tons square inch compression.
1·71 ,, ,, tension.

7. A tube in a bicycle frame is cranked $\frac{3}{4}$ inch, the external diameter

being 0·70 inch, and the internal 0·65 inch. Find how much the stress is increased in the cranked tube over that in a similar straight tube under the same load. *Ans.* Four times (nearly).

8. The section through the back of a hook is a trapezium with the wide side inwards. The narrow side is 1 inch, and the wide side 2 inches; the depth of the section is $2\frac{1}{2}$ inches; the line of pull is $1\frac{1}{2}$ inch from the wide side of the section. Calculate the load on the hook that will produce a tensile skin stress of 7 tons per square inch. *Ans.* 3·87 tons.

9. In the case of a punching-machine, the load on the punch is estimated to be 160 tons. It has a gap of 3 feet from the centre of the punch. The tension flange is 40″ × 3″, and the compression flange 20″ × 2″. There are two 2-inch webs; the distance from centre to centre of flanges is 4 feet. Calculate the stress in the tension flange. *Ans.* 2·05 tons square inch.

CHAPTER XIII.

1. Find the buckling load of a steel strut of 3 inches solid square section with rounded ends, by both Euler's and Gordon's formula, for lengths of 2 feet 6 inches, 9 feet, 15 feet. $E = 30{,}000{,}000$, $f = 49{,}000$, $S = 60{,}000$.
Ans. Euler, 2,250,000, 173,600, 62,500 lbs.
Gordon, 465,500, 176,000, 79,900 lbs.

2. Calculate the buckling load of a 2″ × 1·03″ rectangular section strut of hard cast iron, 16 inches long, rounded ends. Take the tabular values for a and s. *Ans.* 40·3 tons. (Experiment gave 41·6 tons.)

3. Calculate the buckling load of a piece of cast-iron pipe, length, 24 inches; external diameter, 4·4 inches; internal diameter, 3·9 inches; rounded ends. Take the tabular values for a and S.
Ans. 159 tons (hard), 98 tons (soft), 128 tons (mean). (Experiment, 113·2 tons.)

4. Calculate the buckling load of a cast-iron column, 9 feet long, external diameter, 3·56 inches; internal diameter, 2·91 inches; ends flat, but not fixed; flanges 8 inches diameter.
Ans. 78·7 tons (hard), 48·4 (soft), 63·6 (mean). (Experiment, 61 tons.)

5. Ditto, ditto, but with pivoted ends
Ans. 28·4 tons (hard). The iron was very hard, with fine close-grained fracture. (24·9 tons by experiment.)

6. Calculate the safe load for a cast-iron column, external diameter, 3·54 inches; internal, 2·86 inches; pivoted ends; loaded 2 inches out of the centre; safe tensile stress, 1 ton square inch. *Ans.* 2 tons (nearly).
(N.B.—Cast-iron columns loaded thus nearly always fail by tension.)

7. Calculate the buckling load of T iron struts as follows:—

	Area.	a
(1) $r = 8·2$	1·67 sq. ins.	$\frac{1}{467}$
(2) $r = 16·0$	1·67 ,,	$\frac{1}{467}$
(3) $r = 18·0$	1·95 ,,	$\frac{1}{321}$
(4) $r = 14·3$	1·20 ,,	$\frac{1}{461}$
(5) $r = 19·8$	2·04 ,,	$\frac{1}{300}$
(6) $r = 26·8$	1·20 ,,	$\frac{1}{461}$

Ans. Calculated, (1) 26·1 tons; (2) 19·2; (3) 21·5; (4) 14·8; (5) 20·3; (6) 8·4.
By experiment, 29·0 ,, 18·5 19·7 15·9 19·3 12·7.

8. A hollow circular mild-steel column is required to support a load of 50 tons, length, 28 feet; ends rigidly held; external diameter, 6 inches; factor of safety, 4. Find the thickness of the metal. *Ans.* 0·9 inch.

9. A solid circular-section cast-iron strut is required for a load of 20 tons, length, 15 feet; rounded ends; factor of safety, 6. Find the diameter. *Ans.* 6·25 inches.

(This is most easily arrived at by trial and error, by assuming a section, calculating the buckling load, and altering the diameter until a suitable size is arrived at.)

CHAPTER XIV.

1. (S. and A., 1897.) Find the diameter of a wrought-iron shaft to transmit 90 H.P. at 130 revolutions per minute. If there is a bending moment equal to the twisting moment, what ought to be the diameter? Stress = 5000 lbs. per square inch. *Ans.* 3·54 inches; 4·75 inches.

2. A winding drum 20 feet diameter is used to raise a load of 5 tons. If the driving shaft were in pure torsion, find the diameter for a stress of 3 tons per square inch. *Ans.* 12·68 inches.

3. Find the diameter for the above case if the load is accelerated at the rate of 40 feet per second per second when the winding engine is first started. *Ans.* 16·6 inches.

4. Find the diameter of shaft for a steam-engine having an overhung crank. Diameter of cylinder, 18 inches; steam pressure, 130 lbs. per square inch; stroke, 2 feet 6 inches; overhang of crank, *i.e.* centre of crank-pin to centre of bearing, 18 inches; stress, 7000 lbs. per square inch. *Ans.* 10 inches.

5. Find the diameter of a hollow shaft required to transmit 5000 H.P. at 70 revolutions per minute; stress, 7500 lbs. per square inch; the external diameter being twice the inner; maximum twisting moment = $1\frac{1}{2}$ times the mean. *Ans.* 16·9 inches.

6. A 4-inch diameter shaft, 30 feet long, is found to spring 6·2° when transmitting power; revolutions per minute, 130. Find the horse-power transmitted. G = 12,000,000. *Ans.* 187.

7. (Victoria, 1898.) If a round bar, 1 inch in diameter and 40 inches between supports, deflects 0·0936 inch under a load of 100 lbs. in the middle, and twists through an angle of 0·037 radian when subjected to a twisting moment of 1000 lbs.-inches throughout its length of 40 inches, find E, G, and K; Young's modulus, the moduli of distortion, and volume.

Ans. E = 29,020,000, G = 11,020,000, K = 26,350,000, all in pounds per square inch.

8. A square steel shaft is required for transmitting power to a 30-ton overhead travelling crane. The load is lifted at the rate of 4 feet per minute. Taking the mechanical efficiency of the crane gearing as 35 °/₀, calculate the necessary size of shaft to run at 160 revolutions per minute. The twist must not exceed 1° in a length equal to 30 times the side of the square. G = 13,000,000. *Ans.* 2 inches square.

9. A horse tram-car, weighing 5 tons, when travelling at 8 miles an hour, is pulled up by brakes. Find what weight of helical springs of

solid circular section would be required to store this energy. Stress in springs, 60,000 lbs. per square inch. *Ans.* 0·48 ton.

10. Calculate the amount a helical spring having the following dimensions will compress under a load of one ton. Number of coils, 20·5; mean diameter of spring, 2·5 inches; coils of square section steel, 0·51 inch side. *Ans.* 4·61 inches. (Experiment gave 4·57 inches.)

11. Calculate the stretch of a helical spring under a load of 112 lbs. Number of coils, 22; mean diameter, 0·75 inch. Diameter of wire, 0·14 inches. *Ans.* 1·80 inches. (Experiment gave 1·82 inches.)

CHAPTER XV.

1. Find the forces, by means of a reciprocal diagram, acting on the members of such a roof as that shown in Fig. 488, with an evenly distributed load of 16 lbs. per square foot of covered area. Span, 50 feet; distance apart of principals, 12 feet, which are fixed at both ends. Calculate the force *pa* by taking moments about the apex of the roof. Find the force *tu* by the method of sections. See how they check with the values found from the reciprocal diagram.

2. In the case given above, find the forces when one of the ends is mounted on rollers, both when the wind is acting on the roller side and on the fixed side of the structure, taking a horizontal wind-pressure of 30 lbs. per square foot on a vertical surface.

3. Construct similar diagrams for Fig. 489.

4. Construct a polygon and a reciprocal diagram for the Island Station roof shown in Fig. 490. Check the accuracy of the work by seeing whether the force F is equal to *li*.

5. (Victoria 1896.) A train of length T, weighing W tons per foot run, passes over a bridge of length l greater than T, which weighs w tons per foot run. Find an expression for the maximum shear at any part of the structure, and sketch the shear diagram.

Let y = the distance of the part from the middle of the structure.

$$\text{Shear} = \frac{\text{WT}}{2l}(l - \text{T} + 2y) + wy$$

N.B.—As far as the structure is concerned, the length of the train T is only that portion of it actually upon the structure at the time. This expression becomes that on p. 431, when the length of the train is not less than the length of the structure.

6. Find an expression for the focal length X of a bridge which may be entirely covered with a rolling load of W tons per foot run. The dead load on the bridge = w tons per foot run.

Let $\text{M} = \dfrac{w}{\text{W}}$.

$$x = l(1 - 2\sqrt{\text{M} + \text{M}^2} + 2\text{M})$$

7. A plate girder of 50 feet span, 4 feet deep in the web, supports a uniformly distributed load of 4 tons per foot run. Find the thickness of web required at the ends. Shear stress in web and rivets, 4 tons square

inch. Also find the pitch of rivets, $\frac{3}{4}$ inch diameter, for shearing and bearing stress. Bearing pressure, 9 tons square inch.

Ans. Thickness, $\frac{1}{2}$ inch. Pitch for single row in shear, 2·52 inches; better put two rows of say 5-inch pitch. Ditto for bearing, 2·44 inches and say 5 inches.

8. Find the skin stress due to change of curvature in a two-hinged arch rib, on account of its own dead weight, which produces a mean compressive stress f_c of 6 tons per square inch. $E = 12{,}000$ tons square inch. Span, 550 feet; rise, 114 feet; depth of rib, 15 feet. Also find the deflection due to a stationary test load which produces a further mean compressive stress f_c of 1 ton per square inch.

Ans. Stress, 0·8 ton square inch; $\delta = 0{\cdot}79$ inch.

(NOTE.—The above arch is that over the Niagara River. Some of the details have been assumed, but are believed to be nearly accurate. When tested, the arch deflected 0·81 inch.)

9. Bridge members are subjected to the following loads. State for what static loads you would design the members: (i.) Due to a dead load on the structure of 10 tons in tension, and a live load of 30 tons in tension. (ii.) Dead, 100 tons tension; live, 30 tons compression. (iii.) Dead, 80 tons tension; live, 60 tons compression. (iv.) Dead, 50 tons tension; live, 70 tons compression.

Ans. (i.) 70 tons tension. (ii.) 100 tons tension; when under the action of the live load, the stress is diminished. (iii.) Either for 80 tons tension or 40 tons compression, whichever gave the greatest section. (iv.) 90 tons compression.

CHAPTER XVI.

1. Taking the weight of 1 cubic foot of water at 62·5 lbs. at atmospheric pressure and at 60° Fahr., calculate the weight of one cubic foot when under a pressure of 3 tons per square inch and at a temperature of 100° Fahr. $K = 140$ tons square inch. *Ans.* 63·48 lbs.

2. Find the depth of the centre of pressure of an inclined rectangular surface making an angle of 30° with the surface, length, 10 feet; bottom edge 15 feet below surface and horizontal. *Ans.* 12·6 feet.

3. (Victoria, 1897.) Find the horizontal pull on a chain fixed to the top of a dock gate to keep it from overturning, there being no resistance at the sides of the gate, which is hinged horizontally at the bottom. Height of gate, 40 feet; depth of water on one side, 35 feet. Depth of water on the other side, 23 feet; width of gate, 70 feet; weight of salt water, 64 lbs. per cubic foot. *Ans.* 290 tons.

4. The height of a dam such as that shown in Fig. 521 is 8 feet, and the width 3 feet 6 inches; the back stay is inclined at an angle of 45°. Calculate the pressure on the stay. *Ans.* 9950 lbs.

5. Calculate the number of cubic feet per hour that will pass through a plain orifice in the bottom of a tank; diameter, 1 inch; head, 8·2 inches; $K = 0{\cdot}62$. *Ans.* 80·6.

6. Find the size of a circular orifice required in the bottom of a tank to pass 10,000 gallons per hour. $h = 3$ feet; $K = 0{\cdot}62$.

Ans. 3·07 inches diameter.

7. Calculate the coefficient of discharge for a pipe orifice having a slightly rounded mouth, the coefficient of contraction for which is 0·75

when running a clear stream without touching the sides of the pipe. The discharge is reduced 5 per cent. by friction. *Ans.* 0·90.

8. How many gallons per minute will be discharged through a short pipe 2 inches diameter leading from and flush with the bottom of a tank? Depth of water, 25 feet. *Ans.* 268.

9. In the last question, what would be the discharge if the pipe were made to project 6 inches into the tank, the depth of water being the same as before? *Ans.* 172.

10. In an experiment with a diverging mouthpiece, the vacuum at the throat was 18·3 inches of mercury, and the head of water over the mouthpiece was 30 feet, the diameter of the throat 0·6 inch. Calculate the discharge through the mouthpiece in cubic feet per second. *Ans.* 0·112.

11. (I.C.E., October, 1897.) A weir is 30 feet long, and has 18 inches of head above the crest. Taking the coefficient at 0·6, find the discharge in cubic feet per second. *Ans.* 265·2.

12. A rectangular weir, for discharging daily 10 million gallons of compensation water, is arranged for a normal head over the crest of 15 inches. Find the length of the weir. Take a coefficient of 0·7. *Ans.* 2·36 feet.

13. Find the number of cubic feet of water that will flow over a right-angled V notch per second. Head of water over bottom of notch, 4 inches; $K = 0·6$. *Ans.* 0·165.

14. (I.C.E., February, 1898.) The miner's inch is defined as the flow through an orifice in a vertical plane of 1 square inch in area under an average head of 6½ inches. Find the water-supply per hour which this represents. *Ans.* 1·5 cubic foot; per minute when $K = 0·61$.

15. (Victoria, 1898.) Find the discharge in gallons per hour from a circular orifice 1 inch in diameter under a head of 2 feet, the pipe leading to the orifice being 6 inches in diameter. *Ans.* 885.

16. A horizontal pipe 4 inches diameter is reduced very gradually to half an inch. The pressure in the 4-inch pipe is 50 lbs. square inch above the atmosphere. Calculate the maximum velocity of flow in the small portion without any breaking up occurring in the stream. *Ans.* 1·53.

17. A slanting pipe 2 inches diameter gradually enlarges to 4 inches; the pressure at a given section in the 2-inch pipe is 25 lbs. square inch absolute, and the velocity 8 feet per second. Calculate the pressure in the 4-inch portion at a section 14 feet below. *Ans.* 31·2 lbs. square inch.

18. In the last question, calculate the pressure for a sudden enlargement. *Ans.* 30·96 lbs. square inch.

19. (I.C.E., October, 1898.) A pipe tapers to one-tenth its original area, and then widens out again to its former size. Calculate the reduction of pressure at the neck, of the water flowing through it, in terms of the area of the pipe and the velocity of the water.

$$Ans.\ p_1 - p_2 = \frac{wV_1^2}{2g}(100 - 1).$$

N.B.—The 100 is the ratio $\left(\frac{A_1}{A_2}\right)^2$.

20. Find the loss of head due to water passing through a socket in a pipe, the sectional area of the waterway through the socket being twice that of the pipe. Velocity in pipe, 10 feet per second. *Ans* 0·85 feet.

21. Find the loss of head due to water passing through a diaphragm in a pipe, the area being one-third of that of the pipe. Velocity, 10 feet per second. *Ans.* 21·3 feet.

22. A locomotive scoops up water from a trough 1 mile in length. Calculate the number of gallons of water that will be delivered into the tender when the train travels at 40 miles per hour. The water is lifted 8 feet, and is delivered through a 4-inch pipe. The loss of head by friction and other resistances is 30 %. *Ans.* 2130.

23. The head of water in a tank above an orifice is 4 feet. Calculate the time required to lower the head to 1 foot. The area of the surface of the water is 1000 times as great as the sectional area of the jet.

Ans. 250 seconds.

24. (I.C.E., February, 1898.) A canal lock with vertical sides is emptied through a sluice in the tail gates. Putting A for area of lock basin, *a* for area of sluice, H for the lift, find an expression for the time of emptying the lock. (K = coefficient of contraction.) *Ans.* $\dfrac{2A\sqrt{H}}{Ka\sqrt{2g}}$

25. (I.C.E., October, 1898.) A pipe 2 feet diameter draws water from a reservoir at a level of 550 feet above the datum; it falls for a certain distance, and again rises to a level of 500 feet 5 miles from its starting-point; it then falls to a reservoir at a level of 400 feet 1 mile away. Calculate the rate of delivery into the lower reservoir.

Ans. 10 cubic feet per second.

26. (I.C.E., October, 1898.) Taking skin friction to be 0·4 lb. per square foot at 10 feet per second, find the skin resistance in pounds of a ship of 12,000 square feet immersed surface at 15 knots (a knot = 6086 feet per hour). Also the horse-power to overcome skin friction.

Ans. 30,860 lbs. resistance; 1423 H.P.

27. (I.C.E., October, 1898.) A pipe 12 inches diameter connects two reservoirs 5 miles apart, and having a difference of level of 20 feet. Find the velocity of flow and discharge of the pipe, taking the coefficient of friction at 0·01.

Ans. 1·41 feet per second; 1·1 cubic foot per second.

28. (Victoria, 1897.) The jet condenser of a marine engine is 15 feet below the surface of the water. Suppose a barometer connected with the condenser to stand at 5 inches (say 30 inches of vacuum); find the quantity of water discharged per second into the condenser through a pipe 1 inch diameter and 20 feet long. The contraction on entering the pipe may be taken at 0·5. *Ans.* 0 1 cubic feet.

29. (Victoria, 1896.) Water is admitted by a contracting mouthpiece from the bottom of a tank into a 4-inch pipe 600 feet long, and then allowed to escape vertically as a fountain through a 2-inch nozzle into the air, the nozzle being 100 feet below the level of the surface in the tank. Find the height above the nozzle to which the water will rise if the coefficient of resistance of the pipe is 0·005. *Ans.* 30 feet.

30. (Victoria, 1898.) Find the amount of water in gallons per day which will be delivered by a 24 inch cast-iron pipe, 3 miles in total length, when the surface of the water under which it discharges is 175 feet below the surface of the reservoir from which it is drawn, (i.) when the mouth is full open; (ii.) when restricted by a conical nozzle to one-fourth the area.

(NOTE.—When a pipe is provided with a nozzle, the total energy of the water is equal to the frictional resistance due to V, and the kinetic energy of the escaping water at the velocity V_1, together with the resistance at entry into pipe, and resistances due to sudden enlargements, etc., if any.)

Ans. (i.) 12,580,000; (ii.) 12,120,000.

31. Calculate the velocity of discharge by Thrupp's formula in the case of a new cast-iron pipe, 0·2687 feet diameter, having a fall of 28 feet per mile. $n = 1{\cdot}85$, $C = 0{\cdot}005347$, $x = 0{\cdot}67$. *Ans.* 1·81 foot per second.

32. Water is conveyed from a reservoir by a pipe 1 foot diameter and 2 miles long, which contains, (i.) a short sudden enlargement to three times its normal diameter; (ii.) a half-closed sluice-valve; (iii.) a conical nozzle on the bottom end of one-fifth the area of the pipe. The nozzle is 80 feet below the surface of the water in the reservoir. Find the discharge in cubic feet per second. *Ans.* 3·3.

33. A tank in the form of a frustum of a cone is 8 feet diameter at the top, 2 inches at the bottom, and 12 feet high. Find the time required to empty the tank through the bottom hole. $K = 0{\cdot}9$. *Ans.* 430 seconds.

CHAPTER XVII.

1. Calculate the approximate hydraulic and total efficiencies of an overshot water-wheel required for a fall of 40 feet. If 30 cubic feet of water be delivered per second, find the useful horse-power of the wheel.

Ans. 78 per cent. hydraulic efficiency.
70 ,, total ,,
95 H.P.

2. (Victoria, 1895.) Find the diameter of a ram necessary for an accumulator, loaded with 100 tons, in order that 50 H.P. may be transmitted from the accumulator through a pipe 2000 yards long and 4 inches in diameter with a loss of 2 H.P. *Ans.* 19·8 inches.

3. (Victoria, 1896.) The velocity of flow of water in a service pipe 48 feet long is 66 feet per second. If the stop-valve be closed so as to bring the water to rest uniformly in one-ninth of a second, find the (mean) increase of pressure near the valve, neglecting the resistance of the pipe. *Ans.* 384 lbs. square inch.

4. If the water in the last question had been brought suddenly to rest, *i.e.* in an infinitely small space of time, what would have been the resulting pressure. *Ans.* 4190 lbs. square inch.

5. A rotary motor is driven by water from a supply-pipe 200 feet in length. The diameter of the pipe is 3 inches, and the piston of the motor 4 inches; the stroke is 6 inches; length of connecting-rod, 1 foot; revolutions, 120 per minute. Calculate the inertia pressure at the end of the "out" stroke. *Ans.* 247 lbs. square inch.

6. Calculate the horse-power that can be obtained for one minute from an accumulator having a ram of 20 inches diameter, 23-feet stroke, loaded to a pressure of 750 lbs. per square inch. *Ans.* 164.

7. A hydraulic crane, having a velocity ratio of 8 to 1, is required to lift a load of 5 tons. Taking the efficiency of the chain gear at 80 per cent., and the loss of pressure by friction as 90 lbs. per square inch, find

the size of ram required for a pressure in the mains of 700 lbs. per square inch. *Ans.* 15·3 inches.

8. Calculate the side pressure per square foot of projected area on the piers of a bridge standing in the middle of a river, velocity of stream 8 miles per hour, (i.) when the piers present a flat surface to the stream; (ii.) when they are chamfered off at an angle of 45°.

Ans. (i.) 134 lbs.; (ii.) 39 lbs.

9. A jet of water 1½ inch diameter, moving at 50 feet per second, impinges normally on a series of flat vanes moving at a velocity of 20 feet per second. Find the pressure exerted on the vanes. *Ans.* 35·7 lbs.

10. A jet of water, 2 inches in diameter, moving at a velocity of 60 feet per second, glides without shock on to a series of smooth curved vanes moving in a direction parallel to the jet with a velocity of 35 feet per second. The last tip of the vane makes an angle of 60° with the first tip. Find the pressure exerted on the vanes. *Ans.* 31·7 lbs.

11. Find the total efficiency of a Pelton wheel working under the following conditions: Diameter of nozzle, 0·494 inch; diameter of brake wheel, 12 inches; net load on brake, 8·8 lbs.; revolutions per minute, 538; weight of water used per minute, 330 lbs. *Ans.* 66·4 per cent.

12. (I.C.E., February, 1898.) In an inward-flow turbine, the water enters the inlet circumference, 2 feet diameter, at 60 feet per second and at 10° to the tangent to the circumference. The velocity of flow through the wheel is 5 feet per second. The water leaves the inner circumference, 1 foot diameter, with a radial velocity of 5 feet per second. The peripheral velocity of the inlet surface of the wheel is 50 feet per second. Find the angles of the vanes (to the tangent) at the inlet and outlet surfaces.

Inlet, 49°; outlet, 11°.

Chapter XVIII.

1. Find the quantity of water delivered and the horse-power required to drive a single-acting pump working under the following conditions: diameter of pump-barrel, 2 feet; length of stroke, 6 feet; slip, 4 per cent.; head of water on pump, 50 feet, exclusive of friction; speed of flow in main, 3 feet per second; length of main, 1 mile; strokes of pump, 20 per minute; mechanical efficiency, 80 per cent.

Ans. 135,500 gallons per hour; 52·4 H.P.

2. Find the horse-power required to drive a feed-pump for supplying a 100 H.P. boiler working at 130 lbs. per square inch, reckoning 30 lbs. of water per horse-power hour. Mechanical efficiency of pump, 60 per cent.

Ans. 0·76.

3. (I.C.E., February, 1898.) A steam pump is to deliver 1000 gallons of water per minute against a pressure of 100 lbs. per square inch. Taking the efficiency of the pump to be 0·7, what indicated horse-power must be provided? *Ans.* 100.

4. Find the speed of a centrifugal pump having radial vanes in order to lift the water to a height of 20 feet: the outside diameter of the vanes is 18 inches. Neglecting all sources of loss.

Ans. 322 revolutions per minute.

5. Find the speed of a centrifugal pump having curved vanes as in Fig.

609, with no volute, required to lift water to a height of 20 feet, the outside diameter of the vanes being 18 inches. $V_r = \frac{\sqrt{2gH}}{4}$, $V_w = 20$ feet per second, neglecting losses by friction. *Ans.* 563 revolutions per minute.

6. What is the hydraulic efficiency, neglecting friction, of the pump in the last question? *Ans.* 83 per cent.

7. In the case of the pump given in Question 5, calculate the horse-power absorbed by friction on the outside of the two discs. Taking the inner diameter as one-half the outer, and the resistance per square foot at 10 feet per second at 0·8 lb. *Ans.* 1·1.

INDEX

GREEK ALPHABET USED.

THE END.

PRINTED BY WILLIAM CLOWES AND SONS, LIMITED, LONDON AND BECCLES.